# Special Session 68: Entropy-Like Quantities and Applications

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Entropy is a general concept that appears in different settings with different meanings. Thus, it measures disorder in physics, uncertainty in information theory, minimum code length in coding theory, (pseudo-)randomness in measure-preserving dynamical systems, complexity in topological dynamics, and algorithmic complexity in computer science. As for its importance, let us remind that it enters the second axiom of thermodynamics, related to the direction of time in many-particle mechanical systems. In information theory it defines the very concept of information, beside lying at the core of the fundamental results. And, last but not least, in ergodic theory entropy is perhaps the most important invariant of metric and topological conjugacy, which are the equivalence concepts in measure-preserving and topological dynamics, respectively. In the last decades new versions of entropy has come to the fore. Sequence entropy, correlation entropy, permutation entropy, transfer entropy, approximate entropy, sample entropy, etc. are some of the entropy-like quantities proposed by researchers to cope with new challenges in ergodic theory, chaos, synchronization and control, information theory, time series analysis, etc. Along with these new developments, some traditional topics, like the computation (and even computability) of metric and topological entropy, still remain the subject of current research in applied mathematics. Also in physics, entropy is the objective of undiminished research activity, topics ranging from axiomatic aspects to its formulation in non-stationary processes. This Special Session is organized with the scope that researchers on the theoretical or practical aspects of entropy and akin quantities can share their interests and latest results in a multidisciplinary environment. Therefore, participants from all fields of science, mathematics and engineering are very much welcome.

## *The power of the min-max symbols*

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Min-max symbols are a convenient generalization of the well-known kneading symbols in one-dimensional dynamics generated by multimodal mappings. From the computational point of view, min-max symbols require virtually the same effort as the kneading ones, with the plus that they provide additional information about the minimim/maximum character of the iterates of the critical values. We will present some theoretical and practical results which involve min-max symbols, including the computation of topological entropy.

## *Applications of symbolic dynamics in the context of cryptography*

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The relationship between the dynamical systems theory and the principles of cryptography was underlined by Shannon in his seminal work about perfect secrecy. Indeed, it is possible to interpret the main requirements of encryption systems by means of general concepts of dynamical systems theory, as the sensitivity to initial conditions/control parameters and the ergodicity property. However, this relationship could imply security flaws when an attacker can link partial information extracted from the encryption process to a subset of the secret keys of the related cryptosystem. In this work we highlight the outcomes of the applied theory of symbolic dynamics as a tool to detect such a vulnerability.

## *On non-autonomous systems through perturbations*

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A great variety of problems on discrete dynamical systems and difference equations, both depending on parameters, can evolve to non-autonomous when such parameters are perturbed with a perturbation depending on $n$. We are dealing with this problem and give results proving that even when the perturbations has a small amplitude, the dynamical behavior of the system or equation can change drastically In particular, we will use trigonometric and general perturbations trough sinam or cosam Jacobi functions. We will give results on Lyapunov stability, entropy and other dynamics properties.

## *Order patterns: a diagnostic tool for time series*

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Permutation entropy is widely used to evaluate the complexity of time series of moderate length. For long time series, multiscale analysis of all order patterns will give more information, as we shall demonstrate with examples from medicine, wheather, internet traffic, and music. A very interesting case is pink noise where frequencies of order patterns do not depend on the scale, We analyse several models and applications of pink noise and present a simple new model.

## *On the shape of isentropes for multimodal maps.*

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**Sebastian van Strien**

It is well-known that topological entropy. depends monotonically on the parameter in the family of quadratic maps. The same was proved by Milnor and Tresser for cubic maps, and recently monotonicity of entropy for polynomials of arbitrary degree was established by Bruin and Van Strien. This means that the isentropes (i.e., level sets of entropy) are connected subset of parameter, but I will demonstrate in this talk that the topological shape of such sets can be very complicated still, and many questions remain open.

## *Significance Test for Mutual Information*

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A common experimental goal is to detect dependency among multiple time series based on a limited number of observations. Mutual information is an information theoretic measure that is often used to detect dependency. Although the mutual information is theoretically zero for independent processes, estimates derived using finite data sets are imprecise and ambiguous. In this case, one must distinguish nonzero values to determine a likelihood of dependency. A rigorous approach is to use a statistical significance test to assess the null hypothesis of independent processes. In this talk, we present a significant test for mutual information that is accurate for finite (small) data sets. The key development is a method for generating and uniformly sampling surrogates from the set of all sequences that exactly match the $n$-th order properties of the observed data sets. Examples using coupled chaotic maps demonstrate the effectiveness of the test.

## *Interdependencies in climate networks constructed using information measures at different time scales*

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We study global climate networks constructed by means of ordinal time series analysis. Climate interdependencies among the nodes are quantified by mutual information, computed from time series of surface air temperature anomalies, and from their symbolic ordinal representation (OP). This analysis allows identifying topological changes in the network when varying the time-interval of the ordinal pattern. We consider intra-season time-intervals (e.g., the patterns are formed by anomalies in consecutive months) and inter- annual time-intervals (e.g., the patterns are formed by anomalies in consecutive years). We discuss how the network density and topology change with these time scales, and provide evidence of correlations between geographically distant regions that occur at specific time scales. In particular, we find that an increase in the ordinal pattern spacing (i.e., an increase in the timescale of the ordinal analysis), results in climate networks with increased connectivity on the equatorial Pacific area. As the equatorial Pacific is known to be dominated by El Niño-Southern Oscillation (ENSO) on scales longer than several months, our methodology allows constructing climate networks where the effect of ENSO goes from mild (monthly OP) to intense (yearly OP), independently of the length of the ordinal pattern and of the thresholding method employed.

## *g-Entropies: its connections with Shannon and KS entropies and an invariant based on this concept*

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Considering a concave function $g\colon [0,1] \mapsto \mathbb{R}$ vanishing at the origin instead of the Shannon entropy function $\eta(x) = -x \ln x$, $\eta(0) := 0$, leads to the generalizations of dynamical and measure-theoretic entropies . We will show the connections of these generalizations with the dynamical and Kolmogorov-Sinai entropies. We will also introduce the concept of types of $g$-entropy convergence rates introduced by Blume [1] and discuss some results concerning this quantity.

References

[1] F. Blume. Possible rates of entropy convergence *Ergodic Theory & Dynam. Systems* (1997), **17**, 45-70.

## *Finite-time entropy: a probabilistic approach for measuring nonlinear stretching*

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Transport and mixing processes in dynamical systems are often difficult to study analytically and therefore a variety of numerical methods have been developed. Finite-time Lyapunov exponents (FTLEs) or related stretching indicators are frequently used as a means to estimate transport barriers. Alternatively, eigenvectors, singular vectors, or Oseledets vectors of numerical transfer operators find almost-invariant sets, finite-time coherent sets, or time-asymptotic coherent sets, respectively, which are minimally dispersed under the dynamics. While these families of approaches (geometric FTLEs and the probabilistic transfer operator) often give compatible results, a formal link is still missing; here we present a small step toward providing a mathematical link. We propose a new entropy-based methodology for estimating finite-time expansive behaviour along trajectories in autonomous and nonautonomous dynamical systems. We introduce the finite-time entropy (FTE) field as a simple and flexible way to capture nonlinear stretching directly from the entropy growth experienced by a small localised density evolved by the transfer operator. The FTE construction elucidates in a straightforward way the connection between the evolution of probability densities and the local stretching experienced. We develop an extremely simple and numerically efficient method of constructing an estimate of the FTE field. The FTE field is instantaneously calculable from a numerical transfer operator – a transition matrix of conditional probabilities that describes a discretised version of the dynamical system; once one has such a transition matrix, the FTE field may be computed "for free". We also show (i) how to avoid long time integrations in autonomous and time-periodic systems, (ii) how to perform backward time computations by a fast matrix manipulation rather than backward time integration, and (iii) how to easily employ adaptive methods to focus on high-value FTE regions.

## *Entropy-like quantities in the assessment of cardiovascular regulation*

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**Beata Graff, Roberto Monetti, Jose M. Amigo, Krzysztof Narkiewicz**

The interaction between various regulatory mechanisms in healthy persons is still not fully understood. Entropy-like quantities are valuable methods which allow the assessment of the information flow between biological time series. We discuss their application to elucidate the effect of hyperoxia on cardiovascular regulation in healthy subjects.

## *Evolution of random Boolean network toward close to criticality based on local transfer entropy*

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Random Boolean network (RBN) is an abstract model of gene regulation networks. It is known that RBN exhibits a continuous order-disorder phase transition. In recent years, it has been shown that real-world gene regulation networks are working at close to criticality. It has been suggested that optimization of information transfer on gene regulation networks has certain evolutionary advantage because there are evidences that information processing ability of RBN is maximized at criticality. Some authors have been proposed adaptive network models based on simple local rewiring rules and showed that their models evolve toward close to criticality by numerical simulations. However, the role of information transfer in the course of evolution is still unclear in these models because the rewiring rules include no quantities related to information transfer. Here, we propose a new adaptive RBN model whose local rewiring rule involves local information transfer through a single node in the network. Local information transfer is quantified by the local transfer entropy which reflects the quality of information transfer over a single link. We show that our model can evolve toward close to criticality by both numerical simulation and master equation analysis of the in-degree distribution.

## *A theorem in queueing theory with applications to queueing networks*

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**Louis Block, Jo Ann Lee, Lenka Rucka**

Let $T^n$ be an $n-$torus and let $A : T^n \to T^n$ be a hyperbolic automorphism on $T^n$. We give a method with a theoretical basis for using $A$ to generate a sequence of pseudo-random numbers on $[0, 1]^n$. Such a pseudo-random number generator would have useful applications.

## *On determining the Kolmogorov-Sinai entropy*

**Karsten Keller**
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Recent results show that there is a close relation between the Kolmogorov-Sinai entropy and the relatively new concept of Permutation entropy. Here we discuss these results in a broader framework. In particular, we show that under certain separation conditions the distribution of ordinal patterns, which describe the up and down in a system, is sufficient for determining the Kolmogorov-Sinai entropy. Moreover, we adress methods of estimatiing Kolmogorov-Sinai entropy on the base of the given approach and compare them with known methods.

## *Symbolic Correlation Integral*

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We introduce the symbolic correlation integral SC(m), which avoids the noisy parameter "distance" of the classical correlation integral defined by Grassberger-Procaccia. Moreover we provide the asymptotic distribution of SC(m) under the null of i.i.d.. With a MonteCarlo simulation we show the size and the power performance of the new test under linear and nonlinear processes.

## *Periodic points of latitudinal sphere maps*

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For the maps of the two-dimensional sphere into itself that preserve the latitude foliation and are differentiable at the poles, lower estimates of the number of fixed points for the maps and their iterates are obtained. Those estimates give also show that the growth rate of the number of fixed points of the iterates is larger than or equal to the logarithm of the absolute value of the degree of the map.

## *Characterizing thalamocortical information flow using ordinal symbolic measures*

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The thalamus and the cerebral cortex are reciprocally connected brain structures. In the physiological realm, the path of information flow during spontaneous activity is still being debated. Here we investigate information flow within the cortical-thalamo-cortical loop during slow oscillations ($\leq$1Hz), i. e. spontaneous activity emerging during slow wave sleep and anaesthesia. We apply information directionality measures within the framework of the Ordinal Symbolic Dynamics to local field potential recordings simultaneously obtained from the visual thalamus and visual cortex layers 2/3 and 5. We compared across two types of waves, spontaneous and evoked by visual stimulation. In the case of spontaneous activity, we find evidence that supports models triggered by the cortical layer 5. Thus, in the absence of external stimuli cortical layer 5 leads the flow of information towards other cortical layers and thalamus. In the evoked waves, the thalamus plays a more important role, triggering activity on cortical layers 2/3 as expected from the anatomical connectivity. However, our information flow measurements reveal that cortical layer 5 has still a prominent role inducing activity on the remaining cortical layers. We will discuss how this information measurement matches and enhances the current understanding of the anatomical/physiological models of connectivity.

## *Computing the topological entropy of continuous maps using kneading sequences*

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Following the ideas of Block, Keesling, Li and Peterson we introduce an algorithm to compute the topological entropy, with prescribed accuracy, of maps with four pieces of monotonicity. We also show how to extend our algorithm for multimodal maps with at most three different kneading sequences. As an application, we compute the topological entropy of 3–periodic sequences of logistic maps.

## *Temporal symbolic transfer entropy: Measuring information transfer in real-time*

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In nature, dynamical systems with many interacting elements, such as neural networks, often change their modality of couplings over time. Here, we propose an information theoretic measure that effectively monitors these dynamic changes for information transfer. Dealing with time series in real-time inevitably introduces two issues: the impossibility of predefining the range of the state space and of obtaining the probability distribution of occurrence of states, because, in principle, the entire time series are not provided beforehand. We here propose a measure called "temporal symbolic transfer entropy" to overcome these issues in assessing information transfers in real-time. The measure is based on transfer entropy, with two modifications in its setting; one is the use of the permutation partitioning, and second is an introduction of the time evolution scheme of the probability distributions. We will illustrate the power of this measure in a number of experiments using time series data from model systems, neuroscience, and soft robotics by comparison with conventional approaches.

## *Entropy, inverse limits and attractors*

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Works of Barge and Martin provide a nice tool for construction of attractors by inverse limits of one dimensional maps. It was also discovered that topological structure of these inverse limits (attractors) is related to dynamical properties of bonding maps such as structure of periodic orbits or topological entropy. In this talk we will present some more recent result related to that topic, in particular rigidity of topological entropy when the attractor is a hereditarily indecomposable continuum.

## *Identifying dynamical features using ordinal patterns*

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Ordinal pattern statistics is a versatile method for analyzing time series from dynamical systems. In this contribution we shall present and discuss applications such as signal classification, parameter estimation, detection of structural changes, and synchronization analysis.

## *Detecting order in an ordinal pattern*

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In practical time series analysis we are dealing with samples of finite size $T$, often $10 \ll T < 1000$. On an ordinal level, such a sample forms an ordinal pattern (OP). If there are no tied ranks, the OP is a permutation of $(1, 2, \ldots, T)$. If the series is a finite realization of a continuous iid process, each possible OP occurs with the same probability $1/T!$. This is called completely unordered behavior. However, a given process might not generate all these permutations. This holds especially for chaotic time series. In the talk some proposals are made to detect order by a recurrence analysis within an OP leading to entropy–like quantities.

## *Maximum Entropy Complex Networks, and how to measure them*

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**Kevin Judd and Thomas Stemler**

We use an edge-switching algorithm to produce random realisations of a particular network degree distribution. These networks, we claim are maximum entropy realisations. That is, they are random realisations of a given class of networks. When we apply this to the particular case of scale-free complex networks and we find a richer variety of complex networks than typified by (for example) preferential attachment. For weighted (and unweighted) complex networks we define a path-length depedent measure of variability in the structure of the network: this quantity we call the network entropy - as it is defined in an entropy-like manner based on the homogeneity of path-lengths between random nodes. We study the variation in this property across network classes and apply it to the special case of networks generated from time series via an ordinal partitioning of the quantised scalar signal.

## *Ordinal time-series analysis applied to the characterization of a forced excitable system*

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Semiconductor lasers have been used in the last few decades to experimentally explore a variety of dynamical phenomena, among them regular and extreme pulses, multistability, intermittency, quasiperiodicity and chaos. When a semiconductor laser is operated near the threshold current with a moderate level of optical feedback, the output intensity shows sudden, apparently random, dropouts known as low frequency fluctuations (LFF). In the LFF regime semiconductor lasers exhibit excitable behavior and, through the modulation of the injection current, the system can be easily periodically forced. In this work we study the dynamics of such a system using ordinal time-series analysis. This method takes into account the order of consecutive inter-dropout time intervals and allows us to uncover serial correlations in the sequence of intensity dropouts exhibited by the laser. We transform the sequence of inter-dropout intervals into a sequence of ordinal patterns and we analyze the statistics of the patterns and of the transitions between them. We unveil correlations among several consecutive dropouts and we identify clear changes in the dynamics as the modulation amplitude and frequency vary.

## *Conditional entropy of ordinal patterns and its possible applications*

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In this talk we discuss a quantity called conditional entropy of ordinal patterns (CEofOP). It is similar to the celebrated permutation entropy: the latter characterizes the diversity of ordinal patterns themselves, whereas the CEofOP – the average diversity of the ordinal patterns succeeding a given ordinal pattern. We observe that in several relatively simple cases including systems with regular dynamics and Markov shifts over the binary alphabet, the CEofOP for a finite order $d$ coincides with the Kolmogorov-Sinai entropy, while the permutation entropy only asymptotically approaches it. Moreover, we demonstrate that under certain assumptions CEofOP provides a better estimation of the KS entropy than the permutation entropy. Finally we discuss possible applications of the CEofOP to the segmentation of time series.

## *Relationship between permutation entropy and Kolmogorov-Sinai entropy with examples from interval maps*

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In this talk we discuss the relationship between permutation entropy and Kolmogorov-Sinai entropy. The considerations are based on comparing two special partitions of the state space induced by ordinal patterns. An ordinal pattern of order $d$ describes the order relations between the components of a $(d+1)$-dimensional vector. We consider, on the one hand, the partition $\mathcal{P}(d+n-1)$ provided by ordinal patterns of order $d+n-1$ and, on the other hand, the partition $\mathcal{P}(d)_n$ provided by $n$ successive ordinal patterns of order $d$. Due to recent results, the answer to the question of how much more information longer ordinal patterns of order $d+n-1$ provide than $n$ successive shorter ordinal patterns of order $d$ gives an approach to comparing the entropies. We present some extreme examples shedding some new light on the problem of coincidence of the entropies. We also discuss some combinatorial properties regarding the partitions $\mathcal{P}(d+n-1)$ and $\mathcal{P}(d)_n$, which provide a better understanding of the considered problem.

## *Generalized entropies, Large Deviations, and Multifractal Analysis*

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Generalized entropies appear naturally in problems of multifractal analysis, but also in the so-called large deviations rate functions of dynamical systems. This link will be discussed, using standard, as well, as some and novel examples of multifractal spectra.