DIFFERENTIABLE RIGIDITY FOR QUASIPERIODIC COCYCLES IN COMPACT LIE GROUPS

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ABSTRACT. We study close-to-constants quasiperiodic cocycles in $\mathbb{T}^d \times G$, where $d \in \mathbb{N}^*$ and $G$ is a compact Lie group, under the assumption that the rotation in the basis satisfies a Diophantine condition. We prove differentiable rigidity for such cocycles: if such a cocycle is measurably conjugate to a constant one satisfying a Diophantine condition with respect to the rotation, then it is $C^\infty$-conjugate to it, and the KAM scheme actually produces a conjugation. We also derive a global differentiable rigidity theorem, assuming the convergence of the renormalization scheme for such dynamical systems.

1. INTRODUCTION

In the author’s PhD thesis [13], the study of quasiperiodic cocycles in $\mathbb{T}^d \times G$, with $G$ a semisimple compact Lie group, over a Diophantine rotation and satisfying a closeness-to-constants assumption was revisited. Some of the tools that were developed therein are applied in this work in order to obtain a differentiable rigidity theorem for such dynamical systems.

Let us quickly define the objects, after having pointed out that our main reference for the subject is [15]. More detailed definitions are given in Section 3 of the paper. A quasi-periodic cocycle, or shortly a cocycle, is a pair $(\alpha, A(\cdot))$, where $\alpha \in \mathbb{T}^d$ is minimal, and $A(\cdot) : \mathbb{T}^d \to G$, with $G$ a topological group, and in our case a semisimple compact Lie group. The cocycle $(\alpha, A(\cdot))$ acts on $\mathbb{T}^d \times G$ by

$$(\alpha, A(\cdot))(x, S) = (x + \alpha, A(x) \cdot S).$$

The space of such dynamical systems is denoted by $SW^r_\alpha(\mathbb{T}^d, G)$, where $r \in [0, \infty]$ represents the regularity of the mapping $A(\cdot)$, which will be equal to $\infty$ throughout this work. Explicit iteration of the system is impossible, unless the mapping $A(\cdot)$ takes values in an abelian subgroup of $G$. For this reason, a large part of the study of such dynamical systems is devoted to the problem of reducibility: a cocycle is called reducible to a simpler model iff it is conjugate (by a fibered transformation of the phase space) to the model. Ideally, the conjugation should have the same regularity as the mapping $A(\cdot)$.

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Naturally, the simplest models are those given by a constant mapping $A(\cdot) \equiv A \in G$. It is a classical fact that if $G \equiv T$ is a torus and the rotation $\alpha$ is Diophantine (see §2.4 for the definition), then every cocycle is smoothly reducible to a constant one. However, as it turns out, so long as the group $G$ is not Abelian, non-reducible cocycles are expected to be abundant (see [6]). One then has to examine the existence of smooth cocycles that are non-smoothly conjugate to constant ones, as well as eventually the cocycles that are not conjugate to constants via a transfer function of any regularity. A natural question that arises in this framework is whether some reasonable assumptions force a conjugation of low regularity (say measurable) to be in fact $C^\infty$ smooth. A second one, not treated in this work, is the density of reducible cocycles, which in turn can open the door to the study of non-reducible cocycles by approximation.

Concerning this last question of density, the corresponding local theorem is proved in [15] in the $C^\infty$ category. Here, the word local is used to indicate that the mapping $A(\cdot)$ is assumed to be close to a constant, and the proof is by means of a KAM scheme. The iterative procedure produces a product of conjugations that is expected to diverge, but nonetheless drives the cocycle ever closer to a constant one. If this were exact, it would be the content of almost reducibility (cf. Definition 3.2). However, the periods of the conjugations are expected to grow, and this is a quite unsatisfactory technical complication of the scheme. In [15], the problem of loss of periodicity was settled outside the iterative step of the scheme, after a sufficient number of iterations. A reducible cocycle was obtained as an arbitrarily small perturbation of the given one by means of an embedding into a one-parameter family and then using the reducibility theorem in a positive measure set of parameters.

We were able to deal with this unnatural complication by proving a more efficient local conjugation lemma, in which the phenomenon of longer periods is no longer present, and which can serve as an iterative step of a KAM scheme. This improved scheme can be used in the proof of a local differentiable rigidity theorem which improves the one obtained in [10]. Differentiable rigidity is exactly the phenomenon where the assumption of the existence of a measurable conjugation forces the conjugation to be in fact $C^\infty$ smooth, under some relevant assumptions.

Before stating the theorem, let us make some remarks on the necessity of its hypotheses. Firstly, already when $G$ is a torus, a Diophantine condition on $\alpha$ is necessary, see, e.g., [8]. Such a condition, defined in 2.1, quantifies the badness of approximation of $\alpha$ by rational numbers. For the same reasons, a similar condition should be expected to be imposed to the “rotation number in the fibers”, which in this case is $a_d$, the argument of the eigenvalues of the constant $A_d$ to which the cocycle is conjugated. This is a Diophantine condition relative to $\alpha$, and quantifies the badness of the approximation of $a_d$ by iterates of $\alpha$. Such a condition, denoted by $DC_\alpha$ and defined in 5.1, is of full Haar measure in $G$ for every fixed $\alpha$. It is in fact proven to be necessary in our later work [11].
We recall that $G$ stands for a semisimple compact Lie group and $g$ for its Lie algebra, and state the following local differentiable rigidity theorem.

**Theorem 1.1.** Let $\alpha \in DC(\gamma, \tau)$, $A \in G$ a constant, and $(\alpha, Ae^{F(\cdot)}) \in SW_{c}^{\infty}(\mathbb{T}^{d}, G)$ with $F(\cdot): \mathbb{T}^{d} \to g$, $C^{\infty}$ smooth and small enough so that the KAM scheme can be initiated. Suppose, moreover, that there exists $D(\cdot): \mathbb{T}^{d} \to G$, measurable, such that

$$\text{Conj}_{D(\cdot)}(\alpha, A(\cdot)) = (\alpha, A_{d}),$$

where $A_{d} \in DC_{\alpha} \subset G$. Then, the KAM scheme can be made (with an appropriate adjustment of a parameter) to produce a $C^{\infty}$ conjugation. In particular, $(\alpha, Ae^{F(\cdot)})$ is $C^{\infty}$ reducible.

The proof is carried out in $\mathbb{T}^{d} \times SU(2)$, then extrapolated to a general compact Lie group $G$ by means of an appropriate embedding $SU(2) \hookrightarrow G$, so that no background on Lie group theory is needed for the greatest part of the note. The improvement in comparison with [10] consists partly in the more general algebraic context of our theorem, but mainly in the fact that the smallness of the perturbation in our theorem is not related with the measurable conjugation, but only with the applicability of the KAM machinery.

Subsequently, we briefly discuss the opposite phenomenon observed for “generic” cocycles smoothly conjugate to Liouville constant cocycles, where the KAM scheme cannot be made to produce a conjugacy by a simple adjustment of a parameter, and direct the reader to [11] for an improvement of the scheme which settles this problem.

Finally, we use the convergence of renormalization, as well as the measurable invariance of the degree (see [13]) in order to obtain a global differentiable rigidity theorem, without any assumption of closeness to constants, but valid only for one-frequency cocycles ($d = 1$).

**Theorem 1.2.** We suppose that $\alpha \in RDC$ and that $(\alpha, A(\cdot)) \in SW_{c}^{\infty}(\mathbb{T}, G)$ satisfies, for some measurable $D(\cdot): \mathbb{T} \to G$ and some $A_{d} \in DC_{\alpha}$,

$$\text{Conj}_{D(\cdot)}(\alpha, A(\cdot)) = (\alpha, A_{d}).$$

Then, $(\alpha, A(\cdot))$ is $C^{\infty}$ reducible.

For the proof of this theorem, we will need to define the notion of a degree of a cocycle, in the special case where $\text{deg}(\alpha, A(\cdot)) = 0$. This is essentially equivalent to

$$\frac{1}{n} \partial A_{n}(\cdot) \to 0 \text{ in } L^{2}(\mathbb{T}, TG)$$

($TG$ is the tangent bundle of $G$). This condition assures that renormalization (see [16, 2, 13]) converges to constants. This fact, combined with the assumption $\alpha \in RDC$ (of full measure in $\mathbb{T}$), implies that there exists $\tilde{D}(\cdot) \in C^{\infty}(\mathbb{T}, G)$ such that $\text{Conj}_{\tilde{D}(\cdot)}(\alpha, A(\cdot))$ satisfies the assumptions of Theorem 1.1. We give a brief outline of the proof, and let the readers fill in the details by referring them to the articles cited above.
Outline of the proof. By the measurable invariance of the degree, cf. [13, Proposition 6.7], and the fact that the degree of a constant cocycle is 0, we have deg(α, A(·)) = 0, which implies that the cocycle (α, A(·)) is renormalized close to a constant one ([13, Theorem 6.5]). Then, the assumption that α ∈ RDC implies that
\[ (α, A(·)) \sim (α, Ae^{F(·)}) \]
with F(·) small enough so that the KAM scheme can be initiated (see Chapter 9 of the reference). In this context, Theorem 1.1 applies and this proves the theorem.

2. Facts from algebra, calculus and arithmetics

2.1. The group SU(2). The matrix group \( G = SU(2) \) is the multiplicative group of unitary 2 × 2 matrices of determinant 1.

Let us denote the matrix \( S \in G, S = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \), where \((z, w) \in \mathbb{C}^2 \) and \(|z|^2 + |w|^2 = 1\), by \((z, w)_G\), and the subscript will be omitted unless necessary. The manifold \( G = SU(2) \) is thus naturally identified with \( S^3 \subset \mathbb{C}^2 \). When coordinates in \( \mathbb{C}^2 \) are fixed, the circle \( S^1 \) is naturally embedded in \( G \) as the group of diagonal matrices, which is a maximal torus (i.e., a maximal abelian subgroup) of \( G \). The center of \( G \), denoted by \( Z_G \), is equal to \( \{± \text{Id} \} \).

The Lie algebra \( g = su(2) \) is naturally isomorphic to \( \mathbb{R}^3 \approx \mathbb{R} \times \mathbb{C} \) equipped with its vector and scalar product. It will be denoted by \( g \). The element \( s = \begin{pmatrix} it & u \\ -\bar{u} & -it \end{pmatrix} \) will be denoted by \( \{t, u\}_g \in \mathbb{R} \times \mathbb{C} \). Mappings with values in \( g \) will be denoted by
\[ U(·) = \{U_t(·), U_z(·)\}_g = U_t(·) h + U_z(·) j \]
in these coordinates, where \( U_t(·) \) is a real-valued and \( U_z(·) \) is a complex-valued function. The vectors \( h, j, ij \) form an orthonormal and positively oriented basis for \( su(2) \).

The adjoint action of \( h \in su(2) \) on itself is pushed-forward to twice the vector product:
\[ ad_{\{1,0\}}[0, 1] = \{[1, 0], [0, 1]\} = 2[0, i] = 2ij \]
plus cyclic permutations, and the Cartan-Killing form, normalized by \( \langle h, h' \rangle = -\frac{1}{8\pi} tr(ad(h) \circ ad(h')) \) is pushed-forward to the scalar product of \( \mathbb{R}^3 \). The preimages of \( Z_G \) in the maximal toral (i.e., abelian) algebra of diagonal matrices are points with coordinates in the lattice \( \pi \mathbb{Z} \).

The adjoint action of the group \( SU(2) \) on its algebra is pushed-forward to the action of \( SO(3) \approx SU(2)/\pm \text{Id} \) on \( \mathbb{R} \times \mathbb{C} \). In particular, for diagonal matrices of the form \( S = \exp([2\pi s, 0]_g), s \in \mathbb{R}, Ad(S)(\{t, u\} = \{t, e^{i\pi s} u\} \}

2.2. General compact groups. For the notation introduced in this section (as well as the previous), we follow [13]. Therein, one can find a more detailed summary of the subject, which is based in turn on [4, 5, 3, 9] and also [15].
The only fact that we will need from the theory of semisimple groups is the decomposition of the Lie algebra $g$ of such a group $G$ in factors isomorphic to $su(2)$. If $g$ is such an algebra, the orthogonal complement of $t$, a maximal abelian algebra (i.e., a subalgebra on which the restriction of the Lie bracket vanishes identically) admits a decomposition into a direct sum of mutually orthogonal 2-dimensional real subspaces $E_{\rho}$, $\rho \in \Delta_+$, for which the following holds. For each $E_{\rho}$, there exists $h_{\rho} \in t$ such that $\mathbb{R} h_{\rho} \oplus E_{\rho} \cong su(2)$. One can choose a vector $j_{\rho} \in E_{\rho}$, such that the vectors $h_{\rho}, j_{\rho}, i j_{\rho}$ form a basis of $\mathbb{R} h_{\rho} \oplus E_{\rho}$, satisfying the same relations as the basis of $su(2)$.

There exists, moreover, $\tilde{\Delta} \subset \Delta_+$, such that $(h_{\rho})_{\tilde{\Delta}}$ is a basis for $t$, for which there exists $P \in \mathbb{N}^*$ and $m_{\rho, \rho'} \in \mathbb{N}$ such that, for all $\rho' \in \Delta_+$,

$$h_{\rho'} = \frac{1}{P} \sum_{\rho \in \Delta} m_{\rho, \rho'} h_{\rho}.$$  

For $\rho \in \Delta_+$, and for all $a \in t$, we have $[a, j_{\rho}]= 2 i \pi \rho(a) j_{\rho} = 2 i \pi \alpha_{\rho} j_{\rho}$. The adjoint action of $G$ on $t$ (and therefore on the vectors $h_{\rho}$) induces an action of $G$ on $\Delta_+$.

The Lie algebra $g$ is thus decomposed into

$$c \oplus t \oplus \left( \bigoplus_{\rho \in \Delta_+} E_{\rho} \right) = c \oplus \bigoplus_{\rho \in \Delta} \mathbb{R} h_{\rho} \bigoplus_{\rho \in \Delta_+} C j_{\rho}.$$ 

Here, $c$ is the center of the algebra, which is trivial if the algebra is semisimple. This decomposition is referred to as “the root space decomposition with respect to the toral algebra $t$.”

The lattice of preimages of $Z_G$ in $t$ will be denoted by $\mathcal{Z}$. Finally, we will abuse the notation $A^*$ for $A^{-1}$, since, in the more familiar and important examples of compact Lie groups, the unitary and special unitary groups, the Hermitian transpose coincides with the inverse. Moreover, in the KAM part of the theory, even for a general semisimple compact Lie group $G$, its natural unitary representation on the Lie algebra $g$ plays a central role.

2.3. Calculus and functional spaces. We will consider the space $C^\infty(\mathbb{T}^d, g)$ endowed with the standard maximum norms

$$\|U\|_s = \max_{0 \leq \sigma \leq s} \| \partial^\sigma U(\cdot) \|_\infty$$

for $s \geq 0$, and the Sobolev norms

$$\|U\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\hat{U}(k)|^2,$$

where $\hat{U}(k) = \int U(\cdot) e^{-2i\pi k x} \: dx$ are the Fourier coefficients of $U(\cdot)$. The fact that the injections $H^{s + d/2}(\mathbb{T}, g) \hookrightarrow C^s(\mathbb{T}^d, g)$ and $C^s(\mathbb{T}^d, g) \hookrightarrow H^s(\mathbb{T}^d, g)$ are continuous, for all $s \geq 0$, is classical.

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1 The finite set $\Delta_+$ is called the set of positive roots of $g$, and it is a subset of $t^*$. 
We recall the convexity or Hadamard-Kolmogorov inequalities (see [14]) for $U \in C^\infty(T, g)$:

$$\|U(\cdot)\|_\sigma \leq C_{s,\sigma} \|U\|_0^{1-\sigma/s} \|U\|_s^{\sigma/s},$$

where $0 \leq \sigma \leq s$, and the inequalities concerning the composition of functions (see [15]):

$$\|\phi \circ (f + u) - \phi \circ f\|_2 \leq C_\sigma \|\phi\|_{s+1} (1 + \|f\|_0)^3 (1 + \|f\|_s) \|u\|_s.$$

We will use the truncation operators for mappings $T^d \to g$ defined by

$$T_N f(\cdot) = \sum_{|k| \leq N} \hat{f}(k) e^{2\pi i k \cdot \cdot},
\hat{T}_N f(\cdot) = T_N f(\cdot) - \hat{f}(0)
\hat{R}_N f(\cdot) = \sum_{|k| > N} \hat{f}(k) e^{2\pi i k \cdot \cdot}.$$

These operators satisfy the estimates

$$(2) \quad \|T_N f(\cdot)\|_{C^s} \leq C_{s} N^{d/2} \|f(\cdot)\|_{C^s},
(3) \quad \|\hat{R}_N f(\cdot)\|_{C^s} \leq C_{s,\sigma} N^{\sigma-\sigma/s} d/2 \|f(\cdot)\|^s_{C^s}.$$

The Fourier spectrum of a function will be denoted by $\hat{\sigma}(f) = \{k \in \mathbb{Z}^d, \hat{f}(k) \neq 0\}$.

2.4. Arithmetics and continued fraction expansion. The following notion is essential in KAM theory. It is related to the quantification of the closeness of rational numbers to certain classes of irrational numbers.

**Definition 2.1.** We will denote by $DC(\gamma, \tau)$ the set of numbers $\alpha$ in $T^d$ such that, for any $k \in \mathbb{Z}^d - \{0\}$,

$$|\alpha \cdot k|_Z \geq \gamma^{-1} \frac{1}{|k|\tau}.$$

Such numbers are called Diophantine.

The set $DC(\gamma, \tau)$ for $\tau > d + 1$ fixed and $\gamma \in \mathbb{R}_+^*$ is of positive Haar measure in $T^d$. If we fix $\gamma$ and let $\tau$ run through the positive real numbers, we obtain $\bigcup_{\tau > 0} DC(\gamma, \tau)$ which is of full Haar measure. The numbers that do not satisfy any Diophantine condition are called Liouvillian. They form a residual set of 0 Lebesgue measure.

The following definition relates the Diophantine condition satisfied by an irrational number in $T$ with the continued fractions algorithm. We stress that, here, $d = 1$.

**Definition 2.2.** We will denote by $RDC(\gamma, \tau)$ the set of recurrent Diophantine numbers, i.e., the numbers $\alpha$ in $T \sim Q$ such that $G^n(\alpha) \in DC(\gamma, \tau)$ for infinitely many $n$.

Here, $G(\alpha) = \{\alpha^{-1}\}$ is the Gauss map (\{\cdot\} stands for “fractional part”). The set $RDC$ is also of full measure, since the Gauss map is ergodic with respect to a smooth measure with everywhere positive density.
In contexts where the parameters $\gamma$ and $\tau$ are not significant, they will be omitted in the notation of both sets.

3. Cocycles in $\mathbb{T}^d \times G$

3.1. The dynamics. Let $\alpha \in \mathbb{T}^d \equiv \mathbb{R}^d / \mathbb{Z}^d$, $d \in \mathbb{N}^*$, be an irrational rotation, so that the translation $x \mapsto x + \alpha \mod (\mathbb{Z}^d)$ is minimal and uniquely ergodic. The translation will sometimes be denoted by $R_\alpha$. We also remind that $G$ is a semi-simple compact Lie group.

If we also let $A(\cdot) \in C^\infty(\mathbb{T}^d, G)$, the couple $(\alpha, A(\cdot))$ acts on the fibered space $\mathbb{T}^d \times G \to \mathbb{T}^d$ defining a diffeomorphism by

$$(\alpha, A(\cdot)).(x, S) = (x + \alpha, A(x).S).$$

We will call such an action a quasiperiodic cocycle over $R_\alpha$ (henceforth, simply a cocycle). The space of such actions is denoted by $SW^\infty_\alpha(\mathbb{T}^d, G) \subset \text{Diff}^\infty(\mathbb{T}^d \times G)$. Most times we will abbreviate the notation to $SW^\infty_\alpha$. Cocycles are a class of fibered diffeomorphisms, since fibers of $\mathbb{T}^d \times G$ are mapped into fibers, and the mapping from one fiber to another depends, in general, on the base point. The number $d \in \mathbb{N}^*$ is the number of frequencies of the cocycle.

If we consider a representation of $G$ on a vector space $E$, the action of the cocycle can be also defined on $\mathbb{T}^d \times E$, simply by replacing $S$ by a vector in $E$ and multiplication in $G$ by the action. The particular case which will be important in this article is the representation of $G$ on $g$, and the resulting action of the cocycle on $\mathbb{T}^d \times g$.

The $n$-th iterate of the action is given by

$$(\alpha, A(\cdot))^n.(x, S) = (n\alpha, A_n(\cdot))(x, S) = (x + n\alpha, A_n(x).S),$$

where $A_n(\cdot)$ represents the quasiperiodic product of matrices equal to

$$A_n(\cdot) = \begin{cases} A(\cdot + (n-1)\alpha) \cdots A(\cdot), & n > 0 \\ Id, & n = 0 \\ A^*(\cdot + n\alpha) \cdots A^*(\cdot - \alpha), & n < 0 \end{cases}$$

where we recall the abuse of notation $A^* = A^{-1}$.

3.2. Classes of cocycles with simple dynamics, conjugation. The cocycle $(\alpha, A(\cdot))$ is called a constant cocycle if $A(\cdot) = A \in G$ is a constant mapping. In that case, the quasiperiodic product reduces to a simple product of matrices, $(\alpha, A)^n = (n\alpha, A^n)$.

The group $C^\infty(\mathbb{T}^d, G) \equiv SW^\infty_0(\mathbb{T}^d, G)$ acts by dynamical conjugation: Let $B(\cdot) \in C^\infty(\mathbb{T}^d, G)$ and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}^d, G)$. Then we define

$$\text{Conj}_{B(\cdot)}(\alpha, A(\cdot)) = (0, B(\cdot)) \circ (\alpha, A(\cdot)) \circ (0, B(\cdot))^{-1} = (\alpha, B(\cdot + \alpha).A(\cdot).B^*(\cdot))$$
which is in fact a change of variables within each fiber of the product $\mathbb{T}^d \times G$. The dynamics of $\text{Conj}_{B(\cdot)}(\alpha, A(\cdot))$ and $(\alpha, A(\cdot))$ are essentially the same, since

\[
\{\text{Conj}_{B(\cdot)}(\alpha, A(\cdot))\}^n = \{n\alpha, B(\cdot + n \alpha).A_n(\cdot), B^*(\cdot)\}.
\]

**Definition 3.1.** Two cocycles $(\alpha, A(\cdot))$ and $(\alpha, \tilde{A}(\cdot))$ in $SW^\infty_\alpha$ are $H^s$-conjugate, $0 \leq s \leq \infty$, iff there exists $B(\cdot) \in H^s(\mathbb{T}^d, G)$ such that $(\alpha, \tilde{A}(\cdot)) = \text{Conj}_{B(\cdot)}(\alpha, A(\cdot))$. We will use the notation $(\alpha, A(\cdot)) \sim (\alpha, \tilde{A}(\cdot))$ to state that the two cocycles are conjugate to each other. When $s = \infty$ we will simply say that the cocycles are conjugate.

Since constant cocycles are a class whose dynamics can be analyzed, we give the following definition.

**Definition 3.2.** A cocycle will be called reducible iff it is conjugate to a constant.

Due to the fact that not all cocycles are reducible (e.g., generic cocycles in $\mathbb{T} \times \mathbb{S}^1$ over Liouvillean rotations, but also cocycles over Diophantine rotations, even though this is harder to obtain, see [6], [16]) we also need the following concept, which is crucial in the study of such dynamical systems.

**Definition 3.3.** A cocycle $(\alpha, A(\cdot))$ is said to be almost reducible iff there exists a sequence of conjugations $B_n(\cdot) \in C^\infty$, such that $\text{Conj}_{B_n(\cdot)}(\alpha, A(\cdot))$ becomes arbitrarily close to constants in the $C^\infty$ topology, i.e., iff there exists $(A_n)$, a sequence in $G$, such that

\[
A^*_n\{B_n(\cdot + \alpha)A(\cdot)B^*_n(\cdot)\} = e^{F_n(\cdot)} C^\infty \to Id.
\]

This property will, herein, be established in a KAM constructive way, making it possible to measure the rate of convergence against the explosion of the conjugations. Almost reducibility then comes along with obtaining that

\[
\text{Ad}(B_n(\cdot)).F_n(\cdot) = B_n(\cdot).F_n(\cdot).B^*_n(\cdot) \xrightarrow{C^\infty} 0.
\]

If this additional condition is satisfied, almost reducibility in the sense of the definition above and almost reducibility in the sense that “the cocycle can be conjugated arbitrarily close to reducible cocycles” are equivalent.

We can now recall the local almost reducibility theorem. It is used in the proof of the local density theorem, already proved in [15]. We will follow the proof in [13], since it implies a useful corollary (Corrolary 4.2) in a more direct manner than the previously existing proofs.

**Theorem 3.4.** Let $\alpha \in DC(\gamma, \tau) \subset \mathbb{T}^d$, $d \geq 1$, and $G$ a semisimple compact Lie group. Then, there exist $s_0 \in \mathbb{N}^*$ and $\epsilon > 0$ such that, if $(\alpha, Ae^{F(\cdot)}) \in SW^\infty_\alpha(\mathbb{T}^d, G)$ with $\|F(\cdot)\|_0 < \epsilon$ and $\|F(\cdot)\|_{s_0} < 1$, then $(\alpha, Ae^{F(\cdot)})$ is almost reducible.

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4. Almost reducibility

In this section we present the basic steps of the proof of Theorem 3.4. For the next subsection was 'paragraph', $G = SU(2)$, and the proof of the local conjugation lemma when $G$ is an arbitrary compact Lie group will be hinted in the next one.

4.1. Local conjugation in $\mathbb{T}^d \times SU(2)$. Let

$$(a, Ae^{F(\cdot)}) = (a, A_1 e^{F_1(\cdot)}) \in SW^\infty(\mathbb{T}^d, SU(2))$$

be a cocycle over a Diophantine rotation with $F(\cdot)$ satisfying some smallness conditions to be made more precise later on. Without any loss of generality, we can also suppose that $A_1 = [e^{2\pi i a_i}, 0]$ is diagonal. The goal is to conjugate the cocycle ever closer to constant cocycles by means of an iterative scheme. This is obtained by iterating the following lemma, for the detailed proof of which we refer to [15], [6] or [13]. This lemma is the cornerstone of the procedure, since it represents one step of the scheme. The rest of this paragraph is devoted to a summary of its proof, for the sake of completeness.

**Lemma 4.1.** Let $\alpha \in DC(\gamma, \tau)$ and $K \succeq C\gamma N^T$. Let, also, $(a, Ae^{F(\cdot)}) \in SW^\infty(\mathbb{T}^d, G)$ with

$$c_{1,0}K N^{s_0} \varepsilon_0 < 1$$

for some $s_0 \in \mathbb{N}^*$ depending on $d, \gamma, \tau$, and where $\varepsilon_s = \|F\|_s$. Then, there exists a conjugation $G(\cdot) \in C^\infty(\mathbb{T}, G)$ such that

$$(4) \quad G(\cdot + \alpha).Ae^{F(\cdot)}.G^*(\cdot) = A'e^{F'(\cdot)}.$$  

The mappings $G(\cdot)$ and $F'(\cdot)$ satisfy the following estimates

$$\|G(\cdot)\|_s \leq c_{1,5}(N^s + K N^{\tau + 1/2} \varepsilon_0)$$

$$\varepsilon_s' \leq c_{2,5} K^2 N^{2\tau + d} (N^s \varepsilon_0 + \varepsilon_s) \varepsilon_0 + C_{s'e} N^{s' + 2\tau + d} \varepsilon_s.$$  

**Sketch of the proof.** If we suppose that $Y(\cdot) : \mathbb{T}^d \to g$ can conjugate $(a, Ae^{F(\cdot)})$ to $(a, A'e^{F'(\cdot)})$ with $\|F'(\cdot)\| \ll \|F(\cdot)\|$, then it must satisfy the functional equation

$$A^* e^{Y(\cdot + \alpha)} A e^{-Y(\cdot)} = A' e^{F'(\cdot)}.$$  

Linearization of this equation, under the assumption that all $C^0$ norms are smaller than 1, gives

$$(5) \quad Ad(A^*) Y(\cdot + \alpha) + F(\cdot) - Y(\cdot) = \exp^{-1}(A^* A')$$

which we will write in coordinates, separating the diagonal from the non-diagonal part.

The equation for the diagonal coordinate reads $Y_\ell(\cdot + \alpha) - Y_\ell(\cdot) = -F_\ell(\cdot)$. For reasons well known in KAM theory, we have to truncate at an order $N$ to be determined by the parameters of the problem and obtain a solution to the equation

$$Y_\ell(\cdot + \alpha) - Y_\ell(\cdot) = -T_N F_\ell(\cdot) = -T_N F_\ell(\cdot) + \tilde{F}_\ell(0)$$
satisfying the estimate \( \|Y_t(\cdot)\|_s \leq \gamma C_z N^{s+\tau+d/2} \|F_t(\cdot)\|_0 \). The rest satisfies the estimate of equation (3). The mean value \( \hat{F}_t(0) \) is an obstruction and will be integrated in \( \exp^{-1}(A^* A') \).

As for the equation concerning the non-diagonal part, it reads

\[
e^{-4i\pi a} Y_{\epsilon}(\cdot + \alpha) - Y_{\epsilon}(\cdot) = -F_{\epsilon}(\cdot)
\]
or, in the frequency domain,

\[
(e^{2i\pi k\cdot} - 1) \hat{Y}_z(k) = -\hat{F}_z(k), \ k \in \mathbb{Z}^d.
\]

Therefore, the Fourier coefficient \( \hat{F}_z(k_r) \) cannot be eliminated with good estimates if

\[|k_r \cdot \alpha - 2a|_2 < K^{-1}\]

for some \( K > 0 \) big enough. If \( K = N^\nu \), with \( \nu > \tau \), then we know by [6] that, if such a \( k_r \) exists (called a resonant mode) and satisfies \( 0 \leq |k_r| \leq N \), it is unique in \( \{k \in \mathbb{Z}^d, |k - k_r| \leq 2N\} \). Therefore, if we call \( T_{2N} \) the truncation operator projecting on the frequencies \( 0 < |k - k_r| \leq 2N \) if \( k_r \) exists (or on \( |k| \leq N \) if it does not, but this case is easier and follows from the one that we treat), the equation

\[
e^{-4i\pi a} Y_{\epsilon}(\cdot + \alpha) - Y_{\epsilon}(\cdot) = -T_{2N} F_{\epsilon}(\cdot)
\]
can be solved and the solution satisfies \( \|Y_{\epsilon}(\cdot)\|_s \leq C_z N^{s+\nu+d/2} \|F_{\epsilon}(\cdot)\|_0 \). We will define the rest operator by projection on modes satisfying \( |k - k_r| > 2N \).

In total, the equation that can be solved with good estimates is

\[
Ad(A^*) Y(\cdot + \alpha) - Y(\cdot) = -F(\cdot) + \{\hat{F}_t(0), \hat{F}_z(k_r)e^{2i\pi k_r \cdot} + \{R_N F_t(\cdot), R_{2N}^k F_z(\cdot)\}
\]

with \( \|Y(\cdot)\|_s \leq C_z N^{s+\nu+1/2} \|F(\cdot)\|_0 \). Under the smallness assumptions of the hypothesis, the linearization error is small and the conjugation thus constructed satisfies

\[
e^Y(\cdot + a) A e^{F(\cdot)} e^{-Y(\cdot)} = [e^{2i\pi (a + \hat{F}_t(0)), 0}]_G e^{0, \hat{F}_z(k_r)e^{2i\pi k_r \cdot}} e^{\hat{F}(\cdot)}
\]

with \( \hat{F}(\cdot) \) a “quadratic” term. We remark that, a priori, the obstruction

\[\{0, \hat{F}_z(k_r)e^{2i\pi k_r \cdot}\}\]

is of the order of the initial perturbation and, thus, what we call \( \exp^{-1}(A^* A') \) in equation (5) is not constant in the presence of a non-zero resonant mode.

Therefore, if \( k_r \) exists and is non-zero, the application of the lemma cannot be iterated. On the other hand, the conjugation \( B(\cdot) = \{e^{-2i\pi k_r \cdot/2}, 0\} \) is such that, if we denote

\[
F''(\cdot) = Ad(B(\cdot)) F(\cdot) = \{F_t(\cdot), e^{-2i\pi k_r \cdot} F_z(\cdot)\}
\]

\[
Y''(\cdot) = Ad(B(\cdot)) Y(\cdot)
\]

\[A'' = B(\alpha) A = \{e^{2i\pi (\alpha - k_r \cdot/2)}, 0\},\]

then

\[
Ad((A'')^*) Y'(\cdot + \alpha) - Y'(\cdot) + F''(\cdot) = \{\hat{F}_t(0), \hat{F}_z(k_r)\} + \{R_{2N} F_t(\cdot), e^{-2i\pi k_r \cdot} R_{2N}^k F_z(\cdot)\}
\]

\[= \{F''(0), \hat{F}_z''(0)\} + \{R_{2N} F''(\cdot), R_{2N}^k F_z(\cdot)\}.
\]
Here, \( R_{Z_N}^k \) is a decentered rest operator, whose spectral support is outside of \([-N, N]^d \cap \mathbb{Z}^d\), and can therefore be estimated like a classical rest operator \( R_N \). The equation for primed variables can be obtained from equation (6) by applying \( Ad(B(\cdot)) \) and using that \( B(\cdot) \) is a morphism and commutes with \( A \). The passage from one equation to the other is (essentially, i.e., up to the error of linearization) equivalent to the fact that

\[
Conj_{B(\cdot)}(\alpha, A.\exp([\hat{F}_t(0), \hat{F}_{\varepsilon}(k_r)e^{2i\pi k_r}])) = (\alpha, B(\alpha)A.\exp([\hat{F}_t(0), \hat{F}_{\varepsilon}(k_r)])) = (\alpha, \hat{A}),
\]

that is, \( B(\cdot) \) reduces the initial constant perturbed by the obstructions to a cocycle close to \((\alpha, \pm \text{Id})\). There is a slight complication, as \( B(\cdot) \) may be 2-periodic (if \((i\pi k_r, 0) \in g \) is a preimage of \(-\text{Id}\)). If it is so, we can conjugate a second time with a geodesic of minimal length \( C(\cdot) : 2\mathbb{T} \rightarrow G \) such that \( C(1) = -\text{Id} \in Z_G \) and commuting with \( \hat{A} \). The cocycle that we obtain in this way is 1-periodic and close to \((\alpha, e^{i\pi a}, 0)_G\), and the conjugation is also 1-periodic.

Summing up, if we denote \( G(\cdot) = C(\cdot)B(\cdot)e^{\hat{F}(\cdot)} \) and \( A' = C(\alpha)\hat{A} \), then there exists \( F(\cdot) \) satisfying the estimates of the lemma and such that equation (4) is verified.

4.2. Local conjugation in general Lie groups. In this subsection, we briefly revise the generalization of the proof of the local conjugation lemma to a general Lie group \( G \).

Local conjugation in a compact group \( G \) is in fact a multi-dimensional version of the lemma of the previous paragraph. If we still denote by \((\alpha, A(\cdot)) \in SW^\infty_\alpha(\mathbb{T}^d, G)\) the given cocycle, the statement is the same, just replacing \( SU(2) \) by a compact group \( G \), and the steps of the proof are the same.

The first one is solution of the linear equation, which is done in \((\alpha, t)\) as in the diagonal coordinates here above, and in each \( E_\rho \) as in equation (7). We recall that \( \mathfrak{c} \) and \( \mathfrak{t} \) stand for the center of the Lie algebra \( g \) and the Lie algebra of a torus \( \mathcal{T} \) of \( G \) such that \( A \in \mathcal{T} \), respectively. Since \( A \) commutes with all elements in \((\alpha, t)\), the equation is as in the diagonal coordinate of the \( SU(2) \) case. Moreover, the action of \( A \) on each \( E_\rho \) is of the same nature as the action of a diagonal constant in \( SU(2) \) on the non-diagonal part, and one only has to take care of the linear dependence relations between the action on the different \( E_\rho \).

More precisely, if \( j_\rho \in E_\rho \), then there exists \( a_\rho \in \mathbb{R} \) such that \( Ad(A).j_\rho = e^{2i\pi a_\rho}j_\rho \). The terms \((a_\rho)_{\rho \in \Delta} \) are not linearly independent, in general.

The procedure of reducing resonant modes in each \( E_\rho \) produces a “vector”\(^3\) \((k_\rho)_{\rho \in \Delta}, \in \bigcup_{\Delta} \mathbb{Z} \) of resonant modes. They are reduced in the second step, where

\(^2\)\( Z_G \) is the center of the group \( G \), equal to \( \pm \text{Id} \) when \( G = SU(2) \).

\(^3\)Some entries may be \( 0 \), if there is no resonant mode in the corresponding eigenspace. Otherwise, they are equal to the corresponding frequency in \( \mathbb{Z} \).
we construct a vector $H$ such that

$$\text{Conj}_{\exp(H)}(\alpha, A.\exp(\{\text{Ob}\cdot(0) + \text{Ad}(e^H).\text{Ob}\cdot(\cdot)\})) = (\alpha, \tilde{A}).$$

Here, $\text{Ob}$ stands for projection on the resonant modes, $\sum_{\rho \in \Delta_+} \tilde{F}(k_p)e^{2i\pi \cdot j_p}$. A vector $J \in \frac{1}{\tau}\mathbb{Z}$ is constructed by solving the linear system of equation (1) for the vector $(k_p)_{p \in \Delta_+}$. Finally, since the mapping $\exp(J)$ may be $1/P_{\mathbb{Z}}$-periodic, we post-conjugate with $C(\cdot)$, a minimal geodesic connecting the $\text{Id}$ with $\exp(J)$ (which measures the failure of $\exp(J)$ to be 1-periodic) and commuting with $\tilde{A}$. We thus obtain $A'$ which is close to an element of $\alpha \frac{1}{\tau}\mathbb{Z}$, where we recall that $\mathbb{Z}$ is the lattice of preimages of the $\mathbb{Z}_G$ in $t$.

4.3. **Some remarks on the local conjugation lemma.** We would like to briefly comment on the conjugation lemmas of the two previous sections. The difference between our lemma and the corresponding one proved in [6] is quite marginal in $SU(2)$. In the work by H. Eliasson, the group is actually $SO(3)$, and the problem of growth of periods of conjugations is not present, due to the fact that $\text{Inn}(SO(3))$, the group which acts by the adjoint action on $so(3)$, is equal to $SO(3)$ itself. On the other hand,

$$\text{Inn}(SU(2)) = SO(3) = SU(2)/\{\pm \text{Id}\}$$

and, if this fact is not taken care of, the periods may double upon introducing a far-from-the-$\text{Id}$ conjugation, $B(\cdot)$. The special case of $SU(m)$ was already treated in [15], but the method works only for such groups. The difference between the two lemmas becomes significant when general groups are considered, but let us give a first description of the difference in $SU(2)$.

In the works by H. Eliasson and R. Krikorian, $B(\cdot)$ is constructed first in a process of reduction of the resonance. In other words, the argument of the resonant eigenvalue which is close to $k_r \alpha$ is driven close to 0 by means of the conjugation by $B(\cdot)$. Then, since “the eigenvalue 0 is not resonant” (i.e., it produces only constant obstructions), the classical local conjugation lemma used herein to construct $Y(\cdot)$ works well (just set $k_r = 0$). Since, however, the resonances live in $\frac{1}{2}\alpha \mathbb{Z}$, $B(\cdot)$ may be 2-periodic, so that, in local notation,

$$(\alpha, \tilde{A} e^{\tilde{F}(\cdot)}) = \text{Conj}_{B(\cdot)}(\alpha, A e^{F(\cdot)})$$

with $\tilde{F}(\cdot)$ possibly 2-periodic$^5$. The close-to-the-$\text{Id}$ conjugation $Y(\cdot)$ will then be 2-periodic, and the period of the perturbation will double every time a resonance is reduced. In the case of a general group $G$, this problem was dealt with by R. Krikorian after having applied the conjugation lemma with “period doubling” for a sufficient number of times so that the perturbation had already become sufficiently small.

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$^4$Intersected with the ball in $g$ where $\exp$ is bijective.

$^5$In fact, this can be seen to be not exact since $\tilde{F}(\cdot) = \text{Ad}(B(\cdot))F(\cdot)$, which is 1-periodic. Since, however, this is due to the particular structure of the $SU(m)$ groups, we do not want to use this argument.
In this work, we follow the construction of [13], which we discuss now in the $SU(2)$ case. Therein, $Y(\cdot)$ is constructed first, and it conjugates $(\alpha, A e^{F(\cdot)})$ to a constant cocycle perturbed by the obstructions plus a second order term. The obstructions are a constant one on the diagonal (which poses no problem) and eventually a non-constant one in the non-diagonal direction (which, in turn, has to be treated). Then, the conjugation $B(\cdot)$ is constructed in the same way as in the works by H. Eliasson and R. Krikorian, but from our point of view it reduces the resonant mode, and not the resonance. Therefore, $B(\cdot)$ is also eventually 2-periodic. If this is the case, it is due to the fact that

$$B(\cdot + 1).B^* (\cdot) = -Id.$$  

This can be easily rectified by introducing the conjugation $C(\cdot)$, which must satisfy the following constraints. Firstly, it is a torus morphism such that $C(1) = -Id$, so that $C(\cdot).B(\cdot)$ is 1-periodic. Secondly, $C(\cdot)$ must commute with the new constant, $\tilde{A}$, so that no new resonant modes are activated.

This construction may seem uselessly complicated when the group is simply $SU(2)$, but this construction can be readily generalized to any semisimple compact Lie group $G$. We think that for someone who is familiar enough with the structure of such groups, the solution of the problems of linear dependence between the different embeddings $SU(2) \hookrightarrow G$ and the resulting linear dependence relations between the corresponding resonant modes is sufficiently straightforward. We just notice that the resonant mode $k_{\rho'}$ of a root $\rho' \in \Delta_+ \sim \Delta$ satisfies

$$k_{\rho'} = \frac{1}{P} \sum_{\rho \in \Delta} m_{\rho, \rho'} k_{\rho}$$

with the same coefficients as in equation (1). In other words, the resonant mode of the linear combination is the linear combination of the resonant modes. This is not true for resonances.

This ends the construction of the vector $J \in g$. The construction of $C(\cdot)$ is hardly more complicated.

We finally refer the reader to [13] for more details on the subject and to subsequent works by the author, [11] and [12], for further development of the theory, and, in particular, for the optimality of the KAM scheme (i.e., the fact that it provides the necessary and sufficient information for constructing a reducing conjugation whenever it exists).

4.4. **The KAM scheme.** Lemma 4.1 can serve as the step of a KAM scheme, with the following standard choice of parameters: $N_{n+1} = N_n^{1+\sigma} = N_n^{(1+\sigma)^{n-1}}$, where $N = N_1$ is big enough and $0 < \sigma < 1$, and $K_n = N_n^v$ for some $v > \tau$. If we suppose that $(\alpha, A_n e^{F_n(\cdot)})$ satisfies the hypotheses of Lemma 4.1 for the corresponding parameters, then we obtain a mapping $G_n(\cdot) = C_n(\cdot).B_n(\cdot)e^{Y_n(\cdot)}$ that conjugates it to $(\alpha, A_{n+1} e^{F_{n+1}(\cdot)})$, and we use the notation $\varepsilon_{n,s} = \|F_n\|_s$.

If we suppose that the initial perturbation is small in small norm, $\varepsilon_{1,0} < \varepsilon < 1$, and not big in some bigger norm, $\varepsilon_{1,s_0} < 1$, where $\varepsilon$ and $s_0$ depend on the choice of parameters, then we can prove (see [13] and, through that, [7]), that the
scheme can be iterated, and moreover
\[ \epsilon_{n,s} = O(N_n^{-\infty}) \] for every fixed \( s \) and
\[ \|G_n\|_s = O(N_n^{s+\lambda}) \] for every \( s \) and some fixed \( \lambda > 0 \).

We say that the norms of perturbations decay exponentially, while conjugations grow polynomially.

4.5. A “KAM normal form”. The product of conjugations \( H_n = G_n \cdots G_1 \), which by construction satisfies \( \text{Conj}_{H_n} (\alpha, A_1 e^{F_1}) = (\alpha, A_{n+1} e^{F_{n+1}}) \), is not expected to converge. In fact, it converges iff \( B_n(\cdot) \equiv \text{Id} \), except for a finite number of steps. Anyhow, we can obtain a “KAM normal form” for cocycles close to constants

**Lemma 4.2.** Under the hypotheses of Theorem 3.4, there exists \( K(\cdot) \in C^\infty(\mathbb{T}^d, G) \) such that, if we call
\[ \text{Conj}_{K(\cdot)} (\alpha, A e^{F(\cdot)}) = (\alpha, A' e^{F(\cdot)}) , \]
then the KAM scheme applied to \( (\alpha, A' e^{F(\cdot)}) \) for the same choice of parameters consists only in the reduction of resonant modes. The resulting conjugation \( H_n(\cdot) \) has the form \( \prod_{n_i} C_{n_i}(\cdot).B_{n_i}(\cdot) \), where \( \{n_i\} \) are the steps in which reduction of a resonant mode took place.

We say that \( (\alpha, A' e^{F(\cdot)}) \) is in KAM normal form.

The proof, for which we refer the reader to [11], is short. It uses the fast convergence of the scheme in order to prove that a certain rearrangement of the conjugations \( Y_n \) and \( B_n \) converges. For a cocycle in normal form, we relabel the indexes as \( (\alpha, A_{n_i} e^{F_{n_i}}) = (\alpha, A_1 e^{F_1}) \).

5. Proof of Theorem 1.1

In this section, we use the KAM scheme outlined above. We begin by a brief study of the rigidity of conjugation between constant cocycles in general compact groups, which is to be compared with section 2.5.e of [15], and shows that conjugation between constant cocycles is rigid, with no assumptions on the arithmetics. We locally set \( d = 1 \) in order to keep notation simple.

5.1. The toy case. Let \( B : \mathbb{T} \to G \) be a measurable mapping, \( \alpha \in \mathbb{T} \) a minimal translation, and \( C_1, C_2 \in G \) such that
\[ B(\cdot + \alpha)C_1 B^*(\cdot) = C_2. \]

By composing \( B \) with a constant if necessary, we can suppose that \( C_1 \) and \( C_2 \) are on the same maximal torus. If, for simplicity in notation, we identify \( G \) and \( \text{Inn}(g) \approx G/ Z_G \), the group which acts by the adjoint action on \( g \), even though the identification is not accurate, we find that
\[ e^{2i\pi \alpha} \hat{B}(k) C_1 = C_2 \hat{B}(k) \]
so that, if \( e^{2i\pi \xi^0} \) are the eigenvalues of the adjoint action of \( C_i \), the equation
\[ \langle e^{2i\pi (\xi + \xi^0)} \hat{B}(k) f_\rho, j_{\rho'} \rangle = \langle \hat{B}(k) j_\rho, e^{2i\pi \xi^0} j_{\rho'} \rangle \]
has at most one non-zero solution in $k$ for each pair of roots $\rho$ and $\rho'$ for which there exists $Z \in G$ such that $\rho' = Z \rho$. Using a similar argument for directions in the torus, we find that $B : \mathbb{T} \to \text{Inn}(g)$, and therefore $B : \mathbb{T} \to G$, has a finite support in the frequency space, and therefore it is $C^\infty$. Finally, inspection of the calculations carried out in 2.5.e of [15] (or those carried out herein) shows that $B(\cdot)$ can be written as a product of two torus morphisms, which do not take values in the same maximal torus.\footnote{And, eventually, a representative of the Weyl group which permutes the different root spaces.}

5.2. Proof of Theorem 1.1. In order to simplify the proof and to avoid the phenomena related to loss of periodicity, we consider a unitary representation of $G$ and we suppose, firstly, that $D(\cdot + \alpha)A_1 e^{F_1(\cdot)} D^*(\cdot) = A_d$ with $D(\cdot) : \mathbb{T}^d \to G \to U(w)$ a measurable mapping and $F_1(\cdot)$ small enough so that the reduction scheme can be applied. The KAM scheme of the previous section, applied in the simpler algebraic context of $U(w)$, produces the sequence of conjugations $H_{n_1}(\cdot) = H_1(\cdot) = B_{n_1}(\cdot) \cdots B_{n_1}(\cdot)$. This product converges if and only if it is finite. It satisfies

$$D(\cdot + \alpha)H_{n}^* (\cdot + \alpha)A_1 e^{F_1(\cdot)} H_1(\cdot) D^*(\cdot) = A_d$$

or, by introducing some obvious notation

$$D_i(\cdot + \alpha)A_i e^{F_i(\cdot)} D_i^*(\cdot) = A_d$$

Finally, by post-conjugating with a constant we can assume that $A_i$ and $A_d$ are in the same maximal torus.

The case $G = SU(2)$. Let us firstly examine the simpler case where $G = SU(2)$, and suppose that $H_{1}(\cdot)$ diverges, i.e., that it is infinite. If we denote $D_i^{1,1}(\cdot) = \langle D_i(\cdot), j \rangle : \mathbb{T}^d \to \mathbb{C}$, we have

$$(e^{2i\pi(k \cdot \alpha + a_i - a_d)} - 1)e^{2i\pi a_d} D_i^{1,1}(k) = O(\varepsilon, n, 0)$$

where $O(\varepsilon, n, 0)$ is bounded independently of $k$. The divergence of $D_i(\cdot)$ in $L^2$ implies that $|a_i|_z \leq K_n^{-1} = N_{n, y}$, where $A_i = \{\exp(2i\pi a_i), 0\}_{SU(2)}$. Therefore,

$$|k \cdot \alpha + a_i - a_d|_z \geq K_{n, y}, \forall 0 < |k| \leq (2\pi^{-1})^{1/\tau} N_{n, y}^{1/\tau}$$

provided that

$$|k \cdot \alpha - a_d|_z \geq \frac{\tilde{\gamma}^{-1}}{|k|^\tau} \tag{8}$$

for $k \in \mathbb{Z}^*$, i.e., iff $a_d \in DC_a(\tilde{\gamma}, \tau)$. This motivates the following definition.

**Definition 5.1.** A constant $A = \exp(a) \in G$ is Diophantine with respect to $a$ iff for all roots $\rho$ of a root-space decomposition with respect to a torus containing $A$ we have $\rho(a) \in DC_a(\tilde{\gamma}, \tau)$, i.e., if equation (8) is satisfied by $\rho(a)$ with the constants $\tilde{\gamma}, \tau$. By abuse of notation we will write $A \in DC_a(\tilde{\gamma}, \tau)$.\footnote{And, eventually, a representative of the Weyl group which permutes the different root spaces.}
Since \( \nu > \tau \), we find that \( N_i' / N_{n_i} \) goes to infinity and, therefore, for \( i \) big enough, the spectral support of \( H_i(\cdot) = D_i(\cdot)D_i^*(\cdot) \) (i.e., of the diverging sequence of conjugations) is contained in \([-N_i', N_i']^d \subset \mathbb{Z}^d \). In a similar way, and with some obvious notation, we find that

\[
(e^{2i\pi(k\cdot a + a_0') - 1}) D_i^{h, j}(k) = O(\varepsilon, n, 0).
\]

Consequently,

\[
\hat{T}_{N_i'/2} D_i(\cdot) = O_L^2(\varepsilon, n, 0).
\]

On the other hand, since \( \sigma(H_i(\cdot)) \subset [-2N_{n_i}, 2N_{n_i}]^d \) and since \( N_{n_i} \ll N_i' \).

\[
T_{N_i'/2} D(\cdot) = T_{N_i'/2}[D_i(\cdot)D_i^*(\cdot)D(\cdot)]
\]

\[
= T_{N_i'/2} [T_{N_i'}(D_i(\cdot)) H_i^*(\cdot)]
\]

\[
= C_i H_i^*(\cdot) + O_L^2(\varepsilon, n, 0).
\]

The \( O_L^2(\varepsilon, n, 0) \) notation means that the mapping is equal to \( C_i H_i^*(\cdot) \) up to a function whose \( L^2 \) norm is \( O(\varepsilon, n, 0) \), and the constant is controlled and not significant.

Since for \( n \) big enough \( \| D(\cdot) - T_{N_i'} D(\cdot) \|_{L^2} \) is small and \( D_i^*(\cdot) D(\cdot) \) takes values in \( \text{Inn}(su(2)) \approx SO(3) \), we can assume that \( C_i \), which is a linear transformation of \( su(2) \), is bounded away from 0, say

\[
|C_i| > \frac{1}{2}
\]

in operator norm. Since \( H_i^*(\cdot) \) diverges in \( L^2 \), we reach a contradiction.

The case of a general compact Lie group. The case of a general compact group is hardly more complicated. If the sequence of conjugations diverges, there exists a root \( \rho \) such that \( |a_\rho(i)|_{Z} \leq K_{n_i}^{-1} \), where \( a_\rho = \rho(a) \). If we fix such a root \( \rho \), we find that for any other positive root \( \rho' \),

\[
(e^{2i\pi(k\cdot a + a_\rho'(a) - a_\rho(a)) - 1}) e^{2i\pi a_\rho(a')} D_i^{h, j'}(k) = O(\varepsilon, n, 0)
\]

\[
(e^{2i\pi(k\cdot a + a_\rho(a)) - 1}) D_i^{h, j}(k) = O(\varepsilon, n, 0)
\]

and, in the same way as before,

\[
\hat{T}_{N_i'} D_i(\cdot) j_\rho = O_L^2(\varepsilon, n, 0).
\]

Since

\[
T_{N_i'} D(\cdot) j_\rho = T_{N_i'} [D_i(\cdot)D_i^*(\cdot)D(\cdot) j_\rho]
\]

\[
= T_{N_i'} [T_{2N_i'}(D_i(\cdot)) D_i^*(\cdot)B(\cdot) j_\rho]
\]

\[
= C_i D_i^*(\cdot) D(\cdot) j_\rho + O_L^2(\varepsilon, n, 0)
\]

and \( D(\cdot) \) is an isometry, we find that \( C_i \in L(g) \), the space of linear mappings of \( g \) into itself, is bounded away from 0, say

\[
|C_i| > \frac{1}{2}
\]
in the operator norm. Now, $D_s^* \cdot D \cdot j \rho$ diverges in $L^2$ as $j \rho$ does not commute with the reduction of resonant modes. This is due to the fact that, by construction of $h_i$ and by the choice of the root $\rho$,

$$[h_i, j \rho] = 2i\pi \frac{k'_\rho}{D}$$

infinitely often, with $k'_\rho \to \infty$. Thus, the hypothesis that the product of conjugation diverges leads us to a contradiction.

Finally, we observe that, if $F(\cdot)$ is small enough so that the KAM scheme can be applied, the constants in the Diophantine condition on $A_d$ become irrelevant. If we suppose that $A_d \in DC(\gamma, \tau')$ with $\tau' > \tau$, then, after a finite number of iterations of the scheme, $F_i(\cdot)$ is small enough so that the scheme can be initiated if we place $\alpha$ in $DC(\gamma, \tau')$, and the argument presented above remains valid. This concludes the proof of the theorem in its full generality.

5.3. **Reducibility to a Liouvillean constant.** A corollary of this proof is, in fact, the optimality of the scheme in the orbits of Diophantine constant cocycles. By its construction, the scheme converges in the smooth category if and only if it converges in $L^2$, and the proof implies that if a measurable conjugation to such a constant exists, then the scheme converges toward it, eventually modulo a conjugation between constant cocycles.

On the other hand, the transposed argument shows that the KAM scheme is highly non-optimal if the dynamics in the fibers are Liouvillean. More precisely, we let $(\alpha, A e^{F_n(\cdot)}) \in SW^\infty(T, SU(2))$ be smoothly conjugate to $(\alpha, A_L)$, where $A_L$ is a Liouvillean constant in $SU(2)$. Application of the scheme produces a sequence of conjugations $F_n(\cdot)$ and a sequence of cocycles $(\alpha, A_n e^{F_n(\cdot)})$ such that

$$A e^{F_n(\cdot)} = H_n(\cdot + \alpha) A_n e^{F_n(\cdot)} H_n^*(\cdot),$$

where $F_n(\cdot) \to 0$ exponentially fast. If we suppose that the sequence of conjugations converges, we find that in the limit

$$A_L = H(\cdot + \alpha) A_\infty H^*(\cdot),$$

where $A_\infty$ (which we suppose diagonal, just as $A_L$) is the limit of $A_n$. Since $A_L$ is non-resonant, $H(\cdot)$ is a torus morphism, so that $A_\infty$ is Liouvillean itself. Since, now, $F_n(\cdot) \to 0$ exponentially fast and, for $n$ big enough, $A_{n+1} = A_n \exp(F_n(0))$, we can rewrite $A_n e^{F_n(\cdot)}$ as $A_\infty e^{\tilde{F}_n(\cdot)}$, where still $\tilde{F}_n(\cdot) \to 0$ exponentially fast. Since $A_\infty$ is Liouvillean, for any $l \in \mathbb{N}$, there exists $k_l$ such that

$$|a_\infty - k_l \alpha| < \frac{1}{|k_l|^l}.$$

Therefore, the Fourier mode $k_l$ is a resonance for the scheme at the $n$-th step provided that

$$\frac{1}{|k_l|^l} < N_n^{-\nu}, \quad |k_l| < N_n$$

or, equivalently,

$$N_n^{\nu/l} < |k_l| < N_n.$$
Therefore, for \( l > \nu \) a reduction of resonant mode must take place, which contradicts the hypothesis that \( H_n(\cdot) \) converges. Since \( l \) can be chosen arbitrarily big, no choice of \( \nu \) can make the scheme converge.

Strictly speaking, this phenomenon appears for “generic” reducible cocycles. Genericity here is in the product topology \( G \times C^\infty(\mathbb{T}^d, G) \), which is very far from the one induced by \( SW^\infty_\alpha \). Genericity in this sense comes from the demand that resonant modes be non-zero infinitely often, in order to avoid trivialities. The optimality of the scheme is in fact saved, since a converging scheme can be constructed, as in [11].

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**References**


