1. Introduction

Let \((M, g)\) be an \(n\)-dimensional smooth compact Riemannian manifold without boundary and write \(\Delta_g\) for the (non-negative) Laplace operator acting on \(L^2(M, g)\). For each \(\lambda \geq 0\) we consider the space

\[
\mathcal{M}_\lambda = \bigoplus_{\mu \in [\lambda, \lambda+1]} \ker(\Delta_g - \mu^2). \tag{1}
\]

We set \(m_\lambda := \dim \mathcal{M}_\lambda\) and fix an orthonormal basis \(\{\varphi_{j,\lambda}\}_{j=1}^{m_\lambda}\) of \(\mathcal{M}_\lambda\) consisting of eigenfunctions for \(\Delta_g\):

\[
\Delta_g \varphi_{j,\lambda} = \mu_{j,\lambda}^2 \varphi_{j,\lambda}, \quad \|\varphi_{j,\lambda}\|_{L^2} = 1. \tag{2}
\]

The purpose of this note is to prove a simple fact about the geometry of the immersions \(\Phi_\lambda : M \to \mathbb{R}^{m_\lambda}\) defined by

\[
\Phi_\lambda(x) := \frac{1}{k_\lambda} (\varphi_{1,\lambda}(x), \ldots, \varphi_{m_\lambda,\lambda}(x)) \quad x \in M, \tag{3}
\]

where

\[
k_\lambda := \frac{\sqrt{2}}{k_\lambda} \frac{\Gamma \left( \frac{m_\lambda+1}{2} \right)}{\Gamma \left( \frac{m_\lambda}{2} \right)}. \tag{4}
\]
That $\Phi_\lambda$ are immersions for $\lambda$ sufficiently large follows from Theorem 2 below. The constants $k_\lambda$ satisfy $k_\lambda^2 = m_\lambda + o(1)$ and, as we show in $\S$2.1, ensure that

$$\lim_{\lambda \to \infty} \|\Phi_\lambda(x)\|_{L^2(\mathbb{R}^{m_\lambda})} = \frac{1}{\sqrt{\text{vol}(M)}}. \quad (5)$$

Each immersion $\Phi_\lambda$ defines a (pseudo-)distance function $d_\lambda : M \times M \to \mathbb{R}$ by restricting the ambient Euclidean $L^2$ distance:

$$d_\lambda(x, y) := \|\Phi_\lambda(x) - \Phi_\lambda(y)\|_{L^2(\mathbb{R}^{m_\lambda})}. \quad (6)$$

Our main result relates $d_\lambda$ to the Riemannian distance $d_g$ on $M$ induced by $g$.

Recall that $(M, g)$ is said to be an aperiodic manifold if for every $x \in M$, the set of vectors $\xi$ in $T_x M$ for which the geodesic with initial conditions $(x, \xi)$ returns to $x$ has Liouville measure 0. Any manifold with strictly negative sectional curvatures is aperiodic. In contrast, Riemannian manifolds whose geodesic flow is periodic are called Zoll manifolds. Examples of Zoll manifolds include the spheres $S^n$ endowed with the round metric.

**Theorem 1.** Let $(M, g)$ be a smooth compact Riemannian manifold without boundary that is either Zoll or aperiodic. There exists a constant $C$ so that for all $x, y \in M$ and all $\lambda \geq 0$

$$d_\lambda(x, y) \leq C \cdot \lambda d_g(x, y). \quad (7)$$

Relation (5) shows that the diameter of $M$ with respect to $d_\lambda$ is bounded so that (7) is a non-trivial statement only when $d_g(x, y)$ is on the order of $\lambda^{-1}$. In this regime, much more can be said about $d_\lambda(x, y)$. Indeed, on manifolds without conjugate points, the authors show in [5] that if $\lambda d_g(x, y)$ is bounded as $\lambda \to \infty$, then

$$d_\lambda^2(x, y) = \frac{2}{\text{vol}(M, g)} \left( 1 - \frac{2^{n/2} \Gamma \left( \frac{n}{2} + 1 \right)}{n(2\pi)^n} \frac{J_{n-2}(\lambda d_g(x, y))}{\left( \lambda d_g(x, y) \right)^{n/2}} \right) \left( 1 + o(\lambda^2 d_g^2(x, y)) \right), \quad (8)$$

where $J_\nu$ denotes the Bessel function of the first kind of order $\nu$. In particular, if $\lambda d_g(x, y) \to 0$ as $\lambda \to \infty$, we have

$$\frac{d_\lambda^2(x, y)}{(\lambda d_g(x, y))^2} = \frac{1}{n \text{vol}(M)(2\pi)^n} + o(1).$$

The proof of the refined result (8) is significantly more delicate than the proof of Theorem 1, and it can not be carried out on Zoll manifolds.

Note that all the even spherical harmonics take the same values at antipodal points so that $d_\lambda$ is only a pseudo-distance function in general. In contrast, the results in [5] prove that $d_\lambda$ is an honest distance function if, for example, $(M, g)$ has negative sectional curvatures. We deduce Theorem 1 from the following estimate of Zelditch, which says that $\Phi_\lambda$ is an almost-isometric immersion for $\lambda$ sufficiently large. Let us write $g_{\text{eucl}}$ for the flat metric on $\mathbb{R}^d$ for any $d$ and introduce the pullback metrics

$$g_\lambda(x) := \Phi_\lambda^*(g_{\text{eucl}})(x) = k_\lambda^{-2} \sum_{j=1}^{m_\lambda} d_x \phi_{j, \lambda}(x) \otimes d_y \phi_{j, \lambda}(y) \bigg|_{x=y} \quad (9)$$

for any $x \in M$. 

Theorem 2 (Zelditch [19]). Let \((M, g)\) be a compact Riemannian manifold that is either Zoll or aperiodic with \(\dim M = n\). Then, for any \(x \in M\),

\[
g_\lambda(x) = \frac{\alpha_n}{(2\pi)^n} \cdot \lambda^2 \, g(x) \,(1 + o(1))
\]

as \(\lambda \to \infty\), where \(\alpha_n\) denotes the volume of the unit ball in \(\mathbb{R}^n\).

Equation (10) allows us to relate \(d_g\) to the distance function \(d_{g_\lambda}\) of \(g_\lambda\). Theorem 1 then follows by observing that \(d_\lambda \leq d_{g_\lambda}\) (see §3 for details).

The maps \(\Phi_\lambda\) are the Riemannian analogs of Kodaira-type projective embeddings \(\Psi_N : M \to \mathbb{P}H^0(L^\otimes N)^\vee\) of a compact Kähler manifold \(M\) into the projectivization of the dual of the space of global sections \(H^0(L^\otimes N)^\vee\) of the \(N^{th}\) tensor power of an ample holomorphic line bundle \(L \to M\). Just as the choice of a Riemannian metric \(g\) gives a weighted \(L^2\) space on \(M\), a Hermitian metric \(h\) on \(L\) induces a weighted \(L^2\) inner product on \(H^0(L^\otimes N)\). This inner product gives rise to a Fubini Study metric \(\omega_{FS}^N\) on \(\mathbb{P}H^0(L^\otimes N)^\vee\), which plays the role of the Euclidean flat metric \(g_{eucl}\) used in the present article.

The holomorphic analog of Theorem 2 is that the pullback metrics \(\Psi_N^* (\omega_{FS}^N)\) converge in the \(C^\infty\) topology to the curvature form of \(h\). This statement is a result of Zelditch in [21], with weaker versions going back to Tian [16]. Theorem 1 in this holomorphic context follows easily from the holomorphic result of Zelditch and was used to study the sup norms of random homomorphic sections by Feng and Zelditch in [7].

1.1. Application to Sup Norms of Random Waves. Our main application of Theorem 1 is to find upper bounds on \(L^\infty\)-norms of random waves on \((M, g)\).

Definition 1. A Gaussian random wave of frequency \(\lambda\) on \((M, g)\) is a random function \(\phi_\lambda \in \mathcal{M}_\lambda\) defined by

\[
\phi_\lambda := \sum_{j=1}^{m_\lambda} a_{j,\lambda} \varphi_{j,\lambda},
\]

where the \(a_{j,\lambda} \sim N(0, k_\lambda^{-2})\) are independent and identically distributed standard real Gaussian random variables and \(\{\varphi_{j,\lambda}\}_j\) is an orthonormal basis for \(\mathcal{M}_\lambda\) consisting of Laplace eigenfunctions as defined in (2).

The choice of \(k_\lambda\) makes \(\mathbb{E} [\|\phi_\lambda\|_2] = 1\). Gaussian random waves were introduced by Zelditch in [19], and, in addition to their \(L^p\)-norms, a number of subsequent articles have studied their zero sets and critical points (cf., e.g., [9, 10, 13] and references therein).

The statistical features of Gaussian random waves of frequency \(\lambda\) are uniquely determined by their so-called canonical distance, which is precisely \(d_\lambda\) (cf. Definition 2). In particular, upper bounds on the expected value of their \(L^\infty\)-norms are related to the metric entropy of their canonical distance via Dudley’s entropy method (see §2.2). We use Theorem 1 to prove in §4 the following result.

Theorem 3. Let \((M, g)\) be a smooth compact boundaryless Riemannian manifold of dimension \(n\). Assume that \((M, g)\) is either aperiodic or Zoll, and let \(\phi_\lambda\) be a random wave of frequency \(\lambda\). Then

\[
\limsup_{\lambda \to +\infty} \frac{\mathbb{E} [\|\phi_\lambda\|_\infty]}{\sqrt{\log \lambda}} \leq 16 \sqrt{\frac{2n}{\text{vol}_g(M)}}.
\]

(11)
Remark 1. If \((M, g)\) is aperiodic, the more precise control of the distance function described in (8) yields a slight improvement of Theorem 3 by reducing the constant on the right hand side to \(16 \sqrt{\frac{n}{\text{vol}_g(M)}}\). We shall indicate how to use (8) to get the improved upper bound in Remark 3.

The upper bounds of order \(\sqrt{\log \lambda}\) are not new. Indeed, a simple computation shows that \(E[\|\phi_\lambda\|_\infty] = E[\|\psi_\lambda\|_\infty]\) if \(\psi_\lambda\) is chosen uniformly at random from the unit sphere \(S_{M_\lambda} := \{f \in M_\lambda : \|f\|_2 = 1\}\) endowed with the uniform probability measure. For the \(L^2\)-normalized random waves \(\psi_\lambda\), the expectation of the \(L^\infty\)-norms has been studied on many occasions. On round spheres \((S^n, g_{\text{round}})\), VanderKam obtained in [18] that \(E[\|\phi_\lambda\|_\infty] = O(\log^2 \lambda)\). Later, Neuheisel in [14] improved the bound to \(E[\|\phi_\lambda\|_\infty] = O(\sqrt{\log \lambda})\). On general smooth, compact, boundaryless Riemannian manifolds, Burq and Lebeau [4] proved the existence of two positive constants \(C_1, C_2\) so that as \(\lambda \to \infty\),

\[C_1 \sqrt{\log \lambda} \leq E[\|\phi_\lambda\|_\infty] \leq C_2 \sqrt{\log \lambda}.
\]

Although an explicit value for \(C_2\) is not stated in [4], Burq relayed to the authors in a private communication how one may be extracted. Our constant \(16 \sqrt{\frac{n}{\text{vol}_g(M)}}\) is larger (i.e., worse) than theirs, which is approximately \(e^{-e} + 1 + \frac{1}{\sqrt{2\pi}}\). The reason for the discrepancy is that Dudley’s entropy method makes no assumption on the structure of the probability space on which the Gaussian field in question is defined, while Burq and Lebeau use explicit concentration results on the spheres \(S_{M_\lambda}\). There is a partial converse to Dudley’s entropy method, called Sudakov minoration, which allows one to obtain lower bounds on the sup norms of Gaussian fields. Even with the refined control (8) on the canonical distance \(d_\lambda\) from [5], the general lower bounds seem to be on the order of \(\lambda^{-1} \sqrt{\log \lambda}\), which are significantly worse than those given by Burq and Lebeau. Finally, we mention that [4] studies more generally upper bounds on \(L^p\) norms, \(p \in [2, \infty)\), of random waves combination of Laplace eigenfunctions. Results on \(L^p\) norms are also contained in the work of Ayache-Tzvetkov [2] and Tzvetkov [17].

2. Preliminaries

The proof of Theorem 1 relies on three ingredients: the local Weyl law (§2.1), Dudley’s entropy method (§2.2), and Zelditch’s almost-isometry result (Theorem 2 in the Introduction). Throughout this section \((M, g)\) denotes a compact Riemannian manifold of dimension \(n\).

2.1. Asymptotics for the Spectral Projector. For \(x, y \in M\), we write

\[E_{(0, \lambda)}(x, y) = \sum_{\lambda_j \in (0, \lambda]} \varphi_j(x)\varphi_j(y)\]

for the Schwartz kernel of the orthogonal projection

\[E_{(0, \lambda)} : L^2(M, g) \to \bigoplus_{\mu \in (0, \lambda]} \ker (\Delta_g - \mu^2)\].
For $\lambda > 0$ we define

$$N(\lambda) := \dim \left[ \text{range} \left( E_{(0,\lambda)} \right) \right]$$

to be the number of eigenvalues of $\Delta_g$ smaller than $\lambda^2$ counted with multiplicity and set

$$\alpha_n := \frac{\omega_n}{(2\pi)^n},$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. We also write

$$E_{(\lambda,\lambda+1)}(x,y) = \sum_{\lambda_j \in (\lambda,\lambda+1]} \varphi_j(x)\varphi_j(y)$$

for the kernel of the orthogonal projection onto the span of the eigenfunctions of $\Delta_g$ whose eigenvalues lie in $(\lambda^2, (\lambda + 1)^2)$. On several occasions, we use the following result.

**Proposition 1** (Local Weyl Law, [6] and [11]). Let $(M,g)$ be a compact Riemannian manifold of dimension $n$ that is either Zoll or aperiodic. Then

$$E_{(0,\lambda]}(x,x) = \alpha_n \lambda^n + o(\lambda^{n-1}) \quad \text{as} \quad \lambda \to \infty$$

and thus

$$E_{(\lambda,\lambda+1]}(x,x) = n \alpha_n \lambda^{n-1} + o(\lambda^{n-1}) \quad \text{as} \quad \lambda \to \infty,$$

with the implied constants being uniform in $\lambda$ and $x \in M$.

Integrating the above expressions, one has that on compact aperiodic (or Zoll) manifolds,

$$N(\lambda) = \alpha_n \text{vol}_g(M) \lambda^n + o(\lambda^{n-1}) \quad \text{as} \quad \lambda \to \infty.$$

Continuing to write $m_\lambda = \dim \mathcal{M}_\lambda$, we see that

$$m_\lambda = N(\lambda+1) - N(\lambda) = n \alpha_n \text{vol}_g(M) \lambda^{n-1} + o(\lambda^{n-1}) \quad \text{as} \quad \lambda \to \infty. \quad (13)$$

We also need Hörmander’s off-diagonal pointwise Weyl law:

**Lemma 2** ([8]). Let $(M,g)$ be a compact Riemannian manifold of dimension $n$. Fix $x,y \in M$. Then

$$E_{(0,\lambda]}(x,y) = O(\lambda^{n-1}). \quad (14)$$

Using the definition (4) of $k_\lambda$, equation (13), and the fact that

$$\Gamma \left( x + \frac{1}{2} \right)/\Gamma(x) = \sqrt{x} + O(1/\sqrt{x}) \quad \text{as} \quad x \to \infty,$$

we have from (13) that

$$k_\lambda^2 = m_\lambda + O(\lambda^{-\frac{n-1}{2}}) = n \alpha_n \text{vol}_g(M) \lambda^{n-1} + o(\lambda^{n-1}). \quad (15)$$

Finally, by combining (12) and (15), we see that for all $x \in M$,

$$\|\Phi_\lambda(x)\|_{L^2(\mathbb{R}^n \lambda)} = \frac{1}{k_\lambda} \sqrt{E_{(\lambda,\lambda+1]}(x,x)} = \frac{1}{\sqrt{\text{vol}_g(M)}} + o(1) \quad (16)$$

as $\lambda \to \infty$, which confirms (5).
2.2. Dudley’s Entropy Method. Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a complete probability space. A measurable mapping \(\phi : \Omega \to \mathbb{R}^M\) is called a random field on \(M\). If for every finite collection \(\{x_j\}_{j=1}^N \in M\) the random vector \(\{\phi(x_j)\}_{j=1}^N\) is Gaussian, then \(\phi\) is said to be a Gaussian field on \(M\). In addition, if \(\mathbb{E}[\phi(x)] = 0\) for all \(x \in M\), then \(\phi\) is said to be centered. The Gaussian random waves of Definition 1 on \((M, g)\) are examples of centered Gaussian random fields on \(M\).

Definition 2. Let \(\phi\) be a centered Gaussian field on \(M\). The canonical distance on \(M\) induced by \(\phi\) is

\[
d_{\phi}(x, y) := \left(\mathbb{E}[(\phi(x) - \phi(y))^2]\right)^{\frac{1}{2}} \quad \text{for } x, y \in M.
\]

The law of any centered Gaussian field is determined completely by its canonical distance function. We note that in general \(d_{\phi}\) turns \(M\) into a pseudo-metric space. Let us define for each \(\varepsilon > 0\) the \(\varepsilon\)-covering number of \((M, d_{\phi})\) as

\[
N_{d_{\phi}}(\varepsilon) := \inf \left\{ \ell \geq 1 : \exists x_1, \ldots, x_\ell \in M \text{ such that } \bigcup_{j=1}^\ell B_\varepsilon(x_j) = M \right\},
\]

where \(B_\varepsilon(x)\) is the ball of radius \(\varepsilon\) centered at \(x\). Dudley’s entropy method says that

\[
\mathbb{E}\left[\sup_{x \in M} \phi(x)\right] \leq 8\sqrt{2} \int_0^{D_{d_{\phi}}} \sqrt{\log N_{d_{\phi}}(\varepsilon)} \, d\varepsilon,
\]

where \(D_{d_{\phi}} = \text{diam}(M, d_{\phi})/2\). We refer the reader to [1, Theorem 1.3.3] for the statement with an unspecified constant and to the notes [3] of Bartlett where the constant \(8\sqrt{2}\) appears. Unfortunately, we are unaware of a refereed article in which the explicit value \(8\sqrt{2}\) is given.

We observe that if \((M, g)\) is a smooth compact Riemannian manifold and \(\phi_\lambda\) is a random wave on \(M\) with frequency \(\lambda\), then it follows from Definition 1 that \(d_{\lambda}\) and \(d_{\phi_\lambda}\) coincide. Indeed, for \(x, y \in M\)

\[
d_{\lambda}^2(x, y) = \|\Phi_\lambda(x) - \Phi_\lambda(y)\|^2
\]

\[
= \frac{1}{4\lambda^2} \left( E_{\lambda, \lambda+1}(x, x) + E_{\lambda, \lambda+1}(y, y) - 2E_{\lambda, \lambda+1}(x, y) \right)
\]

\[
= \mathbb{E}[(\phi_{\lambda}(x) - \phi_{\lambda}(y))^2].
\]

3. Proof of Theorem 1

The starting point for our proof is the following estimate on \(\text{diam}(M, d_{\lambda})\), the diameter of \(M\) with respect to \(d_{\lambda}\).

Proposition 3. Let \((M, g)\) be a smooth compact boundaryless Riemannian manifold of dimension \(n\). Assume that \((M, g)\) is either aperiodic or Zoll. There exists \(\delta > 0\) so that as \(\lambda \to \infty\),

\[
\delta \leq \text{diam}(M, d_{\lambda}) \leq \frac{2}{\sqrt{\text{vol}_g(M)}} + o(1).
\]

Remark 2. The authors’ results in a forthcoming paper [5] allow one to prove matching upper and lower bounds in (21) and give the precise asymptotic:

\[
\text{diam}(M, d_{\lambda}) = \frac{\sqrt{2}}{\sqrt{\text{vol}_g(M)}} + o(1).
\]
The corresponding statement for Zoll manifolds is simpler and follows from the fact that $E_{[0,\lambda]}$ has a complete asymptotic expansion (cf. [20]).

**Proof.** For $x, y \in M$, we have

$$d^2_\lambda(x, y) = \frac{\|\Phi_\lambda(x) - \Phi_\lambda(y)\|^2}{k^2_\lambda}. \quad (22)$$

From (15) and the fact that $\|\Phi_\lambda(x)\|^2 = \Pi_{(\lambda,\lambda+1)}(x, x)$ for all $x \in M$, we conclude

$$d^2_\lambda(x, y) = \frac{\|\Phi_\lambda(x) - \Phi_\lambda(y)\|^2}{m_\lambda + O(1)} \leq \frac{4}{\text{vol}_g(M)} + o(1).$$

Taking the supremum over $x, y \in M$ proves the upper bound in (21). To prove the lower bound in (21), we proceed by contradiction. That is, suppose that $d^2_\lambda(x, y)$ is not bounded below. In virtue of (22) and (15), this means that

$$\|\Phi_\lambda(x) - \Phi_\lambda(y)\| = o(\lambda^{n-1}) \quad (23)$$

for all $x, y \in M$. Consider the map $\Psi_\lambda : M \to \mathbb{R}^{N(\lambda)}$ given by

$$\Psi_\lambda(x) = (\varphi_1(x), \ldots, \varphi_{N(\lambda)}(x)) \quad \text{for } x \in M,$$

where we continue to write $N(\lambda)$ for the number of eigenvalues of $\Delta_g$ in the interval $(0, \lambda^2]$. Note that the difference between $\Psi_\lambda$ and $\Phi_\lambda$ is that $\Psi_\lambda$ includes all the eigenfunctions up to eigenvalue $\lambda^2$. The local Weyl law (14) shows that for $x, y \in M$ with $x \neq y$,

$$\|\Psi_\lambda(x) - \Psi_\lambda(y)\|^2 = E_{(0,\lambda]}(x, x) + E_{(0,\lambda]}(y, y) - 2E_{(0,\lambda]}(x, y) = \Omega(\lambda^n). \quad (24)$$

On the other hand, for any positive integer $k$,

$$\|\Psi_k(x) - \Psi_k(y)\|^2 = \sum_{j=0}^{k-1} \|\Phi_j(x) - \Phi_j(y)\|^2.$$

The assumption (23) shows that

$$\|\Psi_k(x) - \Psi_k(y)\|^2 = o(k^n),$$

which contradicts (24). \hfill \Box

Let us denote by $d_g$ the distance function for the metric $g$ defined in (9). We continue to write $d_g$ for the distance function of the background metric $g$.

Combining (21) with (15), we see that Theorem 1 reduces to showing that for any $K > 2/\sqrt{\text{vol}_g(M)}$ there exist positive constants $C, \lambda_0$ depending on $(M, g)$ such that $d_\lambda(x, y) \leq C\lambda$ for all $\lambda \geq \lambda_0$ and $x, y \in M$ with $d_g(x, y) \leq K/\lambda$. We write

$$d_\lambda(x, y) = \frac{d_\lambda(x, y)}{d_g(x, y)} \cdot \frac{d_g(x, y)}{d_g(x, y)} \cdot \frac{d_g(x, y)}{d_g(x, y)}. \quad (25)$$

Note that $\frac{d_\lambda(x, y)}{d_g(x, y)} \leq 1$. Indeed, let $\gamma : [0, 1] \to M$ be any length minimizing geodesic between two given points $x$ and $y$ with respect to the metric $g_\lambda = \Phi_\lambda^*(g_{\text{eucl}})$ and consider the curve $\Phi \circ \gamma : [0, 1] \to \mathbb{R}^{m_\lambda}$ joining $\Phi_\lambda(x)$ and $\Phi_\lambda(y)$. Then

$$d_\lambda(x, y) = \|\Phi_\lambda(x) - \Phi_\lambda(y)\|_{g_{\text{eucl}}} \leq \int_0^1 \left\| \frac{d}{dt} \Phi_\lambda(\gamma(t)) \right\|_{g_{\text{eucl}}} dt = d_g(x, y). \quad (26)$$
We turn to show that there exists $C > 0$ such that $\frac{d_{\lambda g}(x,y)}{d_g(x,y)} \leq C \lambda$. Fix $x, y \in M$ and a unit speed geodesic $\gamma$ from $x$ to $y$ with respect to the metric $g$. Theorem 2 states that there exists $C > 0$ so that as $\lambda \to \infty$, one has $g_\lambda = C \cdot \frac{1}{m_\lambda} \cdot \lambda^{n+1} g_0 + o(1)$. Therefore, after suitably adjusting $C$,

$$d_{\lambda g}(x,y) \leq \int_0^{d_g(x,y)} \| \frac{d}{dt} \gamma(t) \|_{g_\lambda} \, dt \leq C \sqrt{\frac{\lambda^{n+1}}{m_\lambda}} d_g(x,y). \tag{27}$$

Substituting (26) and (27) into (25) and using the asymptotics (15) for $m_\lambda$ completes the proof of Theorem 1.

**4. Proof of Theorem 3**

Let $\phi_\lambda$ be a Gaussian random wave of $(M, g)$ with frequency $\lambda$. As explained in (20), the distance $d_\lambda$ induced by the immersions and the distance $d_{\phi_\lambda}$ induced by $\phi_\lambda$ coincide. Theorem 1 therefore allows us to control the metric entropy $N_{d_\lambda}$ (see (17)).

**Proposition 4.** Let $(M, g)$ be a compact aperiodic or Zoll Riemannian manifold. There exist positive constants $C$ and $\lambda_0$ such that if $\phi_\lambda$ is a Gaussian random wave of frequency $\lambda$, then

$$N_{d_\lambda}(\varepsilon) \leq C \cdot \frac{\lambda^n}{\varepsilon^n}$$

for all $\varepsilon > 0$ and $\lambda \geq \lambda_0$.

**Proof.** From Theorem 1, it follows that if $x, y \in M$ are such that $d_g(x,y) \leq \frac{\varepsilon}{C}$ and $\lambda$ exceeds some fixed $\lambda_0$, then $d_{\lambda g}(x,y) \leq \varepsilon$. Writing $B_d(x,r)$ for a ball of radius $r$ in the distance $d$, we see that $B_{d_\lambda}(x, \frac{\varepsilon}{C}) \subseteq B_{d_g}(x, \varepsilon)$ for each $x \in M$. Hence,

$$N_{d_\lambda}(\varepsilon) \leq N_{d_g} \left( \frac{\varepsilon}{C} \right) \tag{28}$$

for every $\lambda \geq \lambda_0$. Define $\alpha_g := \inf \{ k_g(x) : x \in M \}$, where $k_g$ denotes the sectional curvature function for $(M, g)$, and set $\pi/\sqrt{\alpha_g} = \infty$ whenever $\alpha_g \leq 0$. For $\lambda$ large enough,

$$\frac{\varepsilon}{C} \leq \min \left\{ \text{inj}_g(M), \frac{\pi}{\sqrt{\alpha_g} \sqrt{2}}, 2\pi \right\},$$

and so we may apply [12, Lemma 4.1] to obtain

$$N_{d_g} \left( \frac{\varepsilon}{C\lambda} \right) \leq \text{vol}_g(M) \frac{2n}{s_{n-1}} \pi^{n-1} \left( \frac{\varepsilon}{C} \right)^{-n}, \tag{29}$$

where $s_{n-1}$ is the volume of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$. The claim follows from combining (28) and (29). \hfill \Box

To complete the proof of Theorem 3, we input the upper bound on $N_{d_\lambda}$ of Proposition 4 into Dudley’s entropy estimate (18) to conclude that for a constant $C$ depending $(M, g)$ and all $\lambda$ exceeding some $\lambda_0$,

$$\mathbb{E} \left[ \sup_{x \in M} \phi_\lambda(x) \right] \leq 8\sqrt{2n} \int_0^{D_\lambda} \sqrt{\log \left( \frac{C^{1/n} \lambda}{t} \right)} \, dt$$

$$= 8D_\lambda \sqrt{2n} \int_0^1 \sqrt{\log \left( \frac{\alpha_\lambda}{t} \right)} \, dt,$$
where
\[ D_\lambda := \frac{\text{diam}(M, d_\lambda)}{2} \leq \frac{1}{\sqrt{\text{vol}_g(M)}} + o(1), \]
and
\[ \alpha_\lambda := \frac{C^{1/n}}{D_\lambda \lambda}. \quad (30) \]
Setting \( a_\lambda := \frac{1}{\log \alpha_\lambda} \), we get
\[ \mathbb{E}\left[ \sup_{x \in M} \phi_\lambda(x) \right] \leq 8\sqrt{2}D_\lambda \sqrt{n} \log \alpha_\lambda \int_0^1 (1 - a_\lambda \log t)^{1/2} \, dt. \quad (31) \]
We now use the estimate
\[ \left| \int_0^1 (1 - a_\lambda \cdot \log t)^{1/2} \, dt - 1 \right| \leq \frac{a_\lambda}{2} \quad \text{as} \quad a_\lambda \to 0, \]
whose proof we give in Claim 5 below. Hence, combining (32) with (31) and the definition (30) of \( \alpha_\lambda \), we have
\[ \mathbb{E}\left[ \sup_{x \in M} \phi_\lambda(x) \right] \leq 8\sqrt{2}D_\lambda \sqrt{n} \sqrt{\log \lambda} + \log \left( \frac{C^{1/n}}{n \text{vol}_g(M)} \right) \left( 1 + \frac{a_\lambda}{2} \right). \quad (33) \]
Since Proposition 3 guarantees that \( \log \left( \frac{C^{1/n}}{n \text{vol}_g(M)} \right) \) is uniformly bounded in \( \lambda \), and \( a_\lambda \to 0 \), we conclude from (33) that
\[ \mathbb{E}\left[ \sup_{x \in M} \phi_\lambda(x) \right] \sqrt{\log \lambda} \leq 8\sqrt{2} \sqrt{\frac{n}{\text{vol}_g(M)}} + o(1). \]
Since \( \phi_\lambda \) is symmetric,
\[ \mathbb{E}\left[ \| \phi_\lambda \|_\infty \right] \leq 2 \mathbb{E}\left[ \sup_{x \in M} \phi_\lambda(x) \right] \leq 16 \sqrt{\frac{2n}{\text{vol}_g(M)}} + o(1). \quad (34) \]
Taking the lim sup as \( \lambda \to \infty \) in (34) completes the proof.

**Remark 3.** The proof of the result stated in Remark 1 for the aperiodic case follows from using the diameter asymptotics in Remark 2, which give \( D_\lambda = \frac{1}{\sqrt{2 \text{vol}_g(M)}} + o(1) \) instead of \( D_\lambda \leq \frac{1}{\sqrt{\text{vol}_g(M)}} + o(1) \).

**Claim 5 (Proof of (32)).** As \( a \to 0 \),
\[ \left| \int_0^1 (1 - a \cdot \log x)^{1/2} \, dx - 1 \right| \leq \frac{a}{2}. \]

**Proof.** Making the change of variables \( u = (1 - a \log x)^{1/2} \), we get
\[ \int_0^1 (1 - a \cdot \log x)^{1/2} \, dx = \frac{2}{a} e^{\frac{a}{2}} \int_1^\infty u^2 e^{-u^2} \, du. \quad (35) \]
Observe that \( -\frac{a}{2u} \frac{\partial}{\partial u} \) preserves \( e^{-u^2/a} \) and integrate by parts in (35) to get
\[ \int_0^1 (1 - a \cdot \log x)^{1/2} \, dx = 1 + e^{\frac{a}{2}} \int_1^\infty e^{-\frac{u^2}{a}} \, du = 1 + e^{\frac{a}{2}} \sqrt{\frac{a}{2}} \int_1^\infty e^{-\frac{t^2}{a}} \, dt \]
Using the classical estimate
\[
\int_{x}^{\infty} e^{-y^2} \, dy \leq \frac{e^{-x^2}}{x}
\]
shows that, as \(a \to 0\),
\[
\left| \int_0^1 (1 - a \cdot \log x)^{1/2} \, dx - 1 \right| \leq \frac{a}{2},
\]
as desired. \(\square\)

References


