Special Session 41: Topological and Variational Methods for Multivalued Differential Equations

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Multivalued differential equations arise naturally in various branches of modern mathematics, indeed they play a significant role in the description of processes in control theory and non-smooth analysis. Due to their wide applicability in physics, biology, chemistry, and economics, there has recently been an increasing interest in this field. The aim of this Special Session is to outline the recent progress on the field of differential inclusions, both in infinite and finite dimensional spaces. In particular, topics of interest are control problems, non-smooth variational analysis, differential equations with discontinuous nonlinearities, functional inclusions, nonlocal problems, and boundary value problems in bounded and unbounded domains. Due to the diversity of applications and the variety of problems, there is a wide range of methods and techniques available. In this Session, both variational methods (e.g. critical point theory, linking theorems) as well as topological methods (e.g. fixed points theorems, lower and upper solutions, topological degree) will be presented and discussed.

Hartman-type conditions for multivalued Dirichlet problem

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The application of the classical Hartman-type conditions, as presented in [Trans. Amer. Math. Soc. 96 (1960), 493–509] i.e. sign conditions w.r.t. the first state variable and growth conditions (sometimes called as the Bernstein–Nagumo–Hartman conditions) w.r.t. the second state variable, will be discussed, on various levels of abstraction, for a multivalued Dirichlet problem. For instance, in abstract spaces, if the right-hand sides are Marchaud (i.e. globally upper-semicontinuous) and condensing, then the growth conditions can be very liberal. On the other hand, for multivalued upper-Carathéodory condensing right-hand sides, the situation is more delicate. Nevertheless, the related obtained criteria can be still not worse than those of Hartman, i.e. as for vector equations in finite-dimensional spaces. The approach is based on the combination of topological degree arguments, bounding (Liapunov-like) functions and a Scorza–Dragoni approximation technique. An illustrative application of the main existence and localization results can concern partial integro-differential equations involving discontinuities in state variables.

Topological methods and degree for compact multivalued perturbations of Fredholm maps in Banach spaces and applications

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We present a construction of an oriented topological degree theory for locally compact multivalued perturbations of Fredholm maps of index zero in Banach spaces. The construction is based on an infinite dimensional notion of orientation for nonlinear Fredholm maps and by an analogous concept of degree for Fredholm maps of index zero. This theory is applied to existence and bifurcation problems for differential inclusions.

Some qualitative properties of solution sets for a fractional integro-differential inclusion

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We study fractional integro-differential inclusions of the form

\[ D^\alpha_0 x(t) \in F(t, x(t), V(x)(t)) \quad \text{a.e. } ([0, T]), \]

\[ x(0) = x_0, \quad x'(0) = x_1, \]

where \( \alpha \in (1, 2] \), \( D^\alpha_0 \) is the Caputo fractional derivative, \( F : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) is a set-valued map and \( x_0, x_1 \in \mathbb{R}, \quad x_0, x_1 \neq 0 \).

\( V : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R}) \) is a nonlinear Volterra integral operator defined by \( V(x)(t) = \int_0^t k(t, s, x(s)) \, ds \) with \( k(\ldots) : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) a given function. We prove the arcwise connectedness of the solution set this problem when the set-valued map is Lipschitz in the second and third variable. Moreover, under such type of hypotheses on the set-valued map, we establish a more general topological property of the solution set of our problem. Namely, we prove that the set of selections of the set-valued map \( F \) that correspond to the solutions of the problem is a retract of \( L^1([0, T], \mathbb{R}) \). Both results are essentially based on Froyzasowski, Bressan and Colombo results concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values.

Multiplicity results for Neumann-type differential inclusion problems with variable exponent

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We study the existence of at least three distinct solutions for the following Neumann-type differential inclu-
sion problem involving the $p(\cdot)$-Laplacian:

$$
\begin{align*}
(\mathcal{N}_p) & \quad -\Delta_{p(x)} u + a(x)|u|^{p(x)-2}u \in \lambda \mathcal{F}(x,u) \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial \Omega$, $a$ is a suitable function belonging to $L^\infty(\Omega)$, $\Delta_{p(x)} u := \text{div}(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplace operator related to a convenient function $p$ of $C(\Omega)$, $\nu$ is the outer normal unit vector to $\partial \Omega$, $\lambda$ is a positive parameter and $\mathcal{F}(x,\xi)$ is the generalized gradient with respect to $\xi$ of a fixed function $F$ defined on $\Omega \times \mathbb{R}$. The results are obtained by using a multiple critical points theorem for locally Lipschitz continuous functionals.

**Fourth-order hemivariational inequality problem**

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In this talk we will deal with a fourth-order hemivariational inequality with boundary constraints. By using the nonsmooth critical point theory we will prove the existence of infinitely many solutions for our problem.

**Variational differential inclusions arising in optimal control**

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In this talk we will discuss second order variational inclusions leading to second order necessary optimality conditions in optimal control. In this talk we discuss second order variational inclusions leading to second order necessary optimality conditions. More precisely, consider a differential inclusion under state constraints on a given time interval $[0,T]$ and its set of solutions $S$. Given $\bar{x} \in S$, it is possible to associate to it a variational differential inclusion whose solutions are tangent to $\bar{S}$ at $\bar{x}$. Denote by $V(\bar{x})$ the set of all such solutions and let $\bar{g}(\cdot) \in V(\bar{x})$. We introduce a second order variational differential inclusion along $(\bar{x},\bar{y})$ whose solutions are second order tangents to $\bar{S}$ at $(\bar{x},\bar{y})$. The presence of state constraints is handled thanks to a recent Neighbouring Feasible Trajectories Theorem. As a consequence, a new pointwise second order condition verified by the adjoint state of the maximum principle is obtained for the Mayer optimal control problem.

**Viable periodic trajectories in totally leaky sets with barriers**

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The talk is devoted to the existence of solutions to the following single-valued or multivalued differential problem $\dot{x}(t) = f(x(t))$ [resp. $\in F(x(t))$] for a.e. $t \geq 0$.

**The generalized Krasnosel'skii formula and bifurcations of closed orbits to semilinear inclusions**

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**Wojciech Kryszewski**

The classical Krasnosel’skii formula relates the Brouwer degree of the right-hand side of a differential equation in $\mathbb{R}^n$ with the Brouwer fixed point index of the corresponding Poincaré operator (or translations along trajectories). We apply the similar idea to the parameterized semilinear differential inclusion

$$
(\ast) \quad \dot{u} \in A u + F_1(t,u), \quad t \in J, u \in E
$$

where $J = [0,\infty)$, $\lambda \in \Lambda \subset \mathbb{R}^k$, $E$ is a Banach space, $A : D(A) \to E$ is the infinitesimal generator of the $C_0$-semigroup of bounded linear operators on $E$ and $F : \Lambda \times J \times E \to \mathbb{R}^k$ is a set-valued weakly upper semicontinuous map with convex weakly compact values. Namely we compare the respective homotopy invariants of the right-hand side and of the operator $\Phi_t$ which assigns to the initial value $x \in E$ the set $\{u(t) ; u$ is a (mild) solution of $\ast$ starting at $x\}$. It allows to detect bifurcations of closed orbits to $\ast$. We also present how one can apply this method to a concrete reaction-diffusion inclusion.

**Multiple solutions to a Neumann differential inclusion via Morse theory**

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**Francesca Colasuonno, Dimitri Mugnai**

We study a partial differential inclusion driven by the $p$-Laplacian operator, with a locally Lipschitz potential and a Neumann boundary condition. The potential is assumed to be $p$-superlinear at infinity. Following Wang (1991), we prove the existence of a positive and a negative solution via variational methods. Then, using the nonsmooth critical groups and the nonsmooth Morse
identity introduced by Corvellec (1995), we prove the existence of a third non-zero solution.

The Krasnosel’skiĭ formula for constrained semilinear differential equations

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In the talk we will study the relations between the fixed point index of the Poincaré translation operator generated by the the following semilinear differential equation

\[ \dot{u} \in Au + F(t,u), \quad t \in J, \quad u \in C, \]

where \( J := [0,T], \ T > 0, \ C \subset E, \ E \) is a Banach space, \( C \) is a closed convex set, \( A : D(A) \to E, \ D(A) \subset E, \) is the generator of a compact \( C_0 \)-semigroup \( S = \{ S(t) \}_{t \geq 0} \) of bounded linear operators on \( E \) and \( F : J \times E \to E \) is a continuous map or a set-valued weakly upper semicontinuous map with convex weakly compact values. It is to be noted that many partial differential equations (or systems of such equations), e.g. of parabolic (in particular nonlinear reaction-diffusion equations) or hyperbolic type (or inclusions), can be transformed so as to have this form. The presence of the constraining set \( C \) is justified by applications. With this problem one associates the Poincare translation operator \( \Phi_t \), where \( t > 0 \), which assigns to each initial value \( x \in E \) the set of values \( u(t) \), where \( u \) the solution starting at \( x \) with sufficiently small \( t > 0 \) which exhibits small \( t > 0 \) which gives the formula relating the Granas fixed point index of \( \Phi_t \) on \( C \) with the appropriately defined constrained topological degree of the right hand side.

Non-smooth critical point theory on closed convex sets and applications

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A unified approach to some classes of evolution equations and systems with nonlocal conditions

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We present the study of the existence of global solutions for a general semilinear evolution equation in a Banach space \( X \) under the effect of a nonlocal condition expressed by a linear continuous mapping \( F : C([0,a];X) \to X \). A transition from Volterra to Fredholm integral operator associated to the problem appears as a consequence of the specific nature of the nonlocal map \( F \). Further, both the classical Cauchy problem and the Byszewski one, where the nonlocal condition is dissipated on the entire interval \([0,a]\), are recovered as special cases. Thanks to a matrix approach, the results are extended to systems of equations in such a way that the system nonlinearities behave independently as much as possible.

Exact non-local controllability of semilinear differential inclusion

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The main aim of our presentation is to study exact non-local controllability of solutions for semilinear inclusion in a reflexive Banach space \( E \). More precisely, we shall establish sufficient conditions for controllability of non-linear evolution system with generalized non-local condition of the form:

\[
\begin{align*}
\dot{y}(t) & \in A(t,y(t))y(t) + F(t,y(t)) + Bu(t), \\
\int_{0}^{T} y(t)\ dt & = M(y),
\end{align*}
\]

where \( F : J \times E \to E \) is a bounded, closed, multivalued map with convex values, \( A(t,y) \) is a linear continuous operator on \( E \), for each \( (t,y) \in J \times E \), control \( u(\cdot) \) is a given function from \( L^2(J, U) \), a space of admissible control functions with \( U \) as reflexive Banach space, and \( B \) is a bounded linear operator from \( U \) to \( E \). The assumptions regarding the operator \( M : C(J,E) \to E \) will be stated explicitly in the presentation. Sufficient conditions are formulated and proved using a fixed point theorem. Finally, we present an example to illustrate application of the proposed method.

Delay evolution inclusions with general nonlocal initial conditions

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We prove and existence result for bounded \( C^0 \)-solutions to a class of nonlinear delay differential evolution inclusions subjected to nonlocal implicit initial conditions. We assume that the m-dissipative part is the infinitesimal generator of a nonlinear compact semigroup of contrac-
tions, while the multi-valued forcing term is nonempty, convex, weakly compact valued and almost strongly-weakly u.s.c. The main difficulty here is that the history constraint function $g$, which is assumed to be nonexpansive, has affine, instead of linear, growth and thus $g(0)$ is allowed to be nonzero.

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