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MAKING THE NUMERICAL ABSCISSA NEGATIVE FOR A CLASS OF NEUTRAL EQUATIONS

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Abstract. We consider the question of exponential stability of the solution semigroup for a class of scalar differential-difference equations of neutral type. Under very weak assumptions on the coefficients in the equation we show how to construct an appropriate inner product on the underlying state space, which guarantees that the numerical abscissa of the infinitesimal generator is negative.

1. **Introduction.** We consider the following initial value problem for a scalar differential-difference equation of neutral type:

$$\begin{aligned} \frac{d}{dt}[x(t)+cx(t-r)] &= ax(t)+bx(t-r), \quad t\geq 0, \\ x(0)+cx(-r)=\eta, \quad x(\theta) &= \phi(\theta), \quad -r\leq\theta<0, \end{aligned} \tag{1}$$

where $r>0$ and the coefficients a, b and c are real. The initial data (η,ϕ) belongs to the product space $\mathbb{C}\times L_2(-r,0)$. If we make the identification $z(t)=(x(t)+cx(t-r),x(t+\theta))$, then in a standard fashion (see [3]) equation (1) can be reformulated as the Cauchy problem

$$\begin{aligned} z'(t) &= Az(t), \\ z(0) &= (\phi(0)+c\phi(-r),\phi), \end{aligned} \tag{2}$$

on the state space $Z=\mathbb{C}\times L_2(-r,0)$ equipped with the norm

$$\|(\eta,\phi)\|_Z^2=|\eta|^2+\int_{-r}^0|\phi(\theta)|^2d\theta,$$

and compatible inner product

$$\langle(\eta,\phi),(\xi,\psi)\rangle_Z=\eta\bar{\xi}+\int_{-r}^0\phi(\theta)\overline{\psi(\theta)}\,d\theta.$$

Here the operator $A:\mathrm{domA}\subset\mathrm{Z}\to\mathrm{Z}$ is defined on the domain

$$\mathrm{domA}=\{(\eta,\phi)\in\mathrm{Z}:\ \phi\in\mathrm{H}^1(-\mathrm{r},0),\ \eta=\phi(0)+\mathrm{c}\phi(-\mathrm{r})\}$$

by

$$A(\eta,\phi)=(a\phi(0)+b\phi(-r),\phi').$$

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It is well known that A is the infinitesimal generator of a C_0-semigroup $T(t)$ on Z (see [3], [7]) which satisfies the spectrum determined growth condition (see [8]).

In this paper we will show that under assumptions (A1) and (A2) below one can construct an inner product $\langle\cdot,\cdot\rangle_e$ which is equivalent to the standard inner product $\langle\cdot,\cdot\rangle_Z$ above, and for which the numerical abscissa $\omega_1(A,\langle\cdot,\cdot\rangle_e)$ is negative. The numerical abscissa of A, which is norm and inner product dependent, is defined by

$$\begin{aligned}\omega_1(A)=\omega_1(A,\langle\cdot,\cdot\rangle) &= \sup_{\lambda\in\Theta(A)} \operatorname{Re}\lambda \\ &= \inf\left\{\omega:\ \operatorname{Re}\langle Az,z\rangle\le\omega\|z\|^2,\quad z\in\ \operatorname{dom}A\right\}.\end{aligned}$$

Here $\Theta(A)$ denotes the numerical range of the operator A, which is the subset of the complex plane defined by

$$\Theta(A)=\Theta(A,\langle\cdot,\cdot\rangle)=\{\langle Az,z\rangle:\ z\in\ \operatorname{dom}A,\ \|z\|=1\}.$$

The 'making the numerical abscissa negative' in the title refers to the fact that for the neutral equations under consideration, the numerical abscissa is zero in the standard inner product but is negative in an appropriately chosen new inner product. An important consequence is that the new inner product can be used to show that certain finite dimensional semidiscrete approximation schemes (see [1], [2] and [9] for discussion of such approximation schemes) preserve the exponential stability of the original problem. We do not pursue these approximation ideas here (they are explored in [6]), but they serve to motivate the construction of the inner products that we discuss. This paper extends and unifies the results in [4], [5] (where the case $c=0$ is considered) and [6] (where $b=0$ is considered).

2. **Main results.** We make the following assumptions on the coefficients in (1):

(A1): $0<|c|<1$,
(A2): $a<-|b|$.

We will show that under these assumptions we can construct an equivalent inner product which makes the numerical abscissa negative. Since this implies that the eigenvalues of A must lie in the left half complex plane strictly bounded away from the imaginary axis, the assumptions (A1) and (A2) can be no weaker than necessary and sufficient conditions for this behavior of the eigenvalues of A. Therefore (A1) and (A2) are 'almost optimal' given the following observations. First, recall that A has a so-called neutral chain of eigenvalues asymptotic to the vertical line $\operatorname{Re}\lambda=\frac{\ln|c|}{r}$, and (A1) is necessary and sufficient for the neutral chain asymptote to be in the left half complex plane. Second, when $c=0$ a necessary and sufficient condition for the eigenvalues of A to lie in a half plane $\operatorname{Re}\lambda<-\sigma<0$ is $a<-|b|$ or $a=b$ and $b<0$.

To proceed, let us first consider the equation

$$a+\frac{|b|}{x}=\frac{2}{r}\ln x. \tag{3}$$

Assumption (A2) guarantees a solution of (3) on $(0,1)$, which we denote c^*. Thus

$$c^*=e^{\frac{r}{2}(a+\frac{|b|}{c^*})}\text{ and }0<c^*<1. \tag{4}$$

We consider two cases, with the technique for constructing the new inner product different for each case.

2.1. **The case $0 < |c| < c^*$.** Consider the function

$$f(x) = a - x + (|b| - |c|x)e^{-xr}. \tag{5}$$

Observe that $f(-\infty) = +\infty$, $f(0) = a + |b| < 0$, and $f'(x) = -1 + (|c|rx - |b|r - |c|)e^{-xr} < 0$ for $-\infty < x < 0$, so f has a unique negative real root γ which satisfies

$$f(\gamma) = a - \gamma + (|b| - |c|\gamma)e^{-\gamma r} = 0. \tag{6}$$

Also

$$f(\frac{\ln|c|}{r}) = a - \frac{2\ln|c|}{r} + \frac{|b|}{|c|} > a - \frac{2\ln c^*}{r} + \frac{|b|}{c^*} = 0,$$

and

$$\begin{aligned} f(\frac{a}{2} + \frac{|b|}{2c^*}) = f(\frac{\ln c^*}{r}) &= a - \frac{\ln c^*}{r} + \frac{1}{c^*}(|b| - \frac{|c|\ln c^*}{r}) \\ &= a + \frac{|b|}{c^*} - \frac{\ln c^*}{r}(1 + \frac{|c|}{c^*}) \\ &= \frac{2\ln c^*}{r} - \frac{\ln c^*}{r}(1 + \frac{|c|}{c^*}) \\ &= \frac{\ln c^*}{r}(1 - \frac{|c|}{c^*}) \\ &< 0. \end{aligned}$$

Thus

$$\frac{\ln|c|}{r} < \gamma < \frac{a}{2} + \frac{|b|}{2c^*} = \frac{\ln c^*}{r} < 0. \tag{7}$$

Notice that $\frac{1}{r}\ln|c| < \gamma$ implies $|c|e^{-\gamma r} < 1$. Now we define the weight function $g(\theta)$ by

$$g(\theta) = ke^{-2\gamma\theta} \tag{8}$$

where

$$k = \gamma - \gamma c^2 e^{-2\gamma r} + bce^{-2\gamma r} - a. \tag{9}$$

The constant k may be rewritten as

$$\begin{aligned} k &= \gamma(1 + |c|e^{-\gamma r})(1 - |c|e^{-\gamma r}) + bce^{-2\gamma r} - a \\ &= (a + |b|e^{-\gamma r})(1 - |c|e^{-\gamma r}) + bce^{-2\gamma r} - a \\ &= (bc - |bc|)e^{-2\gamma r} - a|c|e^{-\gamma r} + |b|e^{-\gamma r} \end{aligned} \tag{10}$$

We claim that $k > 0$. To see this, observe that from (9) we have

$$\begin{aligned} k &\geq \gamma - \gamma c^2 e^{-2\gamma r} - |bc|e^{-2\gamma r} - a \\ &= \gamma - \gamma c^2 e^{-2\gamma r} - |b|e^{-\gamma r}|c|e^{-\gamma r} - a \\ &\geq \gamma - \gamma c^2 e^{-2\gamma r} - |b|e^{-\gamma r} - a \\ &= \gamma - \gamma c^2 e^{-2\gamma r} - \gamma - \gamma|c|e^{-\gamma r} \\ &= -\gamma|c|e^{-\gamma r}(1 + |c|e^{-\gamma r}) \\ &> 0 \end{aligned} \tag{11}$$

since $\gamma < 0$. Now define a new norm by

$$\|(\eta, \phi)\|_e^2 = |\eta|^2 + \int_{-r}^{0} g(\theta)|\phi(\theta)|^2 d\theta.$$

This is clearly a norm on Z which is equivalent to the original norm $\|\cdot\|_Z$. The compatible inner product is

$$\langle(\eta, \phi), (\xi, \psi)\rangle_e = \eta\bar{\xi} + \int_{-r}^{0} g(\theta)\phi(\theta)\overline{\psi(\theta)}\, d\theta. \tag{12}$$

It can be shown that $\omega_1(A, \langle\cdot,\cdot\rangle_Z) = 0$ (for example, by using an argument similar to that of Lemma 1 in [6]). The next result shows that the new norm makes the numerical abscissa negative.

Theorem 1. *The numerical abscissa satisfies* $\omega_1(A, \langle\cdot,\cdot\rangle_e) \le \gamma < 0$.

Proof. It is sufficient to show that the inner product satisfies $\mathrm{Re}\langle Az, z\rangle_e \le \gamma\|z\|_e^2$ for all $z \in \mathrm{dom}A$. We will make use of the integration by parts formula

$$\begin{aligned}\mathrm{Re}\int_{-r}^{0} g(\theta)\phi'(\theta)\overline{\phi(\theta)} &= \frac{1}{2}g(0)|\phi(0)|^2 - \frac{1}{2}g(-r)|\phi(-r)|^2 \\ &\quad -\frac{1}{2}\int_{-r}^{0} g'(\theta)|\phi(\theta)|^2 d\theta. \end{aligned} \tag{13}$$

To proceed, for $(\eta, \phi) \in \mathrm{dom}A$, we have

$$\begin{aligned}\mathrm{Re}\langle A(\eta,\phi),(\eta,\phi)\rangle_e &= \mathrm{Re}\left\{a[\eta - c\phi(-r)]\overline{\eta} + b\phi(-r)\overline{\eta} + \int_{-r}^{0} g(\theta)\phi'(\theta)\overline{\phi(\theta)}d\theta\right\} \\ &= a|\eta|^2 + (b-ac)\,\mathrm{Re}\,\phi(-r)\overline{\eta} + \frac{1}{2}k|\eta - c\phi(-r)|^2 \\ &\quad -\frac{1}{2}g(-r)|\phi(-r)|^2 + \gamma\int_{-r}^{0} g(\theta)|\phi(\theta)|^2 d\theta \\ &= [a + \frac{1}{2}k]|\eta|^2 + [b - ac - ck]\,\mathrm{Re}\,\eta\overline{\phi(-r)} \\ &\quad -\frac{1}{2}[g(-r) - c^2k]|\phi(-r)|^2 + \gamma\int_{-r}^{0} g(\theta)|\phi(\theta)|^2 d\theta. \end{aligned} \tag{14}$$

Set

$$\delta_1 = b - ac - ck, \tag{15}$$

and

$$\delta_2 = g(-r) - c^2k = k(e^{2\gamma r} - c^2) > 0. \tag{16}$$

Now if we continue from (14) and use the generic inequality

$$\pm\delta_1 \mathrm{Re}\, x\overline{y} \le \frac{1}{2\delta_2}\delta_1^2|x|^2 + \frac{\delta_2}{2}|y|^2 \qquad \text{for all } \delta_2 > 0,$$

we have (δ_1, δ_2 are chosen so that the $|\phi(-r)|^2$ terms cancel to zero)

$$\begin{aligned}\mathrm{Re}\langle A(\eta,\phi),(\eta,\phi)\rangle_e &\le \left[a + \frac{1}{2}k + \frac{(b-ac-ck)^2}{2k(e^{2\gamma r} - c^2)}\right]|\eta|^2 + \gamma\int_{-r}^{0} g(\theta)|\phi(\theta)|^2 d\theta \\ &\le \gamma|\eta|^2 + \gamma\int_{-r}^{0} g(\theta)|\phi(\theta)|^2 d\theta \\ &= \gamma\|(\eta,\phi)\|_e^2. \end{aligned} \tag{17}$$

The result follows. □

The inequality in (17) is a consequence of the following technical lemma.

Lemma 1. *The inequality* $a + k/2 + \frac{(b-ac-ck)^2}{2k(e^{2\gamma r}-c^2)} \le \gamma$ *holds.*

Proof. The inequality is equivalent (after multiplying by $2k(e^{2\gamma r} - c^2)$ and simplifying) to

$$(ac - b)^2 e^{-2\gamma r} \le k^2. \tag{18}$$

We shall verify (18) by considering several cases, which depend on the signs of b and c, and by making use of (6) and (9).

Case (i): $b \geq 0$, $c > 0$.

In this case

$$\begin{aligned}(ac-b)e^{-\gamma r} &= a|c|e^{-\gamma r} - |b|e^{-\gamma r} \\ &= a|c|e^{-\gamma r} + a - \gamma - \gamma|c|e^{-\gamma r} \\ &= a - \gamma + (a-\gamma)|c|e^{-\gamma r} \\ &= a - \gamma - (|b| - |c|\gamma)|c|e^{-2\gamma r} \\ &= -k, \end{aligned} \tag{19}$$

so $(ac-b)^2 e^{-2\gamma r} = k^2$.

Case (ii): $b < 0$, $c < 0$.

The argument is similar to (i). We have

$$\begin{aligned}(ac-b)e^{-\gamma r} &= -a|c|e^{-\gamma r} + |b|e^{-\gamma r} \\ &= k, \end{aligned} \tag{20}$$

so $(ac-b)^2 e^{-2\gamma r} = k^2$.

For cases (iii) and (iv) we first observe that from (6) we get

$$\begin{aligned} a+\gamma &= 2\gamma - |b|e^{-\gamma r} + \gamma|c|e^{-\gamma r} \\ &< -|b|e^{-\gamma r} \\ &< -|b|e^{-\gamma r} - \gamma|c|e^{-\gamma r}. \end{aligned} \tag{21}$$

Case (iii): $b \geq 0$, $c < 0$.

In this case it follows from (21) that $a + \gamma < -be^{-\gamma r} - \gamma|c|e^{-\gamma r}$. We have

$$\begin{aligned} (a+\gamma)|c|e^{-\gamma r} &< (-b - \gamma|c|)|c|e^{-2\gamma r} \\ a|c|e^{-\gamma r} + \gamma|c|e^{-\gamma r} + \gamma - a &< -b|c|e^{-2\gamma r} - \gamma c^2 e^{-2\gamma r} + \gamma - a \\ a|c|e^{-\gamma r} + |b|e^{-\gamma r} &< bce^{-2\gamma r} - \gamma c^2 e^{-2\gamma r} + \gamma - a \\ (b-ac)e^{-\gamma r} &< k. \end{aligned} \tag{22}$$

Thus

$$\begin{aligned} -k < (ac-b)e^{-\gamma r} &= -a|c|e^{-\gamma r} - be^{-\gamma r} \\ &\leq -a|c|e^{-\gamma r} - be^{-\gamma r}|c|e^{-\gamma r} \\ &\leq -a|c|e^{-\gamma r} + bce^{-2\gamma r} + |b|e^{-\gamma r}(1 - |c|e^{-\gamma r}) \\ &= -a|c|e^{-\gamma r} + bce^{-2\gamma r} - |bc|e^{-2\gamma r} + |b|e^{-\gamma r} \\ &= k \end{aligned} \tag{23}$$

so $(ac-b)^2 e^{-2\gamma r} \leq k^2$.

Case (iv): $b < 0$, $c > 0$.

In this case it follows from (21) that $a + \gamma < be^{-\gamma r} - \gamma|c|e^{-\gamma r}$. We have

$$\begin{aligned} (a+\gamma)|c|e^{-\gamma r} &< (b - \gamma|c|)|c|e^{-2\gamma r} \\ a|c|e^{-\gamma r} + \gamma|c|e^{-\gamma r} + \gamma - a &< bce^{-2\gamma r} - \gamma c^2 e^{-2\gamma r} + \gamma - a \\ a|c|e^{-\gamma r} + |b|e^{-\gamma r} &< k \\ (ac-b)e^{-\gamma r} &< k. \end{aligned} \tag{24}$$

Thus

$$\begin{aligned} -k < (b-ac)e^{-\gamma r} &= -a|c|e^{-\gamma r} - |b|e^{-\gamma r} \\ &\leq -a|c|e^{-\gamma r} - |b|e^{-\gamma r}|c|e^{-\gamma r} \\ &= -a|c|e^{-\gamma r} + bce^{-2\gamma r} \\ &\leq -a|c|e^{-\gamma r} + bce^{-2\gamma r} + |b|e^{-\gamma r}(1 - |c|e^{-\gamma r}) \end{aligned}$$

$$= \quad k \tag{25}$$

so $(ac-b)^2 e^{-2\gamma r} \le k^2$. The lemma is proved. □

2.2. **The case $c^* \le |c| < 1$.** Define $\hat{a} = \frac{1}{r}\ln|c| < 0$ and $\hat{k} = -a + (b/c)$. Observe that

$$\begin{aligned} \hat{k} &= -a + (b/c) \ge -a - (|b|/|c|) \\ &\ge -a - (|b|/c^*) = -\frac{2\ln c^*}{r}, \end{aligned} \tag{26}$$

so $\hat{k} > 0$. Next define the weight function $\hat{g}(\theta)$ by

$$\hat{g}(\theta) = \hat{k} e^{-2\hat{a}\theta} \tag{27}$$

and define a new norm by

$$\|(\eta,\phi)\|_e^2 = |\eta|^2 + \int_{-r}^{0} \hat{g}(\theta)|\phi(\theta)|^2 d\theta.$$

As for the previous case, this is clearly a norm on Z which is equivalent to the original norm $\|\cdot\|_Z$. The compatible inner product is

$$\langle(\eta,\phi),(\xi,\psi)\rangle_e = \eta\bar{\xi} + \int_{-r}^{0} \hat{g}(\theta)\phi(\theta)\overline{\psi(\theta)}\, d\theta. \tag{28}$$

Observe that

$$\begin{aligned} a + \frac{\hat{k}}{2} &= \frac{1}{2}(a + \frac{b}{c}) \\ &\le \frac{1}{2}(a + \frac{|b|}{|c|}) \\ &\le \frac{1}{2}(a + \frac{|b|}{c^*}) \\ &= \frac{1}{2}\frac{2}{r}\ln c^* \\ &\le \frac{1}{r}\ln|c| \\ &= \hat{a}. \end{aligned} \tag{29}$$

We have the following result.

Theorem 2. *The numerical abscissa satisfies $\omega_1(A, \langle\cdot,\cdot\rangle_e) \le \hat{a} < 0$.*

Proof. We follow an argument similar to that of Theorem 1. It is sufficient to show that the inner product satisfies $\mathrm{Re}\langle Az, z\rangle_e \le \hat{a}\|z\|_e^2$ for all $z \in \mathrm{dom}A$. To proceed, for $(\eta,\phi) \in \mathrm{dom}A$, we have

$$\begin{aligned} \mathrm{Re}\langle A(\eta,\phi),(\eta,\phi)\rangle_e &= \mathrm{Re}\left\{a[\eta - c\phi(-r)]\overline{\eta} + b\phi(-r)\overline{\eta} + \int_{-r}^{0} \hat{g}(\theta)\phi'(\theta)\overline{\phi(\theta)}d\theta\right\} \\ &= a|\eta|^2 + (b-ac)\,\mathrm{Re}\,\phi(-r)\overline{\eta} + \frac{1}{2}\hat{k}|\eta - c\phi(-r)|^2 \\ &\quad -\frac{1}{2}\hat{g}(-r)|\phi(-r)|^2 + \hat{a}\int_{-r}^{0} \hat{g}(\theta)|\phi(\theta)|^2 d\theta \\ &= [a + \frac{1}{2}\hat{k}]|\eta|^2 + [b - ac - c\hat{k}]\,\mathrm{Re}\,\eta\overline{\phi(-r)} \\ &\quad -\frac{1}{2}\hat{k}[e^{2\hat{a}r} - c^2]|\phi(-r)|^2 \end{aligned}$$

$$
\begin{aligned}
& +\hat{a}\int_{-r}^{0}\hat{g}(\theta)|\phi(\theta)|^2 d\theta \\
\leq \quad & \hat{a}\|(\eta,\phi)\|_e^2. \qquad (30)
\end{aligned}
$$

We used the fact that $b - ac - c\hat{k} = 0$ and $e^{2\hat{a}r} - c^2 = 0$. The result follows. □

3. **Conclusion.** This paper extends and refines previous results found in [5] and [6]. We have obtained inner products which make the numerical abscissa negative for a class of scalar neutral equations. The inner products presented here are equivalent to the standard inner product on the space $Z = \mathbb{C} \times L_2(-r, 0)$. These inner products provide an appropriate setting to establish qualitative properties of approximation methods for the hereditary systems discussed in this paper. For example, the inner product not only guarantees exponential stability of the solution semigroup associated with these equations (a result which is already known by other techniques), but it can also be used to show that the solution semigroups for finite dimensional approximations to these equations are exponentially stable as well, uniformly in the discretisation parameter. We have not pursued this here, but it is our main motivation for constructing the inner products found in this paper.

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