PROCEEDINGS OF THE FOURTH INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
May 24 – 27, 2002, Wilmington, NC, USA pp. **234–245**

KELVIN-HELMHOLTZ INSTABILITY WAVES OF SUPERSONIC MULTIPLE JETS

Joshua Z. Du
Department of Mathematics
Kennesaw State University
Kennesaw, Georgia 30144 USA

1. **Introduction.** Jet aircraft were introduced right after the Second World War. Shortly after, jet noise prediction and reduction became an important research topic. During the 1950s, jet noise was considered to be generated mainly by acoustic quadruples, which were thought to be distributed in the jet flow. During the next two decades, significant progress was made in turbulence research. It was discovered that large-scale turbulence structure/instability waves existed in free shear layer and jet flows. The large-scale turbulence structure/instability waves dominated the dynamics and mixing process of this class of flow. This understanding provided the basis of the current instability wave of supersonic jet noise.

Because of the need for large thrust, many high performance military aircraft are propelled by two jet engines housed close to each other. Besides the obvious military applications, this also have relevancy in the design of new generation of civilian aircraft in which an aircraft could possibly be powered by engine designs that arrive from this line of research. Which ever the configuration of these supersonic jets, the equations of conservation of mass, momentum and energy are employed to study the Kelvin-Helmholtz Instability problem of these kinds of supersonic jets. The general solution of the system of equations and the dispersion relation for instability waves of these jets, which are relevant to the radiated noise, are derived.

This paper includes three types of circular supersonic multiple jets:

1. Twin unit jets with distance between two centers S
2. N unit jets in a row, with distance S between centers of two consecutive jets
3. N unit jets whose centers are uniformly distributed on a circle with radius R

The objective of this work is to determine and to study the characteristics of the acoustic wave modes and the Kelvin-Helmholtz instability waves of multiple supersonic jets. These waves constitute the basic elements of the feedback loop responsible for the generation of resonance tones for jets. In the vortex sheet model, the jets are modeled as N cylindrical columns bounded by vortex sheets. Instability wave solutions valid inside and outside of the jets can be found. The enforcement of the kinematic and dynamic boundary conditions along the vortex sheets is facilitated by the use of the addition theorem for Bessel functions as found in [3]. The imposition of these conditions leads to a dispersion relation in the form of determinants with dimension infinity by infinity. Numerical results of the eigenvalues

1991 *Mathematics Subject Classification.* Primary: 58F15, 58F17, 58F11; Secondary: 53C35.

and eigenfunctions for the upstream propagating acoustic waves and the Kelvin-Helmholtz instability waves can be found by the Newton Iteration Method and the Trace Theorem. They are, however, not included in this paper.

Kelvin-Helmholtz instability waves in multiple supersonic jets constitute the basic elements of the feedback loop responsible for the generation of multiple-jet resonance tones. Tam and Hu [1] classified the acoustic wave modes of a single jet by the azimuthally wave index number n, and the radial wave index number m. The n, m'th mode is denoted by (n, m) where $n = 0, 1, 2, \ldots$ and $m = 1, 2, 3, \ldots$. For supersonic twinjets, Morris [2] classified the eigenfunctions of twinjets into four families according to the symmetries of the twinjets.

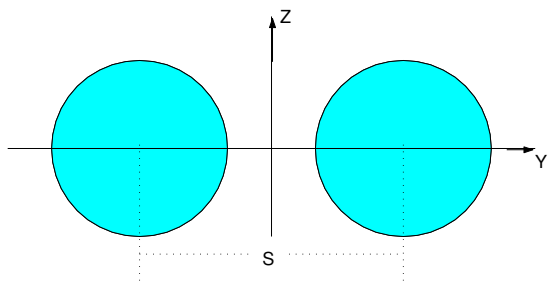
(1). Family 1, symmetric about y-axis and z-axis;
(2). Family 2, symmetric about y-axis and anti-symmetric about z-axis;
(3). Family 3, anti-symmetric about y-axis and symmetric about z-axis;
(4). Family 4, anti-symmetric about y-axis and anti-symmetric about z-axis;

The geometry configuration of multiple jets is far more complicated than twinjets. This paper has extended the idea of Tam and Morris's classification as following. Firstly, treat the general solution of the pressure contributed from all jets as a superposition of many different categories of twinjets according to the distance between two centers of these pairs of twinjets. In each category, there may be several solutions (eigenfunctions), the difference between those solutions in the same category can be seen as a translation (in the case of n jets on a row) or a rotation (in the case n jets on a circle); but all solutions in the same category share the same eigenvalues, hence share the same dispersion relation. Then, in each category, the eigenvalues are still subdivided into 4 families as twinjet.

In the case of N-Jets on a row, the category 1 solutions are all solutions of twinjets with central distance S; category 2 solutions are all those associated with central distance $2S$; category N–1 solution is the unique twinjet formed by two jets from two ends.

In the case of N–jets with all centers on a circle with radius R, the category 1 solutions are all solutions of twinjets with central distance $S = 2R\sin(\pi/N)$; the category l solutions are all solutions of twinjets with central distance

2. **Twinjets.** Two cylindrical coordinate systems, (r_1, θ_1, x), (r_2, θ_2, x) are introduced centered at each jet. The mean velocity and the mean density of each jet are assumed to be $\overline{u}(r)$ and $\overline{\rho}(r)$. The mean pressure, p_∞ is a constant. The whole region of the y z plane can be divided into 3 regions as Figure 1:



Region α: $r_\alpha \leq 1$, $\alpha = 1, 2$
Region 3: $r_1 \geq 1$, $r_2 \geq 1$

The linearized continuity, momentum and energy equations of a compressible inviscid fluid are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\overline{\rho} \mathbf{V}_{\perp} + \rho \overline{u} \overrightarrow{X} \right) = 0 \tag{2.1}$$

$$\overline{\rho} \left[\frac{\partial u}{\partial t} + \overline{u} \frac{\partial u}{\partial x} + \mathbf{V}_{\perp} \cdot \nabla_{\perp} \overline{u} \right] = - \frac{\partial p}{\partial x} \tag{2.2}$$

$$\overline{\rho} \left[\frac{\partial \mathbf{V}_{\perp}}{\partial t} + \overline{u} \frac{\partial \mathbf{V}_{\perp}}{\partial x} \right] = - \nabla_{\perp} p \tag{2.3}$$

$$M_j^2 \left[\frac{\partial p}{\partial t} + \overline{u} \frac{\partial p}{\partial x} \right] + \nabla_{\perp} \cdot \mathbf{V}_{\perp} + \frac{\partial u}{\partial x} = 0 \tag{2.4}$$

Where u is the component of velocity in the x-direction, $\overrightarrow{x}$, $\overrightarrow{y}$, $\overrightarrow{z}$ are unit vectors in the x-, y-, z- directions:

$$\nabla_{\perp} = \frac{\partial}{\partial y} \overrightarrow{y} + \frac{\partial}{\partial z} \overrightarrow{z} \quad , \quad \nabla_{\perp}^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The pressure associated with small amplitude disturbances after separating out the wave factor $e^{i(kx - \omega t)}$ satisfies the equation:

$$\nabla_{\perp}^2 \hat{p} + \left[\frac{2k \nabla_{\perp} \overline{u}}{\omega - k\overline{u}} - \frac{\nabla_{\perp} \overline{\rho}}{\overline{\rho}} \right] \cdot \nabla_{\perp} \hat{p} + \left[\overline{\rho} M_j^2 \left(\omega - \overline{u} k \right)^2 - k^2 \right] \hat{p} = 0 \tag{2.5}$$

where ω and k are the angular frequency and axial wave numbers. The solution of (2.5) in region 1, region 2, and region 3 will be labeled as $\hat{p}_1$, $\hat{p}_2$, and $\hat{p}_3$ respectively.

In region α, where $\alpha = 1, 2$, upon using polar coordinates $(r_\alpha, \theta_\alpha)$, equation (2.5) is reduced to :

$$\frac{\partial^2 \hat{p}_\alpha}{\partial r_\alpha^2} + \left[\frac{1}{r_\alpha} + \frac{2k \frac{\partial \overline{u}}{\partial r_\alpha}}{\omega - \overline{u} k} - \frac{1}{\overline{\rho}} \frac{\partial \overline{\rho}}{\partial r_\alpha} \right] \frac{\partial \hat{p}_\alpha}{\partial r_\alpha} + \frac{1}{r_\alpha^2} \frac{\partial^2 \hat{p}_\alpha}{\partial \theta_\alpha^2} + \left[\overline{\rho} M_j^2 \left(\omega - \overline{u} k \right)^2 - k^2 \right] \hat{p}_\alpha = 0 \tag{2.6}$$

In region 3, the equation (2.5) is reduced to:

$$\nabla_{\perp}^2 \hat{p}_3 + \left[\overline{\rho}_0 M_j^2 \omega^2 - k^2 \right] \hat{p}_3 = 0 \tag{2.7}$$

where

$$\overline{\rho}_0 = \frac{1}{1 + \frac{\gamma - 1}{2} M_j^2}$$

which is the density of the gas outside the jets, γ is the ratio of the specific heats of gas. This may be written in either one of the polar coordinate systems $(r_\alpha, \theta_\alpha)$ as:

$$\frac{\partial^2 \hat{p}_3}{\partial r_\alpha^2} + \frac{1}{r_\alpha} \frac{\partial \hat{p}_3}{\partial r_a} + \frac{1}{r_\alpha^2} \frac{\partial^2 \hat{p}_3}{\partial \theta_a^2} + \left[\overline{\rho} M_j^2 \omega^2 - k^2 \right] p_3 = 0. \tag{2.8}$$

For the vortex sheet model, the mean velocity is $\overline{u}(r) = 1$ inside the jet, and 0 outside the jet. The mean density is $\overline{\rho}(r) = 1$ inside the jet, and $\overline{\rho}_0$ outside the jet. Hence equations (2.6), (2.7), and (2.8) can be reduced to

$$\frac{\partial^2 \hat{p}_\alpha}{\partial r_\alpha^2} + \frac{1}{r_\alpha} \frac{\partial \hat{p}_\alpha}{\partial r_\alpha} + \frac{1}{r_\alpha^2} \frac{\partial^2 \hat{p}_\alpha}{\partial \theta_a^2} + \lambda_i^2 \hat{p}_\alpha = 0 \qquad r_\alpha \leq 1 \tag{2.9}$$

where $\lambda_i = \left[M_j^2 \left(\omega - k \right)^2 - k^2 \right]^{1/2}$ and

$$\frac{\partial^2 \hat{p}_3}{\partial r_\alpha^2} + \frac{1}{r_\alpha} \frac{\partial \hat{p}_3}{\partial r_\alpha} + \frac{1}{r_\alpha^2} \frac{\partial^2 \hat{p}_3}{\partial \theta_a^2} + \lambda_0^2 \hat{p}_3 = 0 \qquad r_\alpha \geq 1 \tag{2.10}$$

where $\lambda_0 = \left(\overline{\rho}_0 M_j^2 \omega^2 - k^2\right)^{1/2}$. The branch cuts of λ_0 and λ_i are chosen so that $0 = \arg \lambda_0$, $\arg \lambda_i < \pi$.

The kinematic and dynamic boundary conditions at the vortex sheets, where $r_\alpha = 1$ are:

$$\hat{p}_\alpha = \hat{p}_3 \text{ if } r_\alpha = 1 \tag{2.11}$$

$$\frac{1}{\overline{\rho}\omega^2}\frac{\partial \hat{p}_3}{\partial r_\alpha} = \frac{1}{(\omega - k)^2}\frac{\partial \hat{p}_\alpha}{\partial r_\alpha} \text{ if } r_\alpha = 1. \tag{2.12}$$

The general solution of the equations (2.9) can be written in the form of Fourier series as

$$\begin{aligned}\hat{p}_3 = \sum_{n=0}^{\infty} \Big[& E_n H_n^{(1)}(\lambda_0 r_1)\cos(n\theta_1) + F_n H_n^{(1)}(\lambda_0 r_1)\sin(n\theta_1) \\ & + G_n H_n^{(1)}(\lambda_0 r_2)\cos(n\theta_2) + G_n K_n^{(1)}(\lambda_0 r_2)\sin(n\theta_2)\Big]\end{aligned} \tag{2.13}$$

where E_n, F_n, G_n, and K_n are Fourier coefficients. The solution of (2.13) can be written as

$$\begin{aligned}\hat{p}_3 = & \sum_{n=0}^{\infty} A_n \left[H_n^{(1)}(\lambda_0 r_1)\cos(n\theta_1) + (-1)^n H_n^{(1)}(\lambda_0 r_2)\cos(n\theta_2)\right] \\ & + \sum_{n=0}^{\infty} B_n \left[H_n^{(1)}(\lambda_0 r_1)\cos(n\theta_1) - (-1)^n H_n^{(1)}(\lambda_0 r_2)\cos(n\theta_2)\right] \\ & + \sum_{n=1}^{\infty} C_n \left[H_n^{(1)}(\lambda_0 r_1)\sin(n\theta_1) + (-1)^n H_n^{(1)}(\lambda_0 r_2)\sin(n\theta_2)\right] \\ & + \sum_{n=1}^{\infty} D_n \left[H_n^{(1)}(\lambda_0 r_1)\sin(n\theta_1) - (-1)^n H_n^{(1)}(\lambda_0 r_2)\sin(n\theta_2)\right]\end{aligned} \tag{2.14}$$

where $E_n = A_n + B_n$, $G_n = (-1)^n (A_n - B_n)$, $F_n = C_n + D_n$, $K_n = (-1)^n (C_n - D_n)$.

There are four sums in the equation (2.14). Each sum represents a linearly independent solution satisfying certain symmetry about the xy - plane and the xz - plane. Let

$$\hat{p}_3^1 = \sum_{n=0}^{\infty} A_n \left[H_n^{(1)}(\lambda_0 r_1)\cos(n\theta_1) + (-1)^n H_n^{(1)}(\lambda_0 r_2)\cos(n\theta_2)\right] \tag{2.15}$$

$$\hat{p}_3^2 = \sum_{n=0}^{\infty} B_n \left[H_n^{(1)}(\lambda_0 r_1)\cos(n\theta_1) - (-1)^n H_n^{(1)}(\lambda_0 r_2)\cos(n\theta_2)\right] \tag{2.16}$$

$$\hat{p}_3^3 = \sum_{n=1}^{\infty} C_n \left[H_n^{(1)}(\lambda_0 r_1)\sin(n\theta_1) + (-1)^n H_n^{(1)}(\lambda_0 r_2)\sin(n\theta_2)\right] \tag{2.17}$$

$$\hat{p}_3^4 = \sum_{n=1}^{\infty} D_n \left[H_n^{(1)}(\lambda_0 r_1)\sin(n\theta_1) - (-1)^n H_n^{(1)}(\lambda_0 r_2)\sin(n\theta_2)\right] \tag{2.18}$$

It is easy to prove that

$$\begin{aligned}
\hat{p}_3^1(r_1,\theta_1,r_2,\theta_2) &= \hat{p}_3^1(r_2,\pi-\theta_2,r_1,\pi-\theta_1),\\
\hat{p}_3^1(r_1,\theta_1,r_2,\theta_2) &= \hat{p}_3^1(r_1,-\theta_1,r_2,-\theta_2),\\
\hat{p}_3^2(r_1,\theta_1,r_2,\theta_2) &= -\hat{p}_3^2(r_2,\pi-\theta_2,r_1,\pi-\theta_1),\\
\hat{p}_3^2(r_1,\theta_1,r_2,\theta_2) &= \hat{p}_3^2(r_1,-\theta_1,r_2,-\theta_2),\\
\hat{p}_3^3(r_1,\theta_1,r_2,\theta_2) &= \hat{p}_3^3(r_2,\pi-\theta_2,r_1,-\theta_1),\\
\hat{p}_3^3(r_1,\theta_1,r_2,\theta_2) &= -\hat{p}_3^3(r_1,-\theta_1,r_2,-\theta_2),\\
\hat{p}_3^4(r_1,\theta_1,r_2,\theta_2) &= -\hat{p}_3^4(r_2,\pi-\theta_2,r_1,\pi-\theta_1),\\
\hat{p}_3^4(r_1,\theta_1,r_2,\theta_2) &= -\hat{p}_3^4(r_2,-\theta_1,r_1,-\theta_2).
\end{aligned}$$

Therefore $\hat{p}_3^1$ represents the solution, which is symmetric about both the xy-plane and the xz-plane. We name $\hat{p}_3^1$ to be the family 1 solution. Similarly, the other solutions are designated as family 2, 3, and 4 solutions respectively.

The family 1 solution in region 3 has the form

$$\hat{p}_3^1 = \sum_{n=0}^{\infty} A_n \left[H_n^{(1)}(\lambda_0 r_1)\cos(n\theta_1) + (-1)^n H_n^{(1)}(\lambda_0 r_2)\cos(n\theta_2)\right]. \tag{2.19}$$

By the addition formulas of Bessel functions, the solution (2.19) can be rewritten as

$$\hat{p}_3^1 = \sum_{n=0}^{\infty}\left[A_n H_n^{(1)}(\lambda_0 r_1) + \left(\sum_{m=0}^{\infty} A_m \overline{C}_{m,n}\right) J_n(\lambda_0 r_1)\right]\varepsilon_n \cos(n\theta_1) \tag{2.20}$$

where

$$\overline{C}_{m,n} = H_{n+m}^{(1)}(\lambda_0 s) + (-1)^m H_{n-m}^{(1)}(\lambda_0 s)$$

where $\varepsilon_0 = 0.5$ and $\varepsilon_n = 1$ for $n = 1, 2, 3....$ General solutions of equations (2.9) and for the vortex sheet model jet which are bounded at the jet axis are:

$$\hat{p}_\alpha = \sum_{n=0}^{\infty} \hat{A}_n J_n(\lambda_i r_\alpha)\cos(n\theta_\alpha). \tag{2.21}$$

On substituting (2.21) for $\alpha = 1, 2$ into the boundary conditions (2.11) and (2.12), it is found that

$$\hat{A}_n J_n(\lambda_i) = \left[A_n H_n^{(1)}(\lambda_0) + \left(\sum_{m=0}^{\infty} A_m \overline{C}_{m,n}\right)\right] J_n(\lambda_0) E_n \tag{2.22}$$

and

$$\frac{\lambda_i \hat{A}_n J_n'(\lambda_i)}{(\omega-k)^2} = \frac{1}{\overline{\rho}_0 \omega^2}\left[A_n \lambda_0 H_n'^{(1)}(\lambda_0) + \left(\sum_{m=0}^{\infty} A_m \overline{C}_{m,n}\right)\right]\lambda_0 \varepsilon_n J_n'(\lambda_0), \tag{2.23}$$

where $n = 0, 1, 2, 3, ...$ and $\varepsilon_0 = 0.5$ and $\varepsilon_n = 1$ for $n = 1, 2, 3....$By eliminating $\hat{A}_n$ from (2.22) and (2.23), the pressure coefficients $A_0, A_1, A_2, \ldots$ of (2.19) are given by an infinite matrix system

$$\sum_{m=0}^{\infty}\left[\overline{c}_{m,n} + \overline{a}_{n,n}\delta_{m,n}\right] A_m = 0, \tag{2.24}$$

where $n = 0, 1, 2, 3 \ldots$, and

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0 \omega^2 \lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\varepsilon_n \left[\overline{\rho}_0 \omega^2 \lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}. \tag{2.25}$$

For nontrivial solutions, the determinant of the coefficients matrix must be equal to zero, in other words,

$$\det |\overline{c}_{m,n} + \overline{a}_{n,n} \delta_{m,n}| = 0$$

where $\delta_{m,n}$ is the Kronecker delta function, $n, m = 0, 1, 2, 3, \ldots$. This is the dispersion relation for the family 1 wave solution.

The dispersion relations for families 2, 3, and 4 can be derived in a similar way. They are

Family 1

$$\det |\overline{c}_{m,n} + \overline{a}_{n,n} \delta_{m,n}| = 0 \tag{2.26}$$

where

$$\overline{c}_{m,n} = H_{n+m}^{(1)}(\lambda_0 s) + (-1)^m H_{n-m}^{(1)}(\lambda_0 s)$$

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0 \omega^2 \lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\varepsilon_n \left[\overline{\rho}_0 \omega^2 \lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}$$

where $n, m = 0, 1, 2, \ldots$, $\varepsilon_0 = 0.5$ and $\varepsilon_n = 1, 2, 3, \ldots$ for $n = 1, 2, 3, \ldots$.

Family 2

$$\det |\overline{c}_{m,n} - \overline{a}_{n,n} \delta_{m,n}| = 0 \tag{2.27}$$

where

$$\overline{c}_{m,n} = H_{n+m}^{(1)}(\lambda_0 s) + (-1)^m H_{n-m}^{(1)}(\lambda_0 s)$$

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0 \omega^2 \lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\varepsilon_n \left[\overline{\rho}_0 \omega^2 \lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}$$

where $n, m = 0, 1, 2, \ldots$, $\varepsilon_0 = 0.5$ and $\varepsilon_n = 1, 2, 3, \ldots$ for $n = 1, 2, 3, \ldots$.

Family 3

$$\det \left[\overline{d}_{m,n} + \overline{a}_{n,n} \delta_{m,n}\right] = 0 \tag{2.28}$$

where

$$\overline{d}_{m,n} = H_{n+m}^{(1)}(\lambda_0 s) + (-1)^{m+1} H_{n-m}^{(1)}(\lambda_0 s)$$

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0 \omega^2 \lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\left[\overline{\rho}_0 \omega^2 \lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}$$

where $n, m = 1, 2, \ldots$

Family 4

$$\det \left[\overline{d}_{m,n} - \overline{a}_{n,n} \delta_{m,n}\right] = 0 \tag{2.29}$$

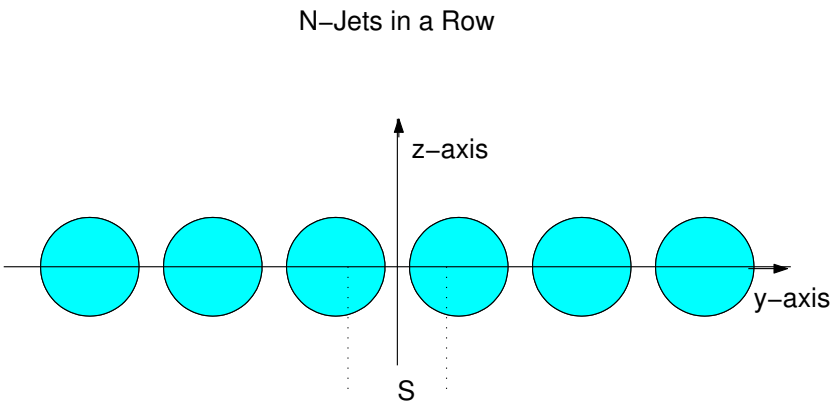
where

$$\overline{d}_{m,n} = H_{n+m}^{(1)}(\lambda_0 s) + (-1)^{m+1} H_{n-m}^{(1)}(\lambda_0 s)$$

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0 \omega^2 \lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\left[\overline{\rho}_0 \omega^2 \lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}$$

where $n, m = 1, 2, \ldots$

3. N **Jets in a Row.** Consider N supersonic jets centered along the y-axis with equal distance S as shown in Figure 2.



For the Vortex Sheet Model, the mean velocity is 1 inside the jets, and 0 outside the jets. The mean density is 1 inside the jets and $\overline{\rho}_0$ outside the jets. Hence equations (2.6) and (2.7) can be reduced as follows:

In Region ℓ (inside jets, $\ell = 1, 2, 3, \ldots, N$), we have

$$\frac{\partial^2 \hat{p}_\ell}{\partial r_\ell^2} + \frac{1}{r_\ell}\frac{\partial \hat{p}_\ell}{\partial r_\ell} + \frac{1}{r_\ell^2}\frac{\partial^2 \hat{p}_\ell}{\partial \theta_\ell^2} + \lambda_\ell^2 \hat{p}_\ell = 0 \qquad r_\ell \leq 1 \tag{3.1}$$

where $\lambda_i = \left[M_j^2 (\omega - k)^2 - k^2\right]^{1/2}$.

In Region $N+1$, the ambient region, we have

$$\frac{\partial^2 \hat{p}_{N+1}}{\partial r_\ell^2} + \frac{1}{r_\ell}\frac{\partial \hat{p}_{N+1}}{\partial r_\ell} + \frac{1}{r_\ell^2}\frac{\partial^2 \hat{p}_{N+1}}{\partial \theta_\ell^2} + \lambda_\ell^2 \hat{p}_{N+1} = 0 \qquad r_\ell \geq 1. \tag{3.2}$$

The general solution of Equation (3.2) is

$$\hat{p}_{N+1} = \sum_{n=0}^{\infty}\sum_{\ell=1}^{N}\left[\overline{\overline{A^\ell}}_n H_n^{(1)}(\lambda_0 r_\ell)\cos(n\theta_\ell) + \overline{\overline{B^\ell}}_n H_n^{(1)}(\lambda_0 r_\ell)\sin(n\theta_\ell)\right]. \tag{3.3}$$

It can also be rewritten as the following $N-1$ categories:

1. Category 1: The interaction between two jets with distance S, total of $N-1$ pairs,

$$\hat{p}_{N+1,1} = \sum_{n=0}^{\infty}\sum_{\ell=1}^{N}\left[\begin{array}{c} \overline{A}_{n,1} H_n^{(1)}(\lambda_0 r_\ell)\cos(n\theta_\ell) + \overline{B}_{n,1} H_n^{(1)}(\lambda_0 r_\ell)\sin(n\theta_\ell) \\ +\overline{C}_{n,1} H_n^{(1)}(\lambda_0 r_{\ell+1})\cos(n\theta_{\ell+1}) + \overline{D}_{n,1} H_n^{(1)}(\lambda_0 r_{\ell+1})\sin(n\theta_{\ell+1}) \end{array}\right].$$

2. Category 2: The Interaction between every two jets with distance $2S$, total of $N-2$ pairs.

$$\hat{p}_{N+1,2} = \sum_{n=0}^{\infty}\sum_{\ell=1}^{N-1}\left[\begin{array}{c} \overline{A}_{n,2} H_n^{(1)}(\lambda_0 r_\ell)\cos(n\theta_\ell) + \overline{B}_{n,2} H_n^{(1)}(\lambda_0 r_\ell)\sin(n\theta_\ell) \\ +\overline{C}_{n,2} H_n^{(1)}(\lambda_0 r_{\ell+2})\cos(n\theta_{\ell+2}) + \overline{D}_{n,2} H_n^{(1)}(\lambda_0 r_{\ell+2})\sin(n\theta_{\ell+2}) \end{array}\right].$$

3. Category 3: The Interaction between every two jets with distance $3S$, total of $N-3$ pairs.

$$\hat{p}_{N+1,3}=\sum_{n=0}^{\infty}\sum_{\ell=1}^{N-2}\left[\begin{array}{c}\overline{A}_{n,3}H_n^{(1)}\left(\lambda_0 r_\ell\right)\cos\left(n\theta_\ell\right)\\ +\overline{B}_{n,3}H_n^{(1)}\left(\lambda_0 r_\ell\right)\sin\left(n\theta_\ell\right)\\ +\overline{C}_{n,3}H_n^{(1)}\left(\lambda_0 r_{\ell+3}\right)\cos\left(n\theta_{\ell+3}\right)\\ +\overline{D}_{n,3}H_n^{(1)}\left(\lambda_0 r_{\ell+3}\right)\sin\left(n\theta_{\ell+3}\right)\end{array}\right].$$

$N-1$ Category $N-1$: The Interaction between every two jets with distance $(N-1)\,S$, total of 2 pairs.

$$\hat{p}_{N+1,3}\quad=\quad\sum_{n=0}^{\infty}\sum_{\ell=1}^{2}\left[\begin{array}{c}\overline{A}_{n,N-1}H_n^{(1)}\left(\lambda_0 r_\ell\right)\cos\left(n\theta_\ell\right)\\ +\overline{B}_{n,N-1}H_n^{(1)}\left(\lambda_0 r_\ell\right)\sin\left(n\theta_\ell\right)\\ +\overline{C}_{n,N-1}H_n^{(1)}\left(\lambda_0 r_{\ell+N-1}\right)\cos\left(n\theta_{\ell+N-1}\right)\\ +\overline{D}_{n,N-1}H_n^{(1)}\left(\lambda_0 r_{\ell+N-1}\right)\sin\left(n\theta_{\ell+N-1}\right)\end{array}\right].$$

N Category N: The Interaction between every two jets with distance NS, total of only 1 pair.

$$\hat{p}_{N+1,3}=\sum_{n=0}^{\infty}\sum_{\ell=1}^{1}\left[\begin{array}{c}\overline{A}_{n,N}H_n^{(1)}\left(\lambda_0 r_\ell\right)\cos\left(n\theta_\ell\right)\\ +\overline{B}_{n,N}H_n^{(1)}\left(\lambda_0 r_\ell\right)\sin\left(n\theta_\ell\right)\\ +\overline{C}_{n,N}H_n^{(1)}\left(\lambda_0 r_{\ell+N}\right)\cos\left(n\theta_{\ell+N}\right)\\ +\overline{D}_{n,N}H_n^{(1)}\left(\lambda_0 r_{\ell+N}\right)\sin\left(n\theta_{\ell+N}\right)\end{array}\right].$$

Considering symmetry, let

$$\overline{A}_{n,\ell}=A_{n,\ell}+B_{n,\ell},\quad \overline{B}_{n,\ell}=C_{n,\ell}+D_{n,\ell},$$
$$\overline{C}_{n,\ell}=(-1)^n\left(A_{n,\ell}-B_{n,\ell}\right),\quad \overline{D}_{n,\ell}=(-1)^n\left(C_{n,\ell}-D_{n,\ell}\right).$$

Each Fourier series of every category can be decomposed as four families of solutions like the case of twinjet.

The dispersion relations for four families of ℓ'th category are as follows:

Family 1 of the ℓ'th **Category**

$$\det\left[\overline{c}_{m,n}^{\ell}+\overline{a}_{n,n}\delta_{m,n}\right]=0 \tag{3.4}$$

where

$$\overline{c}_{m,n}^{\ell}=H_{n+m}^{(1)}\left(\ell\lambda_0 S\right)+(-1)^m H_{n-m}^{(1)}\left(\ell\lambda_0 S\right)$$
$$\overline{a}_{n,m}=\frac{\overline{\rho}_0\omega^2\lambda_i H_n^{(1)}\left(\lambda_0\right)J_n'\left(\lambda_i\right)-(\omega-k)^2\lambda_0 H_n'^{(1)}\left(\lambda_0\right)J_n\left(\lambda_i\right)}{\varepsilon_n\left[\overline{\rho}_0\omega^2\lambda_i J_n\left(\lambda_0\right)J_n'\left(\lambda_i\right)-(\omega-k)^2\lambda_0 J_n'\left(\lambda_0\right)J_n\left(\lambda_i\right)\right]}$$

where $n,m=0,1,2,$, $\varepsilon_0=0.5$ and $\varepsilon_n=1,2,3,\ldots$.

Family 2 of the ℓ'th **Category**

$$\det\left[\overline{c}_{m,n}^{\ell}-\overline{a}_{n,n}\delta_{m,n}\right]=0 \tag{3.5}$$

$$\overline{c}_{m,n}^{\ell}=H_{n+m}^{(1)}\left(\ell\lambda_0 S\right)+(-1)^m H_{n-m}^{(1)}\left(\ell\lambda_0 S\right)$$
$$\overline{a}_{n,m}=\frac{\overline{\rho}_0\omega^2\lambda_i H_n^{(1)}\left(\lambda_0\right)J_n'\left(\lambda_i\right)-(\omega-k)^2\lambda_0 H_n'^{(1)}\left(\lambda_0\right)J_n\left(\lambda_i\right)}{\varepsilon_n\left[\overline{\rho}_0\omega^2\lambda_i J_n\left(\lambda_0\right)J_n'\left(\lambda_i\right)-(\omega-k)^2\lambda_0 J_n'\left(\lambda_0\right)J_n\left(\lambda_i\right)\right]}$$

where $n,m=0,1,2,$, $\varepsilon_0=0.5$ and $\varepsilon_n=1,2,3,\ldots$.

Family 3 of the ℓ'th **Category**

$$\det\left[\overline{d}_{m,n}^{\ell}+\overline{a}_{n,n}\delta_{m,n}\right]=0 \tag{3.6}$$

$$\overline{d}_{m,n}^{\ell} = H_{n+m}^{(1)}(\ell\lambda_0 S) + (-1)^{m+1} H_{n-m}^{(1)}(\ell\lambda_0 S)$$

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0\omega^2\lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega-k)^2 \lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\left[\overline{\rho}_0\omega^2\lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega-k)^2 \lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}$$

where $n, m = 1, 2, \ldots$.

Family 4 of the ℓ'th Category

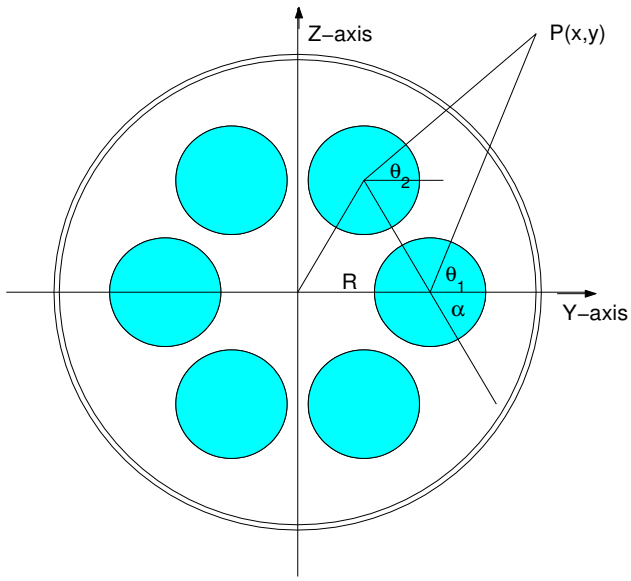
$$\det\left[\overline{d}_{m,n}^{\ell} - \overline{a}_{n,n}\delta_{m,n}\right] = 0 \tag{3.7}$$

$$\overline{d}_{m,n}^{\ell} = H_{n+m}^{(1)}(\ell\lambda_0 S) + (-1)^{m+1} H_{n-m}^{(1)}(\ell\lambda_0 S)$$

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0\omega^2\lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega-k)^2 \lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\left[\overline{\rho}_0\omega^2\lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega-k)^2 \lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}$$

where $n, m = 1, 2, \ldots$.

4. N **Jets on a Circle.** Consider N circular jets on a circle with radius R as shown in Figure 3 Applying the addition formulas in [3],



$$\begin{aligned} H_n^{(1)}(\lambda_0, r_2)\cos(n\theta_2) &= (-1)^n \varepsilon_m \sum_{m=0}^{\infty}\left[H_{n+m}^{(1)}(\lambda_0, S) + (-1)^m H_{n-m}^{(1)}(\lambda_0, S)\right] \\ &\quad J_m(\lambda_0, r_1)\cos(m\theta_1). \end{aligned} \tag{4.1}$$

$$\begin{aligned} H_n^{(1)}(\lambda_0, r_2)\sin(n\theta_2) &= -(-1)^n \varepsilon_m \sum_{m=0}^{\infty}\left[H_{n+m}^{(1)}(\lambda_0, S) + (-1)^m H_{n-m}^{(1)}(\lambda_0, S)\right] \\ &\quad J_m(\lambda_0, r_1)\sin(m\theta_1). \end{aligned} \tag{4.2}$$

to this case, we have

$$\begin{aligned} H_n^{(1)}(\lambda_0, r_2)\cos n(\theta_2+\alpha) &= (-1)^n \varepsilon_m \sum_{m=0}^{\infty}\left[H_{n+m}^{(1)}(\lambda_0, S) + (-1)^m H_{n-m}^{(1)}(\lambda_0, S)\right] \\ &\quad J_m(\lambda_0, r_1)\cos m(\theta_1+\alpha). \qquad (4.3) \end{aligned}$$

$$\begin{aligned} H_n^{(1)}(\lambda_0, r_2)\sin n(\theta_2+\alpha) &= -(-1)^n \varepsilon_m \sum_{m=0}^{\infty}\left[H_{n+m}^{(1)}(\lambda_0, S) + (-1)^m H_{n-m}^{(1)}(\lambda_0, S)\right] \\ &\quad J_m(\lambda_0, r_1)\sin m(\theta_1+\alpha). \qquad (4.4) \end{aligned}$$

where $S = 2R\sin\frac{\pi}{N}$, and $\alpha = \frac{\pi}{2} - \frac{\pi}{N}$. Let $\phi_1 = \theta_1 + \alpha$ and $\phi_2 = \theta_2 + \alpha$, then the addition formulas are

$$\begin{aligned} H_n^{(1)}(\lambda_0, r_2)\cos(n\phi_2) &= (-1)^n \varepsilon_m \sum_{m=0}^{\infty}\left[H_{n+m}^{(1)}(\lambda_0, S) + (-1)^m H_{n-m}^{(1)}(\lambda_0, S)\right] \\ &\quad J_m(\lambda_0, r_1)\cos(m\phi_1). \qquad (4.5) \end{aligned}$$

$$\begin{aligned} H_n^{(1)}(\lambda_0, r_2)\sin(n\phi_2) &= -(-1)^n \varepsilon_m \sum_{m=0}^{\infty}\left[H_{n+m}^{(1)}(\lambda_0, S) + (-1)^m H_{n-m}^{(1)}(\lambda_0, S)\right] \\ &\quad J_m(\lambda_0, r_1)\sin(m\phi_1). \qquad (4.6) \end{aligned}$$

For N-jets, solution of pressure can be classified as $c(N)$ categories according to the distance between two centers $s = s(\ell) = 2R\sin\frac{\ell\pi}{N}$. The number $c(N)$ is a function of N. In each category, solution can be classified as 4 families according to the symmetry. In the same category, pressure shares the same eigenvalues but different eigenfunctions.

For the ℓ'th category, the ambient pressure, $\hat{p}_{N+1}^{\ell,m}$, stands for the eigenfunction of family m in category ℓ, and $D(\omega, k)^{\ell,m} = 0$ stands for the dispersion relation for family m in category ℓ. In category ℓ, the distance s is a function of ℓ.

$$s = s(\ell) = 2R\sin\frac{\ell\pi}{N}.$$

Hence, we have the similar formula

$$\begin{aligned} C_{m,n}(s) &= H_{n+m}^{(1)}(\lambda_0 s) + (-1)^m H_{n-m}^{(1)}(\lambda_0 s) \\ &= H_{n+m}^{(1)}\left(2\lambda_0 R\sin\frac{\ell\pi}{N}\right) + (-1)^m H_{n-m}^{(1)}\left(2\lambda_0 R\sin\frac{\ell\pi}{N}\right) \end{aligned}$$

The corresponding dispersion relation can be summarized as

Family 1 in Category ℓ,

$$D(\omega, k)^{\ell,1} = \det\left[C_{m,n} + a_{n,n}\delta_{m,n}\right]$$

where

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0\omega^2\lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega-k)^2\lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\varepsilon_n\left[\overline{\rho}_0\omega^2\lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega-k)^2\lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}$$

and

$$\begin{aligned} C_{m,n}(s) &= H_{n+m}^{(1)}(\lambda_0 s) + (-1)^m H_{n-m}^{(1)}(\lambda_0 s) \\ &= H_{n+m}^{(1)}\left(2\lambda_0 R\sin\frac{\ell\pi}{N}\right) + (-1)^m H_{n-m}^{(1)}\left(2\lambda_0 R\sin\frac{\ell\pi}{N}\right) \end{aligned}$$

where $n, m = 0, 1, 2, \ldots$, $\varepsilon_0 = 0,5$ and $\varepsilon_n = 1$ for $n = 1, 2, \ldots$.

Family 2 in Category ℓ,

$$D(\omega, k)^{\ell,2} = \det\left[C_{m,n} - a_{n,n}\delta_{m,n}\right]$$

where

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0\omega^2\lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\varepsilon_n\left[\overline{\rho}_0\omega^2\lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}$$

and

$$\begin{aligned} C_{m,n}(s) &= H_{n+m}^{(1)}(\lambda_0 s) + (-1)^m H_{n-m}^{(1)}(\lambda_0 s) \\ &= H_{n+m}^{(1)}\left(2\lambda_0 R\sin\frac{\ell\pi}{N}\right) + (-1)^m H_{n-m}^{(1)}\left(2\lambda_0 R\sin\frac{\ell\pi}{N}\right) \end{aligned}$$

where $n, m = 0, 1, 2, \ldots$, $\varepsilon_0 = 0,5$ and $\varepsilon_n = 1$ for $n = 1, 2, \ldots$.

Family 3 in Category ℓ,

$$D(\omega, k)^{\ell,3} = \det\left[C_{m,n} + a_{n,n}\delta_{m,n}\right]$$

where

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0\omega^2\lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\left[\overline{\rho}_0\omega^2\lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}$$

and

$$\begin{aligned} C_{m,n}(s) &= H_{n+m}^{(1)}(\lambda_0 s) + (-1)^{m+1} H_{n-m}^{(1)}(\lambda_0 s) \\ &= H_{n+m}^{(1)}\left(2\lambda_0 R\sin\frac{\ell\pi}{N}\right) + (-1)^{m+1} H_{n-m}^{(1)}\left(2\lambda_0 R\sin\frac{\ell\pi}{N}\right) \end{aligned}$$

where $n, m = 0, 1, 2, \ldots$.

Family 4 in Category ℓ,

$$D(\omega, k)^{\ell,3} = \det\left[C_{m,n} - a_{n,n}\delta_{m,n}\right]$$

where

$$\overline{a}_{n,m} = \frac{\overline{\rho}_0\omega^2\lambda_i H_n^{(1)}(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 H_n'^{(1)}(\lambda_0) J_n(\lambda_i)}{\left[\overline{\rho}_0\omega^2\lambda_i J_n(\lambda_0) J_n'(\lambda_i) - (\omega - k)^2 \lambda_0 J_n'(\lambda_0) J_n(\lambda_i)\right]}$$

and

$$\begin{aligned} C_{m,n}(s) &= H_{n+m}^{(1)}(\lambda_0 s) + (-1)^{m+1} H_{n-m}^{(1)}(\lambda_0 s) \\ &= H_{n+m}^{(1)}\left(2\lambda_0 R\sin\frac{\ell\pi}{N}\right) + (-1)^{m+1} H_{n-m}^{(1)}\left(2\lambda_0 R\sin\frac{\ell\pi}{N}\right) \end{aligned}$$

where $n, m = 0, 1, 2, \ldots$.*stributions stable et instable differentiables*, J. Amer. Math. Soc. (1) 5 (1992), 33–75.

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ACKNOWLEDGEMENTS

1. It is with great pleasure that I acknowledge the support from the Mentor-Protégé Program, College of Science and Mathematics, Kennesaw State University.
2. I have been fortunate to have had an excellet team of students from the Department of Mathematics at Kennesaw State University, year by year. In particular, I thank Kara N. Keener and Kenneth Dye for their critical help.

Received September 2002; in revised February 2003.

E-mail address: `jdu@kennesaw.edu`