

NEW PERIODIC SOLUTIONS FOR THREE OR FOUR IDENTICAL VORTICES ON A PLANE AND A SPHERE

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Abstract. In this paper we describe new classes of periodic solutions for point vortices on a plane and a sphere. They correspond to similar solutions (so-called choreographies) in celestial mechanics.

Equations of motion and first integrals for vortices on a plane. For n point vortices with Cartesian coordinates (x_i, y_i) and intensities Γ_i , the Hamiltonian equations of the motion are

$$\Gamma_i \dot{x}_i = \frac{\partial \mathcal{H}}{\partial y_i}, \quad \Gamma_i \dot{y}_i = -\frac{\partial \mathcal{H}}{\partial x_i}, \quad 1 \leq i \leq n, \quad (1)$$

where the Hamiltonian is

$$\mathcal{H} = -\frac{1}{4\pi} \sum_{i < j}^n \Gamma_i \Gamma_j \ln |\mathbf{r}_i - \mathbf{r}_j|^2, \quad \mathbf{r}_i = (x_i, y_i). \quad (2)$$

For these equations, the Poisson bracket is $\{x_i, y_j\} = \Gamma_i^{-1} \delta_{ij}$.

The system (1) has three first integrals

$$Q = \sum_{i=1}^n \Gamma_i x_i, \quad P = \sum_{i=1}^n \Gamma_i y_i, \quad I = \sum_{i=1}^n \Gamma_i (x_i^2 + y_i^2), \quad (3)$$

which are not involutive:

$$\{Q, P\} = \sum_{i=1}^N \Gamma_i, \quad \{P, I\} = -2Q, \quad \{Q, I\} = 2P. \quad (4)$$

However, two involutive integrals always exist, for example, $Q^2 + P^2$ and I . With the help of these integrals, we can reduce the system by two degrees of freedom.

Thus, in the case of three vortices the reduced system has one degree of freedom and it is integrable (Gröbli, 1877 [15]; Poincaré, 1893 [21]).

The four-vortex problem is reduced to a system with two degrees of freedom. This system is not integrable (S. Ziglin, 1979 [27]).

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Equations of motion and first integrals for vortices on a sphere \mathbb{S}^2 . For n point vortices moving on a sphere \mathbb{S}^2 , with spherical coordinates (θ_i, φ_i) and intensities Γ_i , the Hamiltonian equations can be written as (Bogomolov, 1977 [4])

$$\dot{\theta}_i = \{\theta_i, \mathcal{H}\}, \quad \dot{\varphi}_i = \{\varphi_i, \mathcal{H}\}, \quad i = 1, \dots, n, \quad (5)$$

with the Poisson bracket $\{\varphi_i, \cos \theta_k\} = \frac{\delta_{ik}}{R^2 \Gamma_i}$ and the Hamiltonian

$$\mathcal{H} = -\frac{1}{4\pi} \sum_{i < k}^n \Gamma_i \Gamma_k \ln \left(4R^2 \sin^2 \frac{\gamma_{ik}}{2} \right). \quad (6)$$

Here, R is the sphere's radius, and γ_{ik} is the angle between the vectors from the center of the sphere to the point vortices i and k so that

$$\cos \gamma_{ik} = \cos \theta_i \cos \theta_k + \sin \theta_i \sin \theta_k \cos(\varphi_i - \varphi_k).$$

Beside the Hamiltonian, the equations (5) have three independent non-involutive integrals

$$F_1 = R \sum_{i=1}^n \Gamma_i \sin \theta_i \cos \varphi_i, \quad F_2 = R \sum_{i=1}^n \Gamma_i \sin \theta_i \sin \varphi_i, \quad F_3 = R \sum_{i=1}^n \Gamma_i \cos \theta_i. \quad (7)$$

The vector $\mathbf{F} = \sum \Gamma_i \mathbf{r}_i$ (\mathbf{r}_i are the radius-vectors of the vortices) with components (F_1, F_2, F_3) is called the *moment or center of vorticity*. The first integrals commute in the following manner: $\{F_i, F_j\} = \frac{1}{R} \varepsilon_{ijk} F_k$. Similar to the case of a plane, we can reduce the system by two degrees of freedom, using involutive integrals, for example, $F_3, |\mathbf{F}|^2$.

So, in the case of three vortices, we have a completely integrable system (P. Newton and R. Kidambi, 1998 [16], 2000 [20]; Borisov and Lebedev, 1998 [9]). The four-vortex problem is reduced to a system with two degrees of freedom and, generally, is not integrable (D. A. and A. A. Bagrets [3]).

Reduction of a system of vortices on a plane. To find new periodic solutions and apply the numerical analysis, we will do the most symmetric reduction of the system by two degrees of freedom. We present here one of the possible ways to do such a reduction, based on new equations of motion for the mutual variables.

Effective reduction of a four-vortex system was done by Khanin [17] (for vortices of the same sign) and by Aref and Pomphrey [2] (for equal vortices). The generalized Jacobi reduction by one degree of freedom in the point vortex dynamics was described in [19].

In our study, we used the universal algebraic method of effective reduction (for an arbitrary number of vortices). It is based on *mutual variables representation* of the motion equations of a system of point vortices [6], [23], [24].

As mutual variables, we introduce the following quantities. Squared distances between the pairs of vortices M_{ij} and oriented areas of triangles Δ_{ijk} ,

$$M_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2, \quad \Delta_{ijk} = (\mathbf{r}_j - \mathbf{r}_k) \wedge (\mathbf{r}_k - \mathbf{r}_i).$$

These variables are due to E. Laura.

We put $\Gamma_i = \Gamma_j = 1, P = Q = 0$. Then the momentum integral I (3) can be written as

$$I = \frac{1}{n} \sum_{i < j}^n M_{ij}, \quad (8)$$

where n is the number of the vortices. Writing the vortices' coordinates in the complex form $z_k = x_k + iy_k$, we obtain the following representation for them:

$$z_k = \frac{1}{n} \sum_{j=k}^n \sqrt{M_{kj}} e^{i\theta_{kj}}, \quad (9)$$

where θ_{kj} is the angle between the vector from the j th vortex to the k th vortex and the positive direction of Ox .

By direct calculation we can derive the following propositions, which describe the behavior of the reduced system of three and four vortices [6].

Proposition 1. *For three identical vortices, the evolution of the mutual distances (with fixed $I = \text{const}$) is described by a Hamiltonian system with one degree of freedom. In terms of the canonical variables (g, G) , this system can be written as*

$$\dot{g} = \frac{\partial \mathcal{H}}{\partial G}, \quad \dot{G} = -\frac{\partial \mathcal{H}}{\partial g}, \quad \mathcal{H} = -\frac{1}{4\pi} \ln M_{12} M_{13} M_{23}, \quad (10)$$

where $M_{12} = 4\left(\frac{I}{2} - G\right)$, $M_{13} = 8G - I + 2\sqrt{12}\sqrt{\left(\frac{I}{2} - G\right)G \cos g}$, $M_{23} = 4\left(\frac{I}{2} - G\right) - 2\sqrt{12}\sqrt{\left(\frac{I}{2} - G\right)G \cos g}$.

Proposition 2. *For four identical vortices, the evolution of the mutual distances is described by a Hamiltonian system with two degrees of freedom. In terms of the canonical variables (g, G, h, H) , this system is*

$$\dot{g} = \frac{\partial \mathcal{H}}{\partial G}, \quad \dot{G} = -\frac{\partial \mathcal{H}}{\partial g}, \quad \dot{h} = \frac{\partial \mathcal{H}}{\partial H}, \quad \dot{H} = -\frac{\partial \mathcal{H}}{\partial h}, \quad (11)$$

$$\mathcal{H} = -\frac{1}{4\pi} \ln M_{12} M_{13} M_{14} M_{23} M_{24} M_{34},$$

where

$$\begin{aligned} M_{12} &= I - G + 2\sqrt{(I-H)(I-G)} \cos h, & M_{34} &= I - G - 2\sqrt{(I-H)(I-G)} \cos h, \\ M_{13} &= I + G + 2\sqrt{(I-H)G} \cos(h+g), & M_{24} &= I + G - H - 2\sqrt{(I-H)G} \cos(h+g), \\ M_{14} &= H + 2\sqrt{(H-G)G} \cos g, & M_{23} &= H - 2\sqrt{(I-G)G} \cos g. \end{aligned}$$

Remark 1. The above canonical variables naturally follow from the Lie-algebraic representation of the equations of motion.

Absolute motion — quadratures and geometric interpretation. According to (9), when $M_{ij}(t)$ are known, one needs to find the angles $\theta_{ij}(t)$ in order to determine the coordinates of the vortices. Clearly, only one of these angles is independent (let it be θ_{12}), the remaining are obtained using the relations

$$\theta_{ij} + \theta_{ik} = \arccos \left(\frac{M_{jk} - M_{ij} - M_{ik}}{2\sqrt{2M_{ij}M_{ik}}} \right), \quad i \neq j, k \neq i. \quad (12)$$

The evolution of θ_{ij} is found by integrating the first order equation:

$$4\pi\dot{\theta}_{ij} = \frac{2}{M_{ij}} \sum_{k=1}^n \Gamma_k + \sum_{k \neq i, j}^n \Gamma_k \left(\frac{1}{M_{ik}} + \frac{1}{M_{jk}} \right) - \frac{1}{M_{ij}} \sum_{k \neq i, j}^n \Gamma_k \left(\frac{M_{jk}}{M_{ik}} + \frac{M_{ik}}{M_{jk}} \right). \quad (13)$$

There is an interesting geometric interpretation of the absolute motion of the periodic solutions of the reduced systems (10), (11).

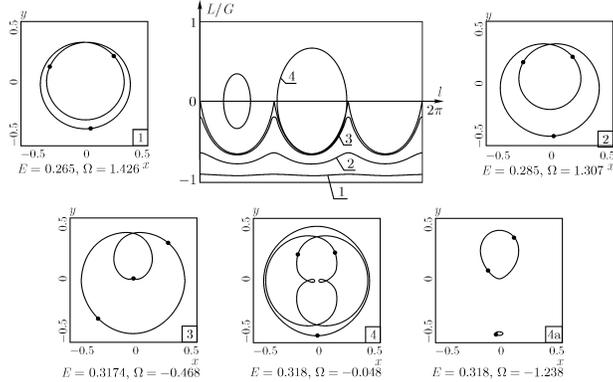


Figure 1. The phase portrait of the reduced system in the three-vortex problem, and the relative choreographies, corresponding to different phase orbits in the portrait.

Proposition 3. Let $\gamma(t)$ be a periodic solution (of period T) of the reduced system. Then

- 1° there exists a frame of reference, rotating with the constant angular velocity Ω_a about the center of vorticity, where each vortex moves along some closed curve $\xi_i(t)$;
- 2° the rotating velocity Ω_a is given by (accurate to $\frac{2\pi p}{Tq}$, $p, q \in \mathbb{Z}$):

$$\Omega_a = \frac{1}{T} \int_0^T \dot{\theta}_{12}(t) dt; \quad (14)$$

3° if Ω_a and $\Omega_o = \frac{2\pi}{T}$ are commensurable (i. e. $\frac{\Omega_a}{\Omega_o} = \frac{p}{q}$, $p, q \in \mathbb{Z}$), then in the fixed frame of reference the vortices also move along closed curves;

4° if any of the curves $\xi_i(t)$ transform into each other by rotation about the center of vorticity by an angle, commensurable to 2π , then there is a (rotating) frame of reference where the corresponding vortices orbit one and the same curve.

Proof. In the right hand side of (13), there is a periodic function of period T . We have a convergent Fourier expansion:

$$4\pi\dot{\theta}_{12} = \sum_{n \in \mathbb{Z}} a^{(n)} e^{i \frac{2\pi n}{T} t}. \quad (15)$$

Integrating (15), and using (12), we find that all the angles θ_{ij} depend on t in the following way:

$$\theta_{ij}(t) = \Omega_{ij} t + g_{ij}(t), \quad (16)$$

where $\Omega_{ij} = a^{(0)} + \frac{2\pi q_{ij}}{T p_{ij}}$, $q_{ij}, p_{ij} \in \mathbb{Z}$, while $g_{ij}(t) = g_{ij}(t + T)$ are T -periodic functions of time.

Substituting the above dependency into (9), we find that the positions of the vortices on the plane are given as follows:

$$z_k(t) = \frac{Q + iP}{\sum \Gamma_i} + u_k(t) e^{i\Omega t}, \quad u_k(t) = u_k(t + T_1) \in \mathbb{C}, \quad \Omega = a^{(0)}. \quad (17)$$

It follows that *in the frame of reference, rotating about the center of vorticity with the angular velocity Ω , all the vortices describe analytic closed curves, given by functions $u_k(t) \in \mathbb{C}$.*

The proofs of 2°, 3°, and 4°, using the relations (15)–(17), are obvious. \square

Remark 2. Without any modification, Proposition 3 is generalized to the case of n vortices if we suppose that $\gamma(t)$ is a periodic solution of a reduced system with $2n-2$ degrees of freedom to the n -vortex problem.

Particular solutions and stationary configurations for n equal vortices.

This section presents most well-known particular solutions for point vortices on a plane and on a sphere.

The case of \mathbb{R}^2 :

1. collinear configurations;
2. polygonal configurations (in particular, polygonal configurations, embedded into each other). An n -polygon (Thomson configuration [25]) is stable if $n \leq 7$, and unstable if $n > 7$ (the final result on the stability of the heptagon was obtained recently in [18]);
3. non-symmetrical stationary configuration (for $n \geq 8$) [13].

The case of \mathbb{S}^2 :

1. Collinear configuration.

The roots θ_i that define a collinear configuration, rotating with velocity Ω , can be found as equilibriums of a system of N particles on a circle with the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{k=1}^n p_k^2 + 4\pi R^2 \Omega \sum_{k=1}^N \cos \theta_k + \frac{1}{2} \sum_{k,i=1}^N \ln \left| \sin \left(\frac{\theta_k - \theta_i}{2} \right) \right|. \quad (18)$$

This system was obtained and studied by A. V. Borisov and V. V. Kozlov (1999 [8]) — a special case with $\Omega = 0$ is called the Dyson system. It is proved that the system (18) is not integrable, but a *quasi-integral* exists, which gives a very good approximation of the behavior of systems at low energy.

2. Analogs of Thomson configurations [5].

In these configurations, the vortices are positioned at the same latitude θ_0 in the vertices of a regular n -polygon and rotate about its center with the angular velocity

$$\omega = \frac{\Gamma(N-1) \operatorname{ctg} \theta_0}{4\pi R^2 \sin \theta_0}.$$

Choreographies. Recently, a new class of periodic solutions to the classical n -body problem has been obtained in celestial mechanics (C. Moore, R. Montgomery, A. Chenciner, C. Simó, and others). For example, they found a remarkable stable periodic orbit in the 3-body problem — namely, a *figure-of-eight* [12]. In this solution, three bodies follow each other along the same curve.

A long and ingenious analytical proof was done using the variational method based on the least action principle. These solutions where all particles move along the same curve are called *choreographies* [11].

Choreographies can be:

- a) *absolute* (in the fixed frame of reference) or *relative* (in rotating frames of reference);
- b) *simple* (a single closed curve) or *complex* (more than one closed curve)

Similar solutions exist for point vortex dynamics on a plane and on a sphere [10]. However, the least action principle cannot be applied to an analytical proof in this case.

Using Proposition 3 and the known analytical solutions [15], one can prove the following theorem for three-vortex systems:

Theorem 1. *If, for the motion of three vortices of equal intensity, the constants of motion, I and \mathcal{H} , satisfy the inequalities*

$$-\ln 3 < \frac{4\pi}{3}\mathcal{H} + \ln I < \ln 2, \tag{19}$$

then such motion is a simple relative choreography (see Fig. 1).

Remark 3. Note also that various properties of the motion of three vortices are discussed in [23] and [24]. For example, the paper [24] focuses on the study of stability of collinear configurations. However, properties of the absolute motion have not actually been studied.

The four-vortex system has a remarkable symmetric analytical solution — Goryachev’s solution [14] (see also [1]) where the vortices form a parallelogram as they move on a plane.

Theorem 2. *If, for the motion of four vortices, the vortices (of equal intensity) form a centrosymmetrical configuration (a parallelogram), while the constants \mathcal{H} and I satisfy the inequalities*

$$-\ln 2 < \frac{2\pi}{3}\mathcal{H} + \ln I < -\ln \frac{144}{5},$$

then the motion is a simple relative choreography (see Fig. 2).

For the complete proofs, see [10].

Let us show that for the system of four identical vortices, there exist choreographies, which do not coincide with Goryachev’s solution. These are *non-analytical choreographies*. These solutions can be obtained by variation of the Hamiltonian near Thomson’s solution.

Indeed, near the Thomson configuration, the Hamiltonian can be written in as

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}^*, \quad \text{deg } \mathcal{H} \geq 3,$$

where \mathcal{H}_2 is a normal form

$$\mathcal{H}_2 = \frac{1}{4\pi} \left(3(x^2 + P_x^2) + 2\sqrt{2}(y^2 + P_y^2) \right),$$

and \mathcal{H}^* is the perturbation function (a term of higher order).

On the level $\mathcal{H}_2 = h_2 = \text{const}$, there are precisely two nondegenerate periodic motions of the Hamiltonian system with the Hamiltonian \mathcal{H}_2 :

$$x = P_x = 0, \quad y^2 + P_y^2 = \frac{\pi}{\sqrt{2}}h_2, \tag{20}$$

$$y = P_y = 0, \quad x^2 + P_x^2 = \frac{4\pi}{3}h_2. \tag{21}$$

According to the Lyapunov theorem, these periodic solutions are conserved under the perturbation \mathcal{H}^* and near Thomson’s solution there are two periodic solutions of the whole system $\mathcal{H}_2 + \mathcal{H}^*$:

solution (20) gives Goryachev’s solution, and

solution (21) gives a non-analytical relative choreography such as shown in Fig. 3.

Remark 4. Here a solution is called non-analytical if it cannot be expressed explicitly in terms of quadratures. As a rule, this means that there are no explicit symmetries for these solutions, since all known symmetrical solutions can usually be found in terms of quadratures.

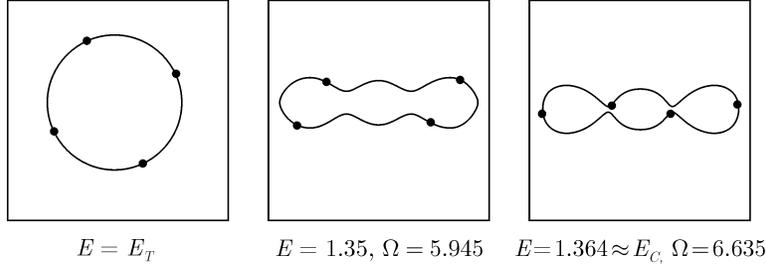


Figure 2. Relative choreographies for Goryachev's solution

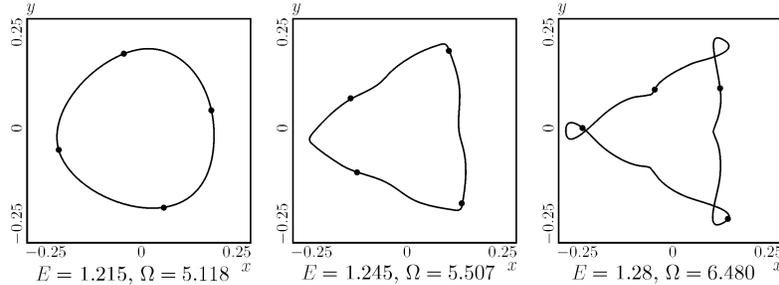


Figure 3. Relative choreographies for the solution (21).

Absolute choreographies. Let us discuss in more detail, whether absolute choreographies exist in the three- and four-vortex problems. According to the above, any relative choreography, corresponding to a periodic solution (of period T) of the reduced system (see (10), (11)) closes in time mT , $m \in \mathbb{N}$. During this interval the vortices pass through the same configurations m times. We denote the corresponding angular velocities of the frames of reference by $\Omega_m^{(k)}$.

For three-vortex choreographies (Theorem 1), there exists a (rotating) frame of reference, where a choreography closes in the smallest possible time T . For the case of four-vortex choreographies (Theorem 2), this time equals $2T$. We denote and the corresponding angular velocities by $\Omega_1^{(0)}$ and $\Omega_2^{(0)}$. The angular velocities of other relative connected choreographies are now given by the following relations.

For three vortices:

$$\Omega_m^{(k)}(E) = \Omega_1^{(0)}(E) + \frac{3k}{m}\Omega_0(E), \text{ where } m \in \mathbb{N}, k \in \mathbb{Z};$$

here $3k$ and m are coprime numbers.

For Goryachev's solution:

$$\Omega_{2m}^{(k)}(E) = \Omega_2^{(0)}(E) + \frac{k}{m}\Omega_0(E),$$

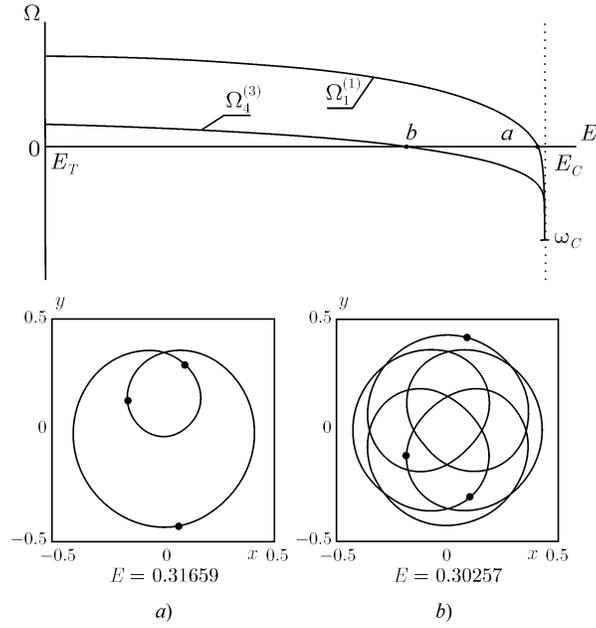


Figure 4. The angular velocities of the frame of reference for relative choreographies of three vortices; two absolute choreographies corresponding to the points a and b .

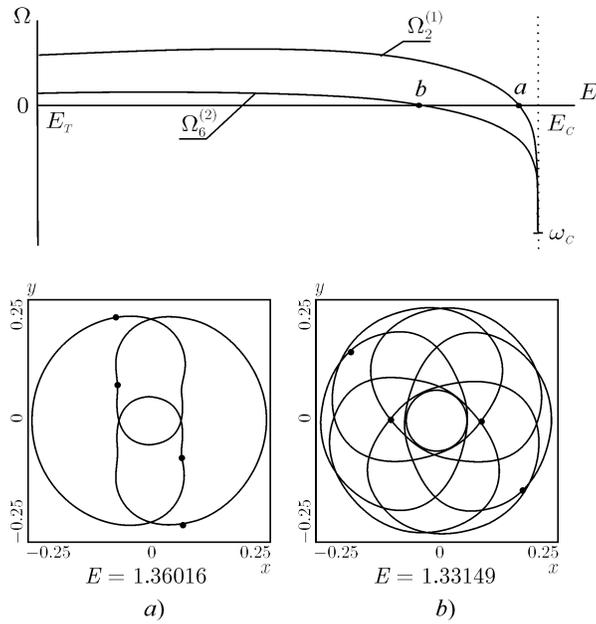


Figure 5. The angular velocities of the frame of reference for relative choreographies of four vortices; two absolute choreographies corresponding to the points a and b .

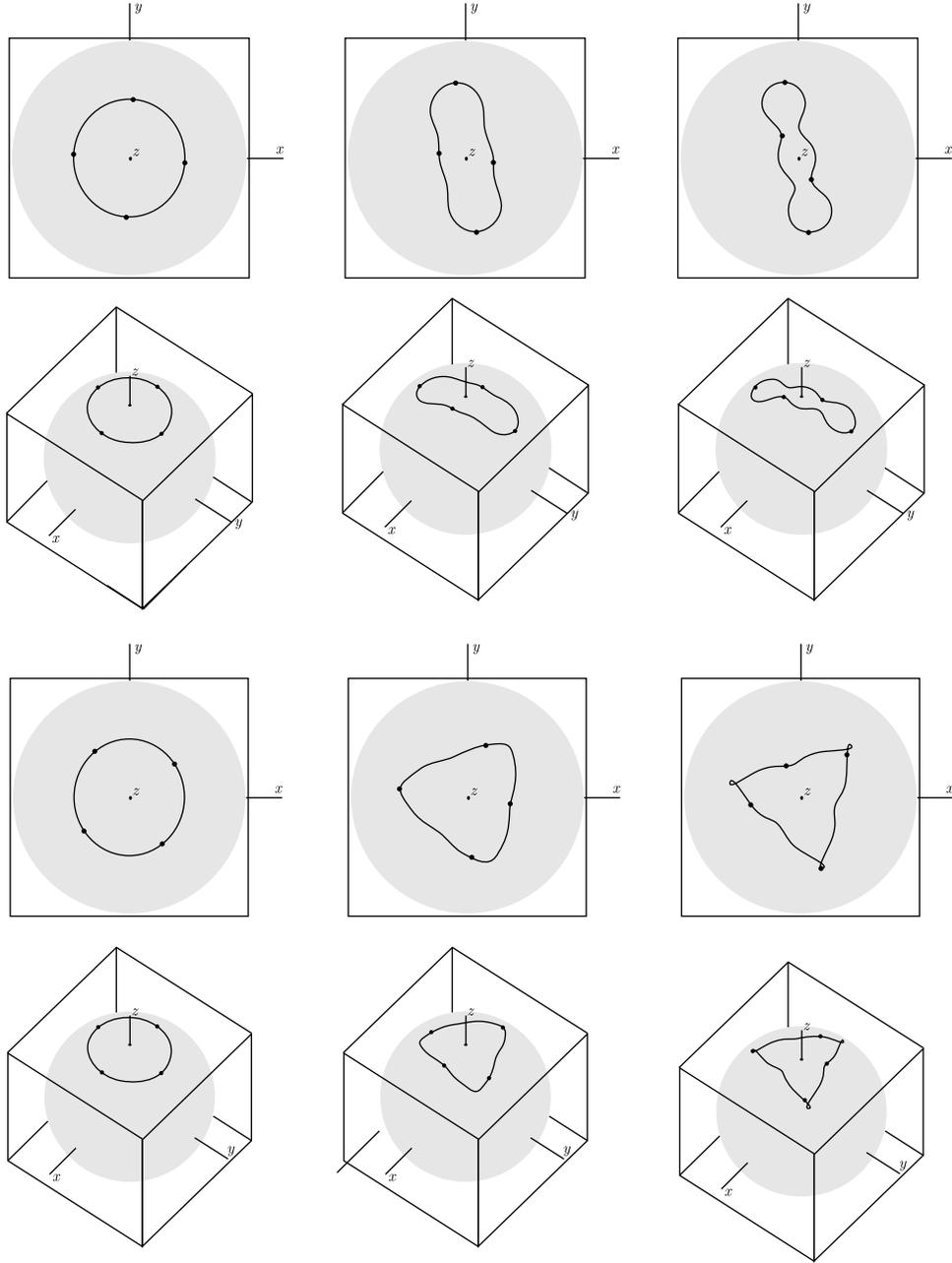


Figure 6. Relative choreographies on a sphere.

where m is odd, $k \in \mathbb{Z}$, the numbers k and m are coprime, while $\Omega_0(E) = \frac{2\pi}{T(E)}$.

Here, the choreography that closes in mT corresponds to the velocity $\Omega_m^{(k)}$ (and the choreography that closes in $2mT$ corresponds to the velocity $\Omega_{2m}^{(k)}$).

Figs. 4 and 5 show the angular velocities in terms of the energy parameter for three and four vortices, respectively. The roots of $\Omega_m^{(k)}(E) = 0$ correspond to the absolute choreographies shown in Figs. 4 a, b and 5 a, b. It can be shown that there is a countable set of energy values such that choreographies described in Theorems 1 and 2 are absolute and simple.

Choreographies on \mathbb{S}^2 . It can be shown, using the above-described methods and the reduction described in [7], that there are similar relative and absolute choreographies in the systems of three and four vortices on a sphere. Several examples of spherical choreographies of four vortices are given in Fig. 6 (they are quite similar to those shown in Figs. 2 and 3).

In [22] and [26] periodic (in a fixed frame of reference) solutions for different numbers of vortices on a sphere are specified, which admit different discrete symmetry groups. These motions are disconnected choreographies. Using the group-theoretic methods, one can construct a large number of disconnected choreographies in vortex dynamics on a sphere, generalizing those specified in [22] and [26]. However, the choreographies that we found cannot be obtained using these methods.

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