Special Session 2: Nonlinear Evolution PDEs and Interfaces in Applied Sciences

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Many issues in applied science can be formulated as interface problems which can be regarded as limiting cases of evolution equations exhibiting transition layers. The study of phase field or diffusive interface problems, Allen-Cahn and Cahn-Hilliard equations have been an active area for the past few decades. This has also been an additional motivation for studying general nonlinear evolution equations. This session will focus on the mathematical properties of these equations including well-posedness, regularity, stability and asymptotic behavior of solutions, as well as their implications for applications.

Cahn-Hilliard Equation with Nonlocal Singular Free Energies

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Stefano Bosia, Maurizio Grasselli

We consider the Cahn-Hilliard equation with a nonlocal free energy. In contrast to previous works the nonlocal part of the free energy is given by a strongly singular integral kernel, which gives rise to an integro-differential operator similar to a fractional power of the Laplacian. Moreover, the homogenous free energy density is singular as well. We prove existence of a unique solution for all times and the existence of a global attractor. Finally, we discuss the boundary regularity of the solutions.

Non-smooth degenerating elliptic equations for damage models

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Francesco Freddi, Antonio Segatti

The thermo-mechanical behaviour of (visco)elastic and special materials is determined by the coexistence of different configurations at the micro-scale, changing due to thermo-mechanical loads (as in the case of degrading mechanical properties in quasi-brittle materials). The continuum damage theory, introduced in the framework of solid-solid phase transitions, turns out to perform a good description of the evolving damage phenomenon in materials losing their stiffness. In particular, an order parameter delineates the state of the cohesion in the body. The resulting PDE system we investigate couples thermo-mechanical actions and phase dynamics and it is strongly nonlinear and possibly degenerating in the elliptic equations for deformations. We discuss existence of a solution in the case of complete damage. This problem is still open in its original formulation, but some answer has been given for suitable weak notion of solutions. We introduce a new notion of solution, accounting for a new “interior” stress which implicitly accounts for the state of damage of the material and write an appropriate free energy. We recover a new PDE system describing complete damage and we show the existence of a weak solution.

Outflow boundary conditions for non-homogeneous flows

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In this talk, I will describe some theoretical results concerning (nonlinear) outflow boundary conditions for the incompressible non-homogeneous Navier-Stokes equations. I will also present a few numerical results showing the influence of the outflow boundary conditions modeling for instance in the framework of diffuse interface models for multiphase flows.

Global weak solution and blow-up criterion of the general Ericksen-Leslie system for nematic liquid crystal flows.

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Elisabetta Rocca, Hao Wu

We investigate the three dimensional general Ericksen-Leslie (EL) system with Ginzburg-Landau type approximation modeling nematic liquid crystal flows. First, we prove existence of global-in-time weak solutions under physically meaningful boundary conditions and suitable assumptions on the Leslie coefficients, which ensures that the total energy of the EL system is dissipated. Moreover, for the EL system with periodic boundary conditions, we prove the local well-posedness of classical solutions under the so-called Parodi’s relation and establish a blow-up criterion in terms of the temporal integral of both the maximum norm of the curl of the velocity field and the maximum norm of the gradient of the liquid crystal director field.

On a generalized Cahn-Hilliard equation with biological applications

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A. Miranville, S. Zelik

We will discuss the asymptotic behavior of a generalization of the Cahn-Hilliard equation with a proliferation term and endowed with Neumann boundary conditions. Such a model has, in particular, applications in biology. We show that either the average of the local density of
cells is bounded, in which case we have a global in time solution, or the solution blows up in finite time. We will also prove that the relevant, from a biological point of view, solutions converge to 1 as time goes to infinity. We will end with some numerical simulations which confirm the theoretical results.

On a variation of the Cahn-Hilliard approach

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A toy problem which results from a variation of the Cahn-Hilliard model will be introduced and some results will be discussed. This results fro a research project with E. Bonetti and G. Tomassetti.

Asymptotic structure of the attractor for processes on time-dependent spaces

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Vittorino Pata

We compare the asymptotic structure of the time-dependent attractor $A_t$ generated by the partial differential equation

$$
\epsilon u_{tt} + \alpha u_t - \Delta u + f(u) = g,
$$

where the positive function $\epsilon = \epsilon(t)$ tends to zero as $t \to \infty$, with the global attractor $A_\infty$ of its formal limit

$$
\alpha u_t - \Delta u + f(u) = g.
$$

We establish an abstract result and we apply it to the proof of the convergence $A_t \to A_\infty$.

On the stability of the weak attractor of the 3D Navier-Stokes equations

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We consider the three-dimensional Navier-Stokes-Voigt (NSV) equations and we analyze, from the asymptotic behavior viewpoint, its Navier-Stokes (NS) limit as the relaxation parameter vanishes. We show that the NSV-attractors converge to the weak NS-attractor in the Hausdorff semidistance induced by the weak $L^2$-metric on the absorbing set of the Navier-Stokes equations. Some results related to the strong topology of $L^2$ are also proved.

On optimal mixing schemes

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Optimal stirring is an important issue in chemical engineering. The underlying optimization problem is the following: given a color function $c$ transported with a solenoidal velocity, what is the velocity $(t,X) \to V(t,X)$ that ensures the quickest mixing? Of course, the answer depends on: - The definition of the mixing criterion - The energy constraint on the velocity Recent works of Mathew et. al. have shown that a good criterion for measuring the mixing of two fluids in the periodic case is the $H^{-\frac{1}{2}}$ norm. An explicit locally-in-time optimal mixing scheme has been suggested in subsequent works from Lin et. al. In this talk we will investigate this mixing scheme both numerically and through a linear stability analysis, in a framework that is as general as possible. In particular we will show the ill-posedness of the linearized model when the energy constraint on the velocity is taken as the kinetic energy, and the well-posedness when this energy constraint is the viscous dissipation energy.

Asymptotic behavior of a generalization of the Caginalp phase-field system

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Monica Conti, Alain Miranville

We discuss the asymptotic behavior of a generalized nonlinear Caginalp phase-field system, based on the theory of heat conduction of type III devised by Green and Naghdi. In the case of two nonlinearities of polynomial critical growth, we prove the existence of global attractors of optimal regularity.

Grisvard’s shift theorem near $L^\infty$ and Yudovich theory in domains with corners

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Roger Temam

The planar Euler equations describe the motion of a 2-D inviscid incompressible fluid, and also arise as a model problem for the study of the barotropic mode (to put it simply, the vertical average) of the Primitive equations of the ocean. It is a result by Yudovich that, in the space-periodic case, there exist a unique weak solution to the Euler system whenever the initial data has bounded vorticity. Relying on a refinement of the sharp Grisvard’s shift theorem on domains with corners, we prove an $L^\infty$ version of Grisvard’s shift theorem for domains with corners, and extend the Yudovich theory of weak solutions for the Euler equations to this class of domains. We also discuss analogous results for the barotropic mode of the Primitive equations. This is partly joint work with Roger Temam and Claude Bardos.

Regularized families of simplified Ericksen-Leslie (RSEL) models

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Theodore Tachim Medjo

We consider a general family of regularized flows for the simplified Ericksen-Leslie (RSEL) model for the hydrodynamics of liquid crystals in 2 and 3-dimensional
compact Riemannian manifolds. The system contains the Navier-Stokes equations, the Navier-Stokes-Voight and the Navier-Stokes alpha-model equations as special cases, and many others. We establish existence, stability, regularity results and singular perturbation results, and we also show the existence of a global attractor and exponential attractor for the general family. Then we establish precise conditions under which each trajectory converges to a single equilibrium by means of a Łojasiewicz-Simon inequality.

**Evolution equations of fast phase transitions in solutions and applications**

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A model of fast phase transitions is analyzed. Examples of transitions described by non-linear evolution equations are given for a number of tasks from material physics. In particular, analytical and numerical solutions of problems on spinodal decomposition, eutectic transformation, rapid solidification, and order-disorder transition are discussed.

**Dynamics of fluid interfaces with surface viscosity and Helfrich forcing**

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In two-phase flow the interface often has important additional aspects. First we discuss the Boussinesq-Scriven surface fluid model and a model for biomembranes due to Arroyo and DeSimone taking curvature elasticity forcing into account. We then present analytical properties of the model and introduce a parametric finite element method for the evolution problem. For the discrete equations we show stability estimates and finally we present numerical computations in two and three space dimensions which show different phenomena stemming from surface viscosity and from forces originating from the Helfrich energy.

**Pullback exponential attractor for a Cahn-Hilliard-Navier-Stokes system in 2D**

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We address the long term behavior of a 2D–Cahn-Hilliard-Navier-Stokes system with polynomial double-well potential, proving that it possesses a pullback exponential attractor. In particular the regularity estimates we obtain depend on the initial data only through fixed powers of their norms and these powers are independent of the growth of the polynomial potential considered in the Cahn-Hilliard equation.

**Well-posedness for the Navier-slip thin-film equation in complete wetting**

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We are interested in the thin-film equation with quadratic mobility, modeling the spreading of a thin liquid film with a Navier-slip condition at the solid substrate. This degenerate fourth-order parabolic equation has the contact line (where liquid, solid, and vapor meet) as a free boundary. There, a zero-contact angle condition is imposed, modeling the so-called “complete wetting” regime. We first argue that the self-similar source-type solution, once its leading order profile is factored-off, is analytic as a function of two variables \((x,x^β)\) with \(β\) irrational, where \(x\) denotes the distance from the contact line. Motivated by this preliminary, we then argue that the full free-boundary problem is well-posed in weighted \(L^2\)-spaces which capture the leading order terms of such \((x,x^β)\) expansion. This is part of a joint project with Manuel V. Gnann, Hans Knüpfer, and Felix Otto.

**On the Cahn-Hilliard equation with dynamic boundary conditions and a dominating boundary potential**

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Several results for the Cahn-Hilliard equation with dynamic boundary conditions are known. In particular, well posedness holds if the potential entering the equations in the domain dominates the potential involved in the boundary condition. In this talk, the opposite case is considered, i.e., the dominating potential is the boundary potential. Well posedness and regularity results obtained in a joint paper with P. Colli and J. Sprekels are presented both for the Cahn-Hilliard equation and for the viscous Cahn-Hilliard equation. Moreover, for the latter, an optimal control problem is discussed.

**The Gel’fand problem for the biharmonic operator**

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We study stable solutions of the equation \(\Delta^2 u = e^u\) in bounded domains \(Ω ⊆ \mathbb{R}^N\). We prove that the extremal solutions are smooth if \(N ≤ 12\).

**On convergent numerical schemes for two-phase flow of incompressible fluids with different mass densities**

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In this talk, we will be concerned with convergence results of numerical schemes for diffuse interface models for two-
phase flow of immiscible, incompressible viscous fluids with different mass densities. In contrast to the case of identical mass densities, for general mass densities only recently diffuse interface models have been suggested which are consistent with thermodynamics and which allow for a solenoidal velocity field (see Abels, Garcke, Grün M3AS 2012). These models consist of a new momentum equation for the velocity field coupled to a Cahn-Hilliard equation for the evolution of the order parameter. A subtle discretization of the convective coupling between the flux of the phase-field and the moment-turn equation allows to formulate a numerical scheme which satisfies a discrete counterpart of the energy estimate. By higher regularity results for discrete solutions of convective Cahn-Hilliard equations, we prove its convergence in two and in three space dimensions. Finally, we shall present numerical simulations to underline the full practicality of our approach and to identify physical settings for which the new coupling term suggested in (Abels, Garcke, Grün, M3AS 2012) seems to be indispensable for numerical stability.

**Convergence to equilibrium for smectic-A liquid crystals in 3D domains**

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In this talk, we focus on a smectic-A liquid crystal model in 3D domains, obtaining three main results: the proof of an adequate Lojasiewicz-Simon inequality in a strong framework, the rigorous proof (via a Galerkin approach) of the existence of global in time weak solutions which are strong (and unique) for large times, and the convergence to equilibrium of the whole trajectory as time goes to infinity. Given any regular initial data, the existence of a unique global in time regular solution (bounded up to infinite time) and the convergence to an equilibrium have been previously proved, but under the constraint of large enough viscosity. Now, all results are obtained without imposing large viscosity.

**The singular limit of an Allen-Cahn equation with a random source term**

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We revisit the singular limit of an Allen-Cahn equation in the case of a random source term which only depends on time; the scaling which we consider is such that the limiting motion equation does not involve the mean curvature of the moving interface. Our main results involve generation and propagation of interface properties.

**A well posedness result for nonlinear viscoelastic equations with memory**

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Monica Conti, Vittorino Pata

In this talk we discuss existence, uniqueness and continuous dependence results for the weak solutions to a nonlinear viscoelastic equation with hereditary memory on a bounded three-dimensional domain. This equation arises in mechanics, in the description of the vibrations of thin rods whose material density is not constant.

**Asymptotic behaviour for a double time-delayed 2D-Navier-Stokes model**

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Julia García-Luengo, Gabriela Planas

In this contribution we analyze the well-posedness and asymptotic behaviour of a double time-delayed 2D-Navier-Stokes model with delay effects in the nonlinear term. Some surprising facts arise in this simple perturbation, for instance about uniqueness of solution, and therefore a suitable approach must be used in order to ensure the existence of attractor. For a suitable choice of the phase-space we prove that the problem is certainly well-posed and have an attractor. This is a joint work with Julia García-Luengo, from Universidad de Sevilla (Spain), and Gabriela Planas, from Universidade Estadual de Campinas (Brazil).

**Analysis of a fully nonlinear reaction-diffusion system describing multicomponent reactive flows**

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We consider combustion problems in the presence of complex chemistry and nonlinear diffusion laws leading to fully nonlinear multispecies reaction-diffusion equations. We establish results of existence of solution and maximum principle, i.e. positivity of the mass fractions, which rely on specific properties of the models. The nonlinear diffusion coefficients are obtained by resolution of the so-called Stefan-Maxwell equations.

**The phase-field transition system endowed with a general regular potential and nonlinear dynamic boundary conditions of non-homogeneous type and non-constant thermal conductivity**

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The work is devoted to the study of a Caginalp phase-field transition system, endowed with a general regular potential and a general class of nonlinear and non-homogeneous dynamic boundary conditions (in both unknown functions), as well as non-constant thermal conductivity. The existence, uniqueness and regularity of solutions is established. This extends previous works, including the already studied boundary conditions, which makes the
A numerical analysis of the Cahn-Hilliard equation with non-permeable walls

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Laurence Cherfils

In this talk we consider the numerical analysis of the Cahn-Hilliard equation in a bounded domain with non-permeable walls, endowed with dynamic-type of boundary conditions. The dynamic-type boundary conditions that we consider were proposed by Goldstein, Miranville and Schimperna in order to describe the interactions with the wall of a binary material. The equation is semi-discretized using a finite element method for the space and error estimates between the exact and the approximation solution are obtained. We also prove the stability of a fully discrete problem based on the backward Euler scheme for the time discretization.

Regularity results for a Cahn-Hilliard-Navier-Stokes system with shear dependent viscosity

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We consider a diffuse interface problem, where the fluid is described by a non-Newtonian type model with shear dependent viscosity of polynomial growth. We will discuss the qualitative behavior of solutions, with particular emphasis on the regularity results.

Entropic solutions to a PDE system for phase transitions and damage in thermo-viscoelastic materials

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We focus on the analysis of a PDE system modelling (non-isothermal) phase transitions and damage in thermo-viscoelastic materials. The model is thermodynamically consistent: in particular, no small perturbation assumption is adopted, which results in the presence of quadratic terms on the right-hand side of the temperature equation. The whole system has a highly nonlinear character. We address the existence for a weak notion of solution, referred to as “entropic”, where the temperature equation is formulated with the aid of an entropy inequality, and of a total energy inequality. This solvability concept reflects the basic principles of thermomechanics and the thermodynamical consistency of the model, and allows us to obtain global-in-time existence theorems without imposing any restriction on the size of the initial data. We prove our results by passing to the limit in a time-discrete scheme carefully tailored to the nonlinear features of the PDE system and of the a priori estimates performed on it.

Strichartz estimates and smooth attractors for wave equations with fractional damping in bounded domains

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We consider Dirichlet problem for a semi-linear wave equation with damping term \((-\Delta)^{\alpha} u_t\), where \(\alpha \in [0, \frac{1}{2}]\), in a bounded smooth domain \(\Omega \subset \mathbb{R}^d\), assuming initial data from usual energy space \(H^1(\Omega) \times L^2(\Omega)\). First, we establish control of \(L^5([0,T];L^{10}(\Omega))\) norm of solutions for corresponding linear non-autonomous problem in terms of energy norm, which does not follow from energy estimate as well as Strichartz estimates for pure wave equation. Then treating semi-linear equation as perturbation of the linear problem we establish its well-posedness in the class of energy solutions with finite \(L_t^3(R_+;L^{10}(\Omega))\) norm. Moreover, we show that solutions from the mentioned class possess smoothing property analogous to solutions of parabolic equation. Finally we show that dynamical system generated by these solutions possesses a smooth global attractor.

From fractional Cahn Hilliard to fractional Porous Medium

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Goro Akagi, Giulio Schimperna

In this talk I will report on a joint work with Goro Akagi (Kobe) and Giulio Schimperna (Pavia) in which we highlight some relations between a fractional Cahn-Hilliard type equation and a fractional Porous Medium equation. In particular, we will start with a fractional Cahn-Hilliard equation (for which we show existence and uniqueness of solutions) and we will rigorously prove that when the fractional order of differentiation tends to zero the solutions will converge to the solution of a fractional Porous Medium equation.
Nonlinear evolution as convex minimization
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I intend to overview some global-in-time variational techniques for evolution equations. In particular, we shall be interested in the possible reformulation of evolution systems, either of dissipative or dispersive type, in terms of minimization problems. The idea is that of moving the successful machinery of the Calculus of Variations (direct method, gamma-convergence, relaxation..) to evolutionary situations. I will in particular present results for gradient and doubly nonlinear flows as well as for nonlinear waves. This is joint work with Goro Akagi.

Finite Number of Determining Parameters for the Navier-Stokes Equations with Applications into Feedback Control and Data Assimilations
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In this talk we will implement the notion of finite number of determining parameters for the long-time dynamics of the Navier-Stokes equations (NSE), such as determining modes, nodes, volume elements, and other determining interpolants, to design finite-dimensional feedback control for stabilizing their solutions. The same approach is found to be applicable for data assimilations. In addition, we will show that the long-time dynamics of the NSE can be imbedded in an infinite-dimensional dynamical system that is induced by an ordinary differential equations, named determining form, which is governed by a globally Lipschitz vector field. The NSE are used as an illustrative example, and all the above mentioned results hold also to other dissipative evolution PDEs.

Flows in karst geometry
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Karst type geometry is a particular type of configuration that consists of both conduit/channel (or vug/chamber) together with porous media. Many important applications involve fluid flows in karstic geometry. Well-known examples include contaminant transport in karst aquifer, oil recovery in karst oil reservoir, proton exchange membrane fuel cell technology, cardiovascular modeling, and carbon-dioxide sequestration among others. The mathematical study of flows in karst geometry is a challenge due to the coupling of the flows in the conduits and flows in the surrounding matrix which are governed by different physical processes, the possibly complex geometry of the network of conduits, the vastly disparate spatial and temporal scales, the strong heterogeneity and the enormous associated uncertainty with natural karst aquifer, and the multi-phase nature of many important applications. In this talk, we will present recent results on the modeling, analysis and simulation of single phase as well as two-phase flows in karstic geometry.

Rayleigh-Taylor instability for the two-phase Navier-Stokes equations with surface tension
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This talk is concerned with the dynamic behaviour of two immiscible and incompressible fluids in a cylindrical container, e.g. a capillary, which are separated by a sharp interface. In case that the heavier fluid overlies the lighter fluid one expects that the heavier fluid sinks down into the lighter one. This effect is known as Rayleigh-Taylor instability. The main result yields the existence of a critical surface tension with the following property. In case that the surface tension of the interface between the two fluids is smaller than the critical surface tension, one has Rayleigh-Taylor instability. On the contrary, if the interface has a greater surface tension than the critical value, the instability effect does not occur and one has exponential stability of the interface. The last part of the talk is concerned with the bifurcation of nontrivial equilibria in multiple eigenvalues. The invariance of the bifurcation equation with respect to rotations and reflections yields the existence of bifurcating subcritical equilibria. Finally it is proven that the bifurcating equilibria are unstable.