

## A NONLINEAR CONTROLLED SYSTEM OF DIFFERENTIAL EQUATIONS DESCRIBING THE PROCESS OF PRODUCTION AND SALES OF A CONSUMER GOOD

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**Abstract.** A nonlinear controlled system of differential equations has been constructed to describe the process of production and sales of a consumer good. This model can be controlled either by the rate of production or by the price of the good. The attainable sets of corresponding controlled systems are studied. It is shown that in both cases the boundaries of these sets are the unions of two two-parameter surfaces. It is proved that every point on the boundaries of the attainable sets is a result of piecewise constant controls with at most two switchings. Attainable sets for different values of parameters of the model will be demonstrated using MAPLE.

**1. Introduction.** One of the fundamental problems of control theory of dynamical systems is the problem of determining or of estimating sets of possible phase states of a system at different moments of time. These sets, which are called attainable sets, play an important role in solving control problems. For example, the exact or approximate knowledge of attainable sets of control systems allows us to estimate the limit possibilities of a control system, to choose an optimal or suboptimal control. This problem reduces to the construction or to the estimation of sets in which the phase vector of a system lies. Moreover, the main question, namely, what mathematical construction can characterize the attainable set was considered in many papers. At present, four basic methods are used to describe attainable sets in the phase space of a dynamical system: a pointwise description of sets, a representation of sets in the form of level lines of smooth functions, the definitions of sets by their support functions and by a parametric description of boundaries. More detailed characteristics of these constructions can be found in [1], [2].

For this study we use the last method applying to building an attainable set for a dynamic microeconomic model of the process of production and sales of a perishable good. Obtaining analytically the type of control that leads a trajectory to the boundary of an attainable set, we wrote a computer program in Maple that builds different shapes of the sets for different parameters of the model. Also this program can be used by real companies to predict the right production plan and price strategy in order to maximize profit.

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**2. Description of the Model.** Let us consider a monopoly producing some perishable consumer good at a net unit cost of one. The process of production and sales of such microeconomic system can be described by the nonlinear system of differential equations below. Since a similar model was first mentioned in [3], we will omit details of its description and how it was constructed.

$$\begin{cases} \dot{x}_1(t) = -n_p(Y - x_2(t))x_1(t) + u(t), \\ \dot{x}_2(t) = n_p(Y - x_2(t))x_1(t) - k_1x_2(t), \\ \dot{x}_3(t) = pn_p(Y - x_2(t))x_1(t) - u(t), \quad t \in [0, T], \\ x_1(0) = x_1^0 \geq 0, \quad x_2(0) = x_2^0 \geq 0, \quad x_3(0) = x_3^0 \geq 0, \end{cases} \quad (1)$$

where  $x_1(t)$  - the amount of the good on the market,  $x_2(t)$  - a quantity of the good unused in consumers' homes,  $x_3(t)$  - a cumulative profit,  $u(t)$  - the rate of production,  $k_1$  - the speed of consumption,  $Y$  - potential demand,  $t$  - time,  $T$  - the end of the time interval,  $p$  - the sales price,  $n_p$  - a coefficient of the rate of sales of the good.

For durable goods, the factor  $n_p$  vanishes or becomes negligibly small as  $x_2 \rightarrow Y$ . However, for perishable products or provision, demand always exists independently of the amount sold of the product. Thus, we assume that  $n_p = n(p)$  is decreasing function when price  $p$  ( $p > 1$ ) is increasing. To describe  $n_p$  we choose the following formula  $n_p = \frac{n_0}{p}$ , where  $n_0$  is some constant.

Also from the economic nature of our variables we assume that  $x_2^0 < Y$ .

Here  $u(t)$  is a control function with the usual Lebesgue measure such that

$$0 < u_1 \leq u(t) \leq u_2, \text{ for almost all } t \in [0, T], \quad (2)$$

Using conclusions similar to our conclusions in [4] we can show the validity of the following statement, describing properties of the variables of (1).

**Lemma 1.** *Let  $u(t)$  be some control function satisfying (2). Then solution of system (1),  $x(t)$ , corresponding to this control exists over the entire interval  $[0, T]$ , and its components  $x_1(t)$ ,  $x_2(t)$  satisfy the following inequalities :*

$$x_1(t) > 0, \quad 0 < x_2(t) < Y, \quad t \in (0, T]. \quad (3)$$

**3. An Attainable Set of the Controlled Dynamical System.** Let  $X(T)$  be an attainable set for system (1) reachable from an initial condition  $x_0 = (x_1^0, x_2^0, x_3^0)$  at moment  $T$ , i. e.,  $X(T)$  is the set of all ends  $x(T) = (x_1(T), x_2(T), x_3(T))$  of trajectories  $x(t) = (x_1(t), x_2(t), x_3(t))$  of system (1) under admissible controls  $u(t)$  satisfying (2). It is proved in work [5] that  $X(T)$  is a compact set in  $\mathbb{R}^3$  (it is closed and bounded).

Next, we study the boundary of  $X(T)$ . Let  $x^*$  be a point on the boundary of  $X(T)$  with the corresponding control  $u_*(t)$  and trajectory  $x^*(t)$ ,  $t \in [0, T]$  of system (1) such that  $x^* = x^*(T)$ . Then following [5] we can state that there exists a nontrivial solution of the adjoint system :

$$\begin{cases} \dot{\psi}_1(t) = n_p(Y - x_2^*(t))(\psi_1(t) - \psi_2(t) - p\psi_3(t)), \\ \dot{\psi}_2(t) = -n_px_1^*(t)(\psi_1(t) - \psi_2(t) - p\psi_3(t)) + k_1\psi_2(t), \\ \dot{\psi}_3(t) = 0, \quad t \in [0, T], \end{cases} \quad (4)$$

for which

$$u_*(t) = \begin{cases} u_1 & , \quad \text{if } L(t) < 0, \\ \forall u \in [u_1, u_2] & , \quad \text{if } L(t) = 0, \\ u_2 & , \quad \text{if } L(t) > 0. \end{cases} \quad (5)$$

Here  $L(t) = \psi_1(t) - \psi_3(t)$ . The function  $L(t)$  is so-called the switching function, its behavior determines the type of control  $u_*(t)$  leading to the boundary of an attainable set.

From the last equation of (4) we obtain that  $\psi_3(t) = \psi_3^0$ , where  $\psi_3^0$  is some constant. Replacing this into the first two equations gives us

$$\begin{cases} \dot{\psi}_1(t) = n_p(Y - x_2^*(t))(\psi_1(t) - \psi_2(t) - p\psi_3^0), \\ \dot{\psi}_2(t) = -n_px_1^*(t)(\psi_1(t) - \psi_2(t) - p\psi_3^0) + k_1\psi_2(t). \end{cases} \tag{6}$$

Also we can rewrite  $L(t)$  as  $L(t) = \psi_1(t) - \psi_3^0$ .

Next, we obtain a differential equation for switching function  $L(t)$ . The following statement is valid.

**Lemma 2.** *The switching function  $L(t)$  satisfies the following nonhomogeneous linear differential equation of the second order with variable coefficients :*

$$L^{(2)}(t) - a(t)L^{(1)}(t) + b(t)L(t) = d(t), \tag{7}$$

where

$$\begin{aligned} a(t) &= n_p(Y - x_2^*(t)) + \frac{k_1Y}{Y - x_2^*(t)}, \quad b(t) = n_pk_1(Y - x_2^*(t)), \\ d(t) &= n_pk_1\psi_3^0(p - 1)(Y - x_2^*(t)). \end{aligned}$$

*Proof.* First, from the definition of function  $L(t)$  we find that  $\dot{L}(t) = \dot{\psi}_1(t)$ . Next, we rewrite system (6) in terms of  $L(t)$ ,  $\psi_2(t)$  and their derivatives:

$$\begin{cases} \dot{L}(t) = n_p(Y - x_2^*(t))(L(t) - \psi_2(t) - (p - 1)\psi_3^0), \\ \dot{\psi}_2(t) = -n_px_1^*(t)(L(t) - \psi_2(t) - (p - 1)\psi_3^0) + k_1\psi_2(t). \end{cases} \tag{8}$$

We express function  $\psi_2(t)$  from the first equation of system (8) and replace it into the second equation of the system. Next, we do necessary differentiation using system (1). Simplifying the obtained expression, we find required equation (7) for function  $L(t)$ . □

Let us investigate the possibility for function  $L(t)$  become zero over some subinterval in  $[0, T]$ . The following statement takes place.

**Lemma 3.** *There is no interval  $\Delta \subset [0, T]$  on which  $L(t) \equiv 0$  for all  $t \in \Delta$ .*

*Proof.* Assume that  $L(t) = 0$  on  $\Delta \subset [0, T]$ . Depending on values of  $\psi_3^0$  we need to consider two cases.

Case 1. If  $\psi_3^0 \neq 0$ , then using (3) we can see that the right side of equation (7) keeps the same sign, but the left side of the equation equals zero. That is a contradiction.

Case 2. If  $\psi_3^0 = 0$ , then  $d(t) = 0$ . By the uniqueness of solution for Cauchy problem we obtain that  $L(t) = 0$  over  $[0, T]$ , then  $\psi_1(t) = 0$ ,  $\psi_2(t) = 0$ , which contradicts the condition of nontriviality of solution to (4).

In both cases we obtained the contradiction to our assumption. □

Therefore, control  $u_*(t)$  leading a trajectory to the boundary of an attainable set is bang-bang control, and (5) can be rewritten as :

$$u_*(t) = \begin{cases} u_1 & , \quad \text{if } L(t) < 0, \\ u_2 & , \quad \text{if } L(t) > 0. \end{cases} \tag{9}$$

At points of discontinuity we define function  $u_*(t)$  by its limit from the left.

Next, we need to estimate the number of zeros of function  $L(t)$ . The following statement is valid.

**Lemma 4.** *Over time interval  $(0, T)$  the switching function  $L(t)$  has at most two zeros.*

*Proof.* Let us consider corresponding to (7) the homogeneous differential equation

$$L^{(2)}(t) - a(t)L^{(1)}(t) + b(t)L(t) = 0, \quad (10)$$

characteristic equation of which

$$\lambda^2(t) - a(t)\lambda(t) + b(t) = 0,$$

has the discriminant

$$D(t) = a^2(t) - 4b(t) = \left(\frac{k_1 Y}{Y - x_2^*(t)} - n_p(Y - x_2^*(t))\right)^2 + 4k_1 n_p x_2^*(t).$$

that is positive for all  $t \in [0, T]$ . Then the roots of this characteristic equation can be evaluated as

$$\lambda_1(t) = \frac{1}{2}(a(t) - \sqrt{D(t)}), \quad \lambda_2(t) = \frac{1}{2}(a(t) + \sqrt{D(t)}).$$

It can be shown that  $\lambda_1(t)$  and  $\lambda_2(t)$  satisfy the inequality :

$$\lambda_1(t) < \sqrt{k_1 n_p Y} < \lambda_2(t), \quad t \in [0, T]. \quad (11)$$

Since (11) states that the constant  $\sqrt{k_1 n_p Y}$  always occurs between the roots  $\lambda_1(t)$  and  $\lambda_2(t)$ , it follows from the work [6] that each solution of equation (10) on the interval  $(0, T)$  has at most one zero. Then the validity of our statement follows from the generalized Rolle's theorem ([7]).  $\square$

Next, using (9) and assertions proved in Lemmas 3 and 4, we can formulate our principal statement.

**Main Theorem.** *Every point  $x^*$  on the boundary of an attainable set  $X(T)$  can be obtained as a result of a piecewise constant control  $u_*(t)$  with at most two switchings.*

From this theorem we conclude that depending on relationships between parameters of our model, the control  $u_*(t)$  leading to the boundary of an attainable set is a piecewise constant function, taking values  $\{u_1, u_2\}$  with at most two switchings over time interval  $(0, T)$ . It can have one of the types below.

Type 1. (without switchings)

$$u_*(t) = u_1, \quad t \in [0, T],$$

or

$$u_*(t) = u_2, \quad t \in [0, T].$$

Type 2. (with one switching)

$$u_*(t) = \begin{cases} u_2 & , \text{ if } 0 \leq t \leq \vartheta, \\ u_1 & , \text{ if } \vartheta < t \leq T \end{cases}$$

or

$$u_*(t) = \begin{cases} u_1 & , \text{ if } 0 \leq t \leq \vartheta, \\ u_2 & , \text{ if } \vartheta < t \leq T. \end{cases}$$

Type 3. (with two switchings)

$$u_*(t) = \begin{cases} u_2 & , \text{ if } 0 \leq t \leq \vartheta_1, \\ u_1 & , \text{ if } \vartheta_1 < t \leq \vartheta_2, \\ u_2 & , \text{ if } \vartheta_2 < t \leq T \end{cases}$$

or

$$u_*(t) = \begin{cases} u_1 & , \text{ if } 0 \leq t \leq \vartheta_1, \\ u_2 & , \text{ if } \vartheta_1 < t \leq \vartheta_2, \\ u_1 & , \text{ if } \vartheta_2 < t \leq T. \end{cases}$$

Above,  $\vartheta$ ,  $\vartheta_1$  and  $\vartheta_2$  are the moments of switchings.

**4. Shapes of Attainable Sets for Different Parameters of the Model.**

Knowing how to get to the boundary of an attainable set for our model, we wrote a computer program in MAPLE that builds an attainable set as a union of two two-parameter surfaces in space  $R^3$ ; parameters are moments of switchings. The points on the first surface that will be shown in white are the ends of trajectories corresponding to control  $u_1 \rightarrow u_2 \rightarrow u_1$ , and in gray to  $u_2 \rightarrow u_1 \rightarrow u_2$ . We choose standard parameters to be  $u_1 = 150$ ,  $u_2 = 300$ ,  $Y = 500$ ,  $p = \$2$ ,  $T = 5$  days, and  $k_1 = 1$ . Corresponding attainable set is shown on Figure 1. We see that during five days the company can make up to \$1800 of profit. However, if the rate of consumption is less than one ( $k_1 = 0.5$ ), the shape of attainable set becomes skinny; the number of units on market increases and the maximum possible profit decreases (Figure 2.). On Figure 3 we see that increasing price ( $p = \$20$ ) increases profit dramatically. For the same five days we can obtain up to \$28000 of profit. On Figure 4, all parameters are standard except the end of the time interval ( $T = 20$  days). During twenty days the maximum possible profit is approximately \$6000 that is obviously higher than in case shown on Figure 1 for five days.

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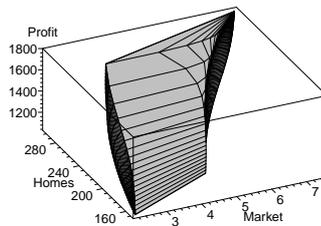


Figure 1.

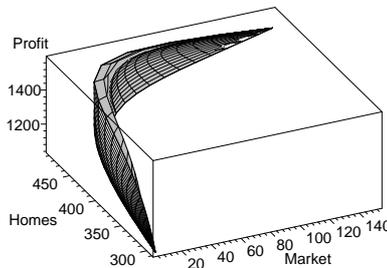


Figure 2.

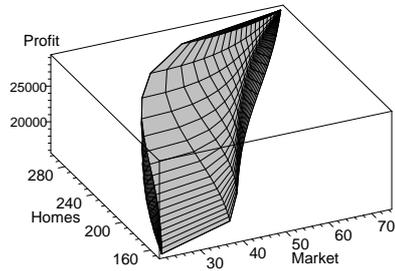


Figure 3.

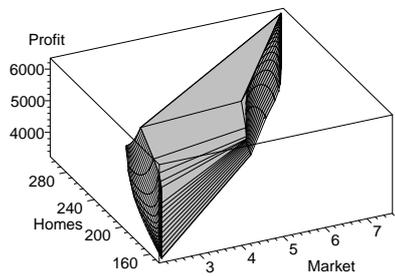


Figure 4.

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