PROCEEDINGS OF THE FOURTH INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
May 24 – 27, 2002, Wilmington, NC, USA pp. 526–535

EXISTENCE AND MULTIPLICITY RESULTS FOR A CLASS OF NONLINEAR ELLIPTIC PROBLEMS IN $\mathbb{R}^N$

M. LAZZO

Università di Bari, Dipartimento di Matematica
via Orabona 4, 70125 Bari, Italy

Abstract. We study the effect of non constant coefficients a and b on the existence and multiplicity of positive solutions for the equation $-\mathrm{div}\ a(x)\nabla u + \lambda u = b(x)\,|u|^{p-2}u$ in $\mathbb{R}^N$.

1. Introduction. We are interested in positive solutions to the problem

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\mathrm{div}\big(a(x)\nabla u\big) + \lambda\, u = |u|^{p-2}u & \text{in } \mathbb{R}^N\,, \\ \quad u(x) \longrightarrow 0 \quad \text{as } |x| \to \infty \end{cases}$$

where $N \geq 2$ and $2 < p < 2^*$ (2^* is the limiting exponent for the Sobolev embedding of $H^1(\mathbb{R}^N)$ into $L^s(\mathbb{R}^N)$).

It is well known that, if a is a positive and constant function, then (P_λ) has one positive solution, for any $\lambda > 0$; this solution is unique up to translations (see [3] and [9]). We investigate the effect of a non constant coefficient a on the existence and multiplicity of positive solutions to (P_λ).

Throughout the paper, we assume that a is a continuous and bounded real function in $\mathbb{R}^N$, and we denote

$$a_0 = \inf_{\mathbb{R}^N} a(x)\,, \qquad a_\infty = \liminf_{|x|\to\infty} a(x)\,.$$

Theorem 1.1. *Assume $0 \leq a_0 < a_\infty$ and suppose that there exist $r > 0$ and $q > Np/(2N + 2p - Np)$ such that*

$$\big\{x \in \mathbb{R}^N \,:\, a(x) = 0\big\} \subset B_r(0)\,, \qquad \frac{1}{a} \in L^q\big(B_r(0)\big)\,. \tag{1.1}$$

Then: there exists $\lambda^* > 0$ such that (P_λ) has one positive solution, for any $\lambda > \lambda^*$.

The additional hypothesis (1.1) is significant when $a_0 = 0$, namely when (P_λ) is a degenerate elliptic problem; it is inspired by a similar condition introduced by Murthy and Stampacchia in [11]; see also [5]. In the "model case", that is if $a(x) \sim |x|^\alpha$ near the origin, (1.1) amounts to $0 < \alpha < (2N + 2p - Np)/p$; we refer to [4] and references therein for some related problems.

Solutions found in Theorem 1.1 arise as constrained minimizers of the energy functional associated with (P_λ). We study the asymptotic behavior of such minimizers, as λ goes to infinity, and prove that they concentrate around global minimum points of a, provided $a_0 > 0$ (see Theorem 4.2 for the precise statement).

AMS 1991 Subject Classification: Primary 35Q55, 35J20, 58E05.

Key words and phrases: nonlinear elliptic equations, degenerate equations, variational methods, Ljusternik–Schnirelman category.

Motivated by this qualitative result, we investigate the effect of the topology of

$$\Omega = \left\{x \in \mathbb{R}^N \,:\, a(x) = a_0\right\} \tag{1.2}$$

on the number of positive solutions to (P_λ). The following notation is used in Theorem 1.2: $\Omega_\delta = \left\{x \in \mathbb{R}^N \,:\, \mathrm{dist}(x,\Omega) \le \delta\right\}$, for any $\delta > 0$.

Theorem 1.2. *Assume $0 < a_0 < a_\infty$. Then: for any $\delta > 0$, there exists $\lambda_\delta > 0$ such that (P_λ) has at least $\mathrm{cat}_{\Omega_\delta}(\Omega)$ positive solutions, for any $\lambda > \lambda_\delta$.*

As usual, $\mathrm{cat}_{\Omega_\delta}(\Omega)$ denotes the Ljusternik–Schnirelman category of Ω in Ω_δ (definitions are recalled in Section 5).

Our existence and multiplicity results can be extended to the problem

$$(\widetilde{\mathrm{P}}_\lambda) \qquad \begin{cases} -\mathrm{div}\left(a(x)\nabla u\right) + \lambda\, u = b(x)\,|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ \quad u(x) \longrightarrow 0 \quad \text{as } |x| \to \infty \end{cases}$$

under mild assumptions on the coefficient b. Some results in this direction are presented in Section 6.

2. Notations and preliminaries.

For any $\Omega \subset \mathbb{R}^N$, let $\mathcal{H}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{H}(\Omega)} = \left(\int_\Omega \left(a(x)|\nabla u|^2 + \lambda\,|u|^2\right)dx\right)^{1/2}.$$

Remark 2.1. When $a_0 > 0$, $\mathcal{H}(\mathbb{R}^N)$ is clearly the same as $H^1(\mathbb{R}^N)$, the usual Sobolev space. If $a_0 = 0$ and (1.1) holds, it is not difficult to see that $\mathcal{H}(B_R(0)) \subset W^{1,m}(B_R(0))$ for any $R > 0$, where $m = 2q/(1+q)$; the choice of q in (1.1) yields $p < Nm/(N-m)$. Moreover, the assumption $a_\infty > 0$ easily gives $\mathcal{H}(\mathbb{R}^N \setminus B_R(0)) \subset H^1(\mathbb{R}^N \setminus B_R(0))$, if $R > 0$ is sufficiently large. Therefore, the classical embedding theorems imply that $\mathcal{H}(\mathbb{R}^N)$ is embedded continuously in $L^p(\mathbb{R}^N)$ and compactly in $L^p_{loc}(\mathbb{R}^N)$ (see Proposition 1 in [5] for details; see also [12]).

Let us define a *weak solution* of (P_λ) as a function $u \in \mathcal{H}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \left(a(x)\nabla u \cdot \nabla\varphi + \lambda\, u\,\varphi\right)dx - \int_{\mathbb{R}^N} |u|^{p-2}u\,\varphi\,dx = 0$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. In view of the homogeneity of (P_λ), weak solutions arise as critical points of the functional

$$E_\lambda(u) = \int_{\mathbb{R}^N} \left(a(x)|\nabla u(x)|^2 + \lambda\,|u(x)|^2\right)dx\,, \tag{2.1}$$

constrained on the manifold $\Sigma = \left\{u \in \mathcal{H}(\mathbb{R}^N) \,:\, \int_{\mathbb{R}^N} |u(x)|^p\,dx = 1\right\}$. We denote

$$m_\lambda = \inf\left\{E_\lambda(u) \,:\, u \in \Sigma\right\}. \tag{2.2}$$

Let us recall some facts about minimization problems with constant coefficients. For any $\lambda, \alpha > 0$, the infimum

$$M_{\lambda,\alpha} = \inf\left\{\int_{\mathbb{R}^N} \left(\alpha\,|\nabla u(x)|^2 + \lambda\,|u(x)|^2\right)dx \,:\, u \in \Sigma\right\}$$

is attained at a positive function $\omega_{\lambda,\alpha} \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$. Such a function is unique up to translations, it is radially symmetric around the origin and it decays exponentially at infinity (see for instance [3,7–9]).

A simple computation gives

$$M_{\lambda,\alpha} = \lambda^\theta \, M_{1,\alpha} \,, \tag{2.3}$$

where $\theta = 1 + N/p - N/2$. Moreover, it is easy to check that the function $\alpha \mapsto M_{1,\alpha}$ is continuous and strictly increasing.

To simplify our notations, throughout the paper we will denote

$$M_0 = M_{1,a_0} \,, \quad \omega_0 = \omega_{1,a_0} \,, \quad M_\infty = M_{1,a_\infty} \,, \quad \omega_\infty = \omega_{1,a_\infty} \,. \tag{2.4}$$

Let us notice that M_0 and ω_0 are well defined if $a_0 > 0$; moreover, $a_0 < a_\infty$ implies $M_0 < M_\infty$.

3. Proof of Theorem 1.1. Throughout this section, the assumptions in Theorem 1.1 are understood.

It is well known that the energy functional associated with subcritical problems in unbounded domains may not satisfy the Palais–Smale condition; this phenomenon depends on the lack of compactness in the embedding $H^1 \subset L^p$. In our case, the Palais–Smale condition holds in a suitable sublevel which depends on the behavior of a at infinity:

Proposition 3.1. *For every $\lambda > 0$, the functional E_λ satisfies the Palais–Smale condition on Σ, at any level $c < M_{\lambda,a_\infty}$.*

Proof. Let $c < M_{\lambda,a_\infty}$ and let $\{u_n\} \subset \Sigma$ be a Palais–Smale sequence for E_λ on Σ at the level c, that is

$$E_\lambda(u_n) = \int_{\mathbb{R}^N} \big(a(x)|\nabla u_n(x)|^2 + \lambda\,|u_n(x)|^2\big)dx = c + o(1) \tag{3.1}$$

$$-\operatorname{div}\big(a(x)\nabla u_n\big) + \lambda\, u_n - \mu_n |u_n|^{p-2} u_n = o(1) \quad \text{in} \quad \mathcal{H}^{-1}(\mathbb{R}^N) \tag{3.2}$$

($\mathcal{H}^{-1}(\mathbb{R}^N)$ is the dual space of $\mathcal{H}(\mathbb{R}^N)$). The sequence $\{u_n\}$ is bounded in $\mathcal{H}(\mathbb{R}^N)$ so, up to a subsequence, it has a weak limit u in $\mathcal{H}(\mathbb{R}^N)$. By Remark 2.1, in order to prove that $u_n \to u$ in $\mathcal{H}(\mathbb{R}^N)$, it suffices to prove that, for any $\delta > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} \big(a(x)|\nabla u_n(x)|^2 + \lambda\,|u_n(x)|^2\big)dx < \delta$$

for any $n \geq R$. By contradiction assume that, for some subsequence u_{n_k} and some $\delta > 0$, we have

$$\int_{\mathbb{R}^N \setminus B_k(0)} \big(a(x)|\nabla u_{n_k}(x)|^2 + \lambda\,|u_{n_k}(x)|^2\big)dx \geq \delta \tag{3.3}$$

for any k. Let $0 < \varepsilon < a_\infty$ be such that $c < M_{\lambda,a_\infty-\varepsilon}$; let $r_\varepsilon > 0$ be such that $a(x) \geq a_\infty - \varepsilon$ for any $|x| \geq r_\varepsilon$. Since $\|u_n\|_{\mathcal{H}(\mathbb{R}^N)}$ is bounded, there exists $\rho > r_\varepsilon$ such that

$$\int_{\{\rho<|x|<\rho+1\}} \big(a(x)|\nabla u_{n_k}(x)|^2 + \lambda\,|u_{n_k}(x)|^2\big)dx < \varepsilon \tag{3.4}$$

for any k (up to a subsequence). Let $\eta \in C^\infty(\mathbb{R}^N;[0,1])$ satisfy $\eta(x)=1$ for $|x|\le\rho$, $\eta(x)=0$ for $|x|\ge\rho+1$ and define $v_k=\eta\, u_{n_k}$, $w_k=(1-\eta)u_{n_k}$. From (3.1), (3.2) and (3.4) it follows $E_\lambda(w_k)=c\int_{\mathbb{R}^N}|w_k(x)|^p\,dx+o(1)+O(\varepsilon)$. As a consequence:

$$\begin{aligned}E_\lambda(w_k)&\ge\int_{\mathbb{R}^N}\big((a_\infty-\varepsilon)|\nabla w_k(x)|^2+\lambda\,|w_k(x)|^2\big)dx\\&\ge M_{\lambda,a_\infty-\varepsilon}\left(\int_{\mathbb{R}^N}|w_k(x)|^p\,dx\right)^{2/p}\\&\ge M_{\lambda,a_\infty-\varepsilon}\left(\frac{E_\lambda(w_k)}{c}\right)^{2/p}+o(1)+O(\varepsilon)\,.\end{aligned}$$

By (3.3), $E_\lambda(w_k)$ is positive, so the previous inequality yields

$$E_\lambda(w_k)\ge\big(M_{\lambda,a_\infty-\varepsilon}\big)^{p/(p-2)}c^{2/(2-p)}+o(1)+O(\varepsilon)\,. \tag{3.5}$$

From (3.1), (3.4) and (3.5) we get

$$\begin{aligned}c+o(1)=E_\lambda(u_{n_k})&=E_\lambda(v_k)+E_\lambda(w_k)+O(\varepsilon)\ge E_\lambda(w_k)+O(\varepsilon)\\&\ge\big(M_{\lambda,a_\infty-\varepsilon}\big)^{p/(p-2)}c^{2/(2-p)}+o(1)+O(\varepsilon)\,;\end{aligned}$$

letting $k\to\infty$ and $\varepsilon\to0$ gives $c\ge M_{\lambda,a_\infty-\varepsilon}$, a contradiction. □

We are able to prove that the minimization problem (2.2) is solvable, if λ is sufficiently large:

Proposition 3.2. *There exists $\lambda^*>0$ such that m_λ is achieved, for any $\lambda>\lambda^*$.*

Proof. Since $a_0<a_\infty$ and a is continuous, there exist $\bar x\in\mathbb{R}^N$ and $\rho,\delta>0$ such that $a(x)\le a_\infty-\delta$ if $|x-\bar x|\le\rho$. Let $u_\lambda(x)=\lambda^{N/2p}\,\omega_\infty\big(\lambda^{1/2}(x-\bar x)\big)$ (ω_∞ was defined in (2.4)). Simple computations give

$$\begin{aligned}\lambda^{-\theta}\,E_\lambda(u_\lambda)&=\int_{\mathbb{R}^N}\Big(a\big(\bar x+\lambda^{-1/2}x\big)|\nabla\omega_\infty(x)|^2+|\omega_\infty(x)|^2\Big)dx\\&=M_\infty+\int_{\mathbb{R}^N}\Big(a\big(\bar x+\lambda^{-1/2}x\big)-a_\infty\Big)|\nabla\omega_\infty(x)|^2\,dx\\&\le M_\infty-\int_{\{|x|\le\sqrt{\lambda}\rho\}}\delta\,|\nabla\omega_\infty(x)|^2\,dx+\int_{\{|x|>\sqrt{\lambda}\rho\}}\|a\|_{L^\infty}|\nabla\omega_\infty(x)|^2\,dx\\&=M_\infty-\delta\|\nabla\omega_\infty\|^2_{L^2}+o(1)\end{aligned}$$

and

$$\int_{\mathbb{R}^N}|u_\lambda(x)|^p\,dx=\int_{\mathbb{R}^N}|\omega_\infty(x)|^p\,dx=1\,.$$

Thus, for λ sufficiently large: $m_\lambda\le E_\lambda(u_\lambda)<\lambda^\theta\,M_\infty=M_{\lambda,a_\infty}$. In a standard way, Proposition 3.1 guarantees that m_λ is achieved. □

Proof of Theorem 1.1. By Proposition 3.2, m_λ is achieved in some $v\in\Sigma$, for λ sufficiently large. By replacing v by $|v|$, if necessary, we can assume $v\ge0$. As a constrained critical point, v is a weak solution of $-\mathrm{div}\big(a(x)\nabla v\big)+\lambda\,v=\mu\,|v|^{p-2}v$, with $\mu=\|v\|^2_{\mathcal{H}(\mathbb{R}^N)}$ as a Lagrange multiplier. It is easy to see that $u=\mu^{\frac{1}{p-2}}v$ is

a weak nonnegative solution of (P_λ). Finally, the maximum principle applies (see [11] for the case $\inf a(x) = 0$), hence $u > 0$ in $\mathbb{R}^N$. $\square$

4. Asymptotic behavior of minimizers.

Lemma 4.1. *Assume $0 < a_0 < a_\infty$. Then:*

$$\lim_{\lambda \to +\infty} \lambda^{-\theta}\, m_\lambda = M_0\,. \tag{4.1}$$

Proof. Let $x_0 \in \Omega$ (cf. (1.2)) and let $u_\lambda(x) = \lambda^{N/2p}\omega_0\big(\lambda^{1/2}(x - x_0)\big)$ (cf. (2.4)). Plainly, $\int_{\mathbb{R}^N} |u_\lambda(x)|^p dx = 1$ and

$$\begin{aligned} E_\lambda(u_\lambda) &= \lambda^\theta \int_{\mathbb{R}^N} \Big(a\big(x_0 + \lambda^{-1/2}x\big)|\nabla\omega_0(x)|^2 + |\omega_0(x)|^2\Big)dx \\ &= \lambda^\theta \left(\int_{\mathbb{R}^N} \Big(a(x_0)|\nabla\omega_0(x)|^2 + |\omega_0(x)|^2\Big)dx + o(1)\right) = \lambda^\theta\big(M_0 + o(1)\big)\,. \end{aligned}$$

Therefore: $\lambda^\theta\, M_0 = M_{\lambda,a_0} \le m_\lambda \le E_\lambda(u_\lambda) \le \lambda^\theta\big(M_0 + o(1)\big)$, and (4.1) follows as $\lambda \to \infty$. $\square$

In the next theorem we investigate the asymptotic behavior of minimizers of E_λ on Σ, as λ goes to infinity. Our result is in the same spirit as in [13], where the author proves that sequences of positive ground states of the equation $-h\,\Delta u + V(x)u = |u|^{p-2}u$ concentrate around global minima of V, as $h \to 0^+$, provided

$$0 < \inf_{\mathbb{R}^N} V(x) < \liminf_{|x| \to +\infty} V(x)\,.$$

Theorem 4.2. *Assume $0 < a_0 < a_\infty$. Let $\lambda_n \to +\infty$ and let $u_n \in \Sigma$ be such that*

$$E_{\lambda_n}(u_n) = m_{\lambda_n}\,. \tag{4.2}$$

Then (up to subsequences) there exist $x_n \in \mathbb{R}^N$ and $w_n \in H^1(\mathbb{R}^N)$ such that

$$u_n(x) = \lambda_n^{N/2p} w_n\big(\lambda_n^{1/2}(x - x_n)\big)\,,$$

and

$$x_n \longrightarrow x_0 \in \Omega\,, \quad w_n \longrightarrow \omega_0 \ \ \textit{strongly in } H^1(\mathbb{R}^N)$$

as $n \to \infty$.

Proof. Throughout the proof, we denote $a_n(x) = a(\lambda_n^{-1/2}x)$.
Let λ_n and u_n be as above and let $v_n(x) = \lambda_n^{-N/2p}u_n(\lambda_n^{-1/2}x)$. We apply Lions' Lemma (cf. [10]) to the sequence of probability measures defined by $\mu_n(X) = \int_X |v_n(x)|^p\, dx$. Vanishing is easily ruled out since both v_n and ∇v_n are bounded in $L^2(\mathbb{R}^N)$ (see for instance Lemma I.1 in [10]). If dichotomy occurs, there exists $\gamma \in (0,1)$ such that, for any $\varepsilon > 0$, v_n splits into the sum of two functions $v_n^1, v_n^2 \in H^1(\mathbb{R}^N)$, which satisfy the following conditions: $\mathrm{supp}(v_n^1) \subset B_R(z_n)$ and $\mathrm{supp}(v_n^2) \subset \mathbb{R}^N \setminus B_{R_n}(z_n)$, for some $R > 0$, $R_n \to \infty$, $z_n \in \mathbb{R}^N$, and

$$\int_{\mathbb{R}^N} |v_n^1(x)|^p\, dx \ge \gamma - \varepsilon\,, \qquad \int_{\mathbb{R}^N} |v_n^2(x)|^p\, dx \ge 1 - \gamma - \varepsilon$$

for any n. Thus

$$\begin{aligned}\lambda_n^{-\theta}\, m_{\lambda_n} = \lambda_n^{-\theta}\, E_{\lambda_n}(u_n) &= \int_{\mathbb{R}^N} \Big(a_n(x)|\nabla v_n(x)|^2 + |v_n(x)|^2\Big)dx \\ &\geq \int_{\mathbb{R}^N} \Big(a_0|\nabla v_n^1(x)|^2 + |v_n^1(x)|^2\Big)dx + \int_{\mathbb{R}^N} \Big(a_0|\nabla v_n^2(x)|^2 + |v_n^2(x)|^2\Big)dx + O(\varepsilon) \\ &\geq M_0 \left[\Big(\int_{\mathbb{R}^N} |v_n^1(x)|^p\, dx\Big)^{2/p} + \Big(\int_{\mathbb{R}^N} |v_n^2(x)|^p\, dx\Big)^{2/p}\right] + O(\varepsilon) \\ &\geq M_0 \left[\big(\gamma-\varepsilon\big)^{2/p} + \big(1-\gamma-\varepsilon\big)^{2/p}\right] + O(\varepsilon)\,.\end{aligned}$$

By letting $\varepsilon \to 0$ and $n \to \infty$, (4.1) gives $1 \geq \gamma^{2/p} + \big(1-\gamma\big)^{2/p}$, a contradiction. As a consequence, the sequence $\{\mu_n\}$ is tight: there exists $\{z_n\} \subset \mathbb{R}^N$ such that, for any $\varepsilon \in]0,1[$, there exists $R_\varepsilon > 0$ such that $\int_{B_{R_\varepsilon}(z_n)} |v_n(x)|^p\, dx \geq 1-\varepsilon$. If we define $w_n(x) = v_n(x+z_n)$, the previous inequality rewrites as

$$\int_{B_{R_\varepsilon}(0)} |w_n(x)|^p\, dx \geq 1-\varepsilon\,. \tag{4.3}$$

Now, let $\eta : \mathbb{R}^N \to [0,1]$ be a C^∞ function such that $\eta_\varepsilon(x) = 1$ for $|x| \leq R_\varepsilon$, $|\nabla\eta_\varepsilon(x)| \leq \varepsilon$ for any x, and $\eta_\varepsilon(x) = 0$ for $|x|$ larger than some $R'_\varepsilon > 0$.
Let $\widetilde{w}_n = \eta_\varepsilon w_n$; it is easy to see that $|\nabla\widetilde{w}_n(x)|^2 \leq (1+\varepsilon)\,\big(|\nabla w_n(x)|^2 + \varepsilon|w_n(x)|^2\big)$, thus

$$\begin{aligned}&\int_{\mathbb{R}^N} \big(a_n(x+z_n)|\nabla w_n(x)|^2 + |w_n(x)|^2\big)dx \geq \\ &\qquad \geq \int_{\mathbb{R}^N} \Big(\frac{a_n(x+z_n)}{1+\varepsilon}\,|\nabla\widetilde{w}_n(x)|^2 + |\widetilde{w}_n(x)|^2 - \varepsilon\, a_n(x+z_n)|w_n(x)|^2\Big)dx \\ &\qquad \geq \frac{1}{1+\varepsilon}\int_{\mathbb{R}^N} \big(a_n(x+z_n)|\nabla\widetilde{w}_n(x)|^2 + |\widetilde{w}_n(x)|^2\big)dx + O(\varepsilon)\,.\end{aligned} \tag{4.4}$$

If the sequence $\{\lambda_n^{-1/2} z_n\}$ is not bounded, we can assume $\lambda_n^{-1/2}|z_n| \to +\infty$ (up to a subsequence). Thus $a_n(x+z_n) \geq a_\infty - \varepsilon$ for any $|x| < R'_\varepsilon$ and n sufficiently large, and

$$\int_{B_{R'_\varepsilon}(0)} a_n(x+z_n)|\nabla\widetilde{w}_n(x)|^2\, dx \geq \int_{\mathbb{R}^N} a_\infty|\nabla\widetilde{w}_n(x)|^2\, dx + O(\varepsilon)\,. \tag{4.5}$$

From (4.4) and (4.5) we get

$$\begin{aligned}&\int_{\mathbb{R}^N} \big(a_n(x+z_n)|\nabla w_n(x)|^2 + |w_n(x)|^2\big)dx \geq \\ &\qquad\qquad \geq \frac{1}{1+\varepsilon}\int_{\mathbb{R}^N} \big(a_\infty|\nabla\widetilde{w}_n(x)|^2 + |\widetilde{w}_n(x)|^2\big)dx + O(\varepsilon) \\ &\qquad\qquad \geq \frac{M_\infty}{1+\varepsilon}\left(\int_{\mathbb{R}^N} |w_n(x)|^p\, dx\right)^{2/p} + O(\varepsilon)\,.\end{aligned} \tag{4.6}$$

From (4.2), the definition of w_n, (4.6) and (4.3) we get

$$\lambda_n^{-\theta} m_{\lambda_n} = \int_{\mathbb{R}^N} \Big(a_n(x+z_n)|\nabla w_n(x)|^2 + |w_n(x)|^2\Big)dx \geq \frac{M_\infty}{1+\varepsilon}(1-\varepsilon)^{2/p} + O(\varepsilon)\,. \tag{4.7}$$

If we let $n \to \infty$ and $\varepsilon \to 0$, (4.1) and (4.7) give $M_0 \geq M_\infty$, a contradiction (cf. Section 2).

As a result, the sequence $x_n = \lambda_n^{-1/2} z_n$ is bounded; up to a subsequence, it converges to some $x_0 \in \mathbb{R}^N$. Again up to a subsequence, w_n converges to some w weakly in $H^1(\mathbb{R}^N)$, strongly in $L^p_{loc}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Furthermore, (4.3) implies that w_n converges to w strongly in $L^p(\mathbb{R}^N)$, so that $w \in \Sigma$.

Now, fix $0 < \varepsilon < 1$, let η_ε and $\widetilde{w}_n$ be as before and $\widetilde{w}_\varepsilon = \eta_\varepsilon w$. Plainly, $\widetilde{w}_n$ tends to $\widetilde{w}_\varepsilon$ weakly in $H^1(B_{R'_\varepsilon}(0))$ and $a_n(x + z_n) \to a(x_0)$ uniformly in $B_{R'_\varepsilon}(0)$. As a consequence:

$$\begin{aligned} M_0 &\geq \liminf_{n\to\infty} \frac{1}{1+\varepsilon} \int_{B_{R'_\varepsilon}(0)} \big(a_n(x+z_n)|\nabla \widetilde{w}_n(x)|^2 + |\widetilde{w}_n(x)|^2\big) dx + O(\varepsilon) \\ &\geq \frac{1}{1+\varepsilon} \int_{B_{R'_\varepsilon}(0)} \big(a(x_0)|\nabla \widetilde{w}_\varepsilon(x)|^2 + |\widetilde{w}_\varepsilon(x)|^2\big) dx + O(\varepsilon) \\ &\geq \frac{M_{1,a(x_0)}}{1+\varepsilon} \left(\int_{B_{R'_\varepsilon}(0)} |\widetilde{w}_\varepsilon(x)|^p \, dx \right)^{2/p} + O(\varepsilon)\,. \end{aligned}$$

As $\varepsilon \to 0$, we get $M_0 \geq \int_{\mathbb{R}^N} \big(a(x_0)|\nabla w(x)|^2 + |w(x)|^2\big)\, dx \geq M_{1,a(x_0)} \geq M_0$, whence $M_{1,a(x_0)} = M_0$, which implies $x_0 \in \Omega$. Since $w \in \Sigma$, from the equality $\int_{\mathbb{R}^N} \big(a_0|\nabla w(x)|^2 + |w(x)|^2\big)\, dx = M_0$ we deduce $w = \omega_0$.

Finally, w_n converges to ω_0 strongly in $H^1(\mathbb{R}^N)$. Indeed,

$$\begin{aligned} M_0 &\leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \Big(a_0|\nabla w_n(x)|^2 + |w_n(x)|^2\Big) dx \\ &\leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big(a_0|\nabla w_n(x)|^2 + |w_n(x)|^2\Big) dx \\ &\leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big(a_n(x+z_n)|\nabla w_n(x)|^2 + |w_n(x)|^2\Big) dx \\ &= \limsup_{n\to\infty} \lambda_n^\theta\, E_{\lambda_n}(u_n) = M_0 \end{aligned}$$

(the last equality follows from (4.2) and Lemma 4.1). □

Remark 4.3. It is not difficult to check that the proof of Theorem 4.2 still works if (4.2) is replaced by the following assumption:

$$E_{\lambda_n}(u_n) \leq M_{\lambda_n, a_0} + \lambda_n^\theta\, k_n\,,$$

where k_n is any sequence which goes to 0^+ as $n \to \infty$.

5. Proof of Theorem 1.2. We begin this section by recalling a definition: if Y is a closed subset of a topological space X, the *Ljusternik–Schnirelman category* $\mathrm{cat}_X(Y)$ is the least number of closed and contractible sets in X which cover Y. We write $\mathrm{cat}(X)$ for $\mathrm{cat}_X(X)$.

Let us outline the proof of Theorem 1.2.

The functional E_λ is bounded below on Σ and it satisfies the Palais–Smale condition in the sublevel $E_\lambda^c = \big\{u \in \Sigma \,:\, E_\lambda(u) \leq c\big\}$, for any $c < m_{\lambda, a_\infty}$. Therefore, a classical result in Ljusternik–Schnirelman Theory implies that the number of critical points of E_λ in E_λ^c is at least $\mathrm{cat}\,(E_\lambda^c)$. Thus, in order to relate the number

of solutions of (P_λ) with the topology of Ω, it is enough to find a suitable sublevel whose category is, in turn, related with the category of Ω. To this aim, the following result is very useful (for the proof, see for instance [2]):

Theorem 5.1. *Let $c > 0$ and let E^* be a closed subset of E_λ^c. Let $\Phi_\lambda : \Omega \to E^*$ and $\beta : E_\lambda^c \to \Omega_\delta$ be continuous maps. Assume that $\beta \circ \Phi_\lambda$ is homotopically equivalent to the embedding $j : \Omega \longrightarrow \Omega_\delta$. Then:* $\mathrm{cat}_{E_\lambda^c}(E^*) \geq \mathrm{cat}_{\Omega_\delta}(\Omega)$.

In some cases, $\mathrm{cat}_{\Omega_\delta}(\Omega) = \mathrm{cat}_\Omega(\Omega)$, at least for δ small. This happens, for instance, if Ω is the closure of a bounded open set with smooth boundary, or a smooth and compact submanifold of $\mathbb{R}^N$. If Ω is a finite set, then $\mathrm{cat}_{\Omega_\delta}(\Omega) = \mathrm{cat}_\Omega(\Omega) =$ cardinality of Ω, for δ small.

Henceforth, we assume $0 < a_0 < a_\infty$. With a view to applying Theorem 5.1, we first define a map $\Phi_\lambda : \Omega \to \Sigma$. For any $x_0 \in \Omega$, and $x \in \mathbb{R}^N$, we let

$$\Phi_\lambda(x_0)(x) = \lambda^{N/2p}\omega_0\big(\lambda^{1/2}(x - x_0)\big)$$

(ω_0 was defined in (2.4)). Simple computations give

$$E_\lambda(\Phi_\lambda(x_0)) = \lambda^\theta\big(M_0 + o(1)\big)\,, \tag{5.1}$$

uniformly in $x_0 \in \Omega$.

Next we define the barycenter map $\beta : H^1(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$\beta(u) = \frac{\int_{\mathbb{R}^N} \chi(x)|u(x)|^p\,dx}{\int_{\mathbb{R}^N} |u(x)|^p\,dx}\,;$$

here $\chi(x) = x$ if $|x| \leq \rho$, $\chi(x) = \rho\, x/|x|$ if $|x| \geq \rho$ and $\rho > 0$ is such that $\Omega \subset B_\rho(0)$.

Proposition 5.2. *For any $\delta > 0$, there exist $\lambda_\delta, k_\delta > 0$ such that, for any $\lambda > \lambda_\delta$, and for any $u \in \Sigma$ with $E_\lambda(u) \leq M_{\lambda,a_0} + \lambda^\theta\, k_\delta$, we have $\beta(u) \in \Omega_\delta$.*

Proof. By contradiction, assume that for some $\delta > 0$ there exist $\lambda_n \to +\infty$, $k_n \to 0^+$, and $u_n \in \Sigma$ such that $E_{\lambda_n}(u_n) \leq M_{\lambda_n,a_0} + \lambda_n^\theta\, k_n$ and $\beta(u_n) \notin \Omega_\delta$ for any n. By Theorem 5.2 and Remark 5.3, up to a subsequence we have $u_n(x) = \lambda_n^{N/2p} w_n\big(\lambda_n^{1/2}(x - x_n)\big)$ for any n, with $x_n \to x_0 \in \Omega$ and $w_n \to \omega_0$ strongly in $H^1(\mathbb{R}^N)$ (and in $L^p(\mathbb{R}^N)$) as $n \to \infty$. Therefore, again up to a subsequence we have

$$\begin{aligned}\beta(u_n) &= \int_{\mathbb{R}^N} \chi(x)|u_n(x)|^p\,dx \\ &= \int_{\mathbb{R}^N} \chi\big(x_n + \lambda_n^{-1/2}x\big)|w_n(x)|^p\,dx \longrightarrow \int_{\mathbb{R}^N} \chi(x_0)|\omega_0(x)|^p\,dx = x_0\end{aligned}$$

as $n \to \infty$, a contradiction since $x_0 \in \Omega$ and $\beta(u_n) \notin \Omega_\delta$. □

As we noticed in the proof of Theorem 1.1, minimizers of E_λ on Σ do not change sign in $\mathbb{R}^N$. Due to the asymptotic estimate (4.1), the same property holds for "low energy" critical points of E_λ on Σ, at least for λ sufficiently large (cf. [1]):

Proposition 5.3. *There exist $k_0, \lambda_0 > 0$ such that, for any $\lambda > \lambda_0$, every critical point u of E_λ on Σ does not change sign, provided*

$$E_\lambda(u) \leq \lambda^\theta\,\big(M_0 + k_0\big)\,. \tag{5.2}$$

Proof. Let $0 < k_0 < \big(2^{(p-2)/p}-1\big)M_0$. Let u be a critical point of E_λ on Σ satisfying (5.2) and assume both $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ not identically zero. Elementary computations give $\|u^\pm\|_{L^p} \geq (m_\lambda/E_\lambda(u))^{\frac{1}{p-2}}$ (see e.g. [6] for some details), thus $1 = \|u^+\|_{L^p}^p + \|u^-\|_{L^p}^p \geq 2\,(m_\lambda/E_\lambda(u))^{\frac{p}{p-2}}$. From (5.2) we get $\lambda^\theta\,\big(M_0 + k_0\big) \geq 2^{\frac{p-2}{p}} m_\lambda$; as $\lambda \to \infty$, (4.1) gives $M_0 + k_0 \geq 2^{(p-2)/p} M_0$, a contradiction. □

Proof of Theorem 1.2. Let $\delta > 0$ be fixed and let $\lambda_\delta, k_\delta > 0$ be as in Proposition 5.2. Let $c_\lambda = \lambda^\theta\,\big(M_0 + k_\delta\big)$, and define $\Sigma_\lambda = \{u \in \Sigma : E_\lambda(u) \leq c_\lambda\}$, $\Sigma_\lambda^+ = \{u \in \Sigma_\lambda : u \geq 0\}$, and $\Sigma_\lambda^- = \{u \in \Sigma_\lambda : u \leq 0\}$.
Plainly, we can assume $M_0 + k_\delta < M_\infty$, so that Palais–Smale condition holds in a sublevel containing Σ_λ (by Proposition 3.1).
If λ is large enough, (5.1) implies $\Phi_\lambda(x_0) \in \Sigma_\lambda^+$ for any $x_0 \in \Omega$, whereas Proposition 5.2 implies $\beta(u) \in \Omega_\delta$ for any $u \in \Sigma_\lambda$. It is easy to see that $\beta \circ \Phi_\lambda$ is homotopically equivalent to the embedding $j : \Omega \to \Omega_\delta$. Thus Theorem 5.1 gives $\mathrm{cat}_{\Sigma_\lambda}(\Sigma_\lambda^+) \geq \mathrm{cat}_{\Omega_\delta}(\Omega)$. By using the map $-\Phi_\lambda$, we also get $\mathrm{cat}_{\Sigma_\lambda}(\Sigma_\lambda^-) \geq \mathrm{cat}_{\Omega_\delta}(\Omega)$, whence $\mathrm{cat}(\Sigma_\lambda) \geq 2\,\mathrm{cat}_{\Omega_\delta}(\Omega)$, for λ large. Ljusternik–Schnirelman Theory applies and we deduce that E_λ has at least $2\,\mathrm{cat}_{\Omega_\delta}(\Omega)$ critical points on Σ, which satisfy $E_\lambda(u) \leq c_\lambda = \lambda^\theta\,\big(M_0 + k_\delta\big)$. Since we can assume $k_\delta < k_0$, Proposition 5.3 implies that the critical points we have found do not change sign. Then, there are at least $\mathrm{cat}_{\Omega_\delta}(\Omega)$ positive critical points. □

6. Further results. We discuss some extensions of our existence and multiplicity results to $(\widetilde{\mathrm{P}}_\lambda)$, where b is a continuous, bounded and positive real function in $\mathbb{R}^N$. We denote

$$b_0 = \sup_{\mathbb{R}^N}, \qquad b_\infty = \limsup_{|x|\to\infty} b(x)\,.$$

We begin with the counterpart of Theorem 1.1:

Theorem 6.1. *Under the assumptions in Theorem 1.1, suppose in addition*

$$b(x) \geq b_\infty \quad \textit{for any } x \in \mathbb{R}^N. \tag{6.1}$$

Then: there exists $\lambda^* > 0$ such that $(\widetilde{\mathrm{P}}_\lambda)$ has one positive solution, for any $\lambda > \lambda^*$.

Proof. For the sake of brevity, we give only some hints of the proof. Solutions to $(\widetilde{\mathrm{P}}_\lambda)$ arise as critical points of E_λ on $\widetilde{\Sigma} = \Big\{u \in \mathcal{H}(\mathbb{R}^N) \,:\, \int_{\mathbb{R}^N} b(x)\,|u(x)|^p\,dx = 1\Big\}$. With minor changes in the proof of Proposition 3.1, we can prove that E_λ satisfies the Palais–Smale condition on $\widetilde{\Sigma}$, at any level $c < \widetilde{M}_{\lambda,a_\infty,b_\infty}$, where

$$\widetilde{M}_{\lambda,a_\infty,b_\infty} = \inf\left\{\int_{\mathbb{R}^N} \big(a_\infty\,|\nabla u(x)|^2 + \lambda\,|u(x)|^2\big)dx \,:\, \int_{\mathbb{R}^N} b_\infty\,|u(x)|^p\,dx = 1\right\}.$$

Therefore, to prove that $\widetilde{m}_\lambda = \inf\big\{E_\lambda(u) \,:\, u \in \widetilde{\Sigma}\big\}$ is attained, it suffices to find some $u_\lambda \in \widetilde{\Sigma}$ such that $E_\lambda(u_\lambda) < \widetilde{M}_{\lambda,a_\infty,b_\infty}$. Under the assumptions in Theorem 6.1, it is not difficult to check that the previous inequality is fulfilled in $u_\lambda(x) = \lambda^{N/2p}\,\widetilde{\omega}_\infty\big(\lambda^{1/2}(x - \bar{x})\big)$, where $\widetilde{\omega}_\infty$ is the unique positive radial solution of the minimization problem for $\widetilde{M}_{1,a_\infty,b_\infty}$, and $\bar{x}$ is chosen as in the proof of Proposition 3.2. □

Remark 6.2. The conclusion in Theorem 6.1 still holds true if the assumptions $a_0 < a_\infty$ and (6.1) are replaced by

(6.2) *there exists* $\hat{x} \in \mathbb{R}^N$ *such that* $a(\hat{x}) < a_\infty$ *and* $b(\hat{x}) > b_\infty$.

Indeed, in this case a good test function is $\hat{u}_\lambda(x) = \lambda^{N/2p}\,\widetilde{\omega}_\infty\big(\lambda^{1/2}(x-\hat{x})\big)$. We omit the easy proof.

If the set $\widetilde{\Omega} = \big\{x \in \mathbb{R}^N \,:\, a(x) = a_0\,,\ b(x) = b_0\big\}$ is topologically nontrivial, we can prove a multiplicity result:

Theorem 6.3. *Assume* $0 < a_0 < a_\infty$. *Then: for any* $\delta > 0$, *there exists* $\lambda_\delta > 0$ *such that* $(\widetilde{\mathrm{P}}_\lambda)$ *has at least* $\mathrm{cat}_{\widetilde{\Omega}_\delta}(\widetilde{\Omega})$ *positive solutions, for any* $\lambda > \lambda_\delta$.

The proof of Theorem 6.3 is based on the counterpart of Theorem 4.2. As λ goes to infinity, minimizers of E_λ on $\widetilde{\Sigma}$ concentrate around the set $\widetilde{\Omega}$. In this case, the underlying profile is given by $\widetilde{\omega}_0$, the unique positive radial solution of the minimization problem

$$\widetilde{M}_{1,a_0,b_0} = \inf\left\{\int_{\mathbb{R}^N}\big(a_0\,|\nabla u(x)|^2 + \lambda\,|u(x)|^2\big)dx \,:\, \int_{\mathbb{R}^N} b_0\,|u(x)|^p\,dx = 1\right\}.$$

REFERENCES

[1] V. Benci and G. Cerami, *The effect of the domain topology on the number of solutions of nonlinear elliptic problems*, Arch. Rat. Mech. Anal. 114 (1991), 79–93.

[2] V. Benci, G. Cerami and D. Passaseo, *On the number of the positive solutions of some nonlinear elliptic problems*, Nonlinear Analysis, A tribute in honour of G. Prodi, Quaderno Scuola Norm. Sup., Pisa 1991, 93–107.

[3] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations, I – Existence of a ground state*, Arch. Rat. Mech. Anal. 82 (1983), 313–375.

[4] P. Caldiroli and R. Musina, *On a variational degenerate elliptic problem*, NoDEA 7 (2000), 187–199.

[5] J. Chabrowski, *Degenerate elliptic equation involving a subcritical Sobolev esponent*, Portugaliae Math. 53 (1996), 167–177.

[6] S. Cingolani and M. Lazzo, *Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations*, Top. Meth. Nonlinear Anal. 10 (1997), 1–13.

[7] S. Coleman, V. Glaser and A. Martin, *Action minima among solutions to a class of Euclidean scalar field equations*, Comm. Math. Phys. 58 (1978), 211–221.

[8] B. Gidas, W.M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^N$, Math. Analysis Appl., Part A, 7 (1981), 369–402.

[9] M.K. Kwong, *Uniqueness of positive solutions of* $\triangle u - \lambda u + u^p = 0$ *in* $\mathbb{R}^N$, Arch. Rat. Mech. Anal. 105 (1989), 243–266.

[10] P.L. Lions, *The concentration-compactness principle in the calculus of variations: the locally compact case. Part I*, Ann. Inst. H. Poincaré Analyse Non Linéaire 1 (1984), 109–145.

[11] M.K.V. Murthy and G. Stampacchia, *Boundary value problems for some degenerate elliptic operators*, Ann. Mat. Pura Appl. 80 (1968), 1–122.

[12] N.S. Trudinger, *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat., Ser. III., XXVII (1973), 265–308.

[13] X. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations*, Commun. Math. Phys. 153 (1993), 229–244.

Received September 2002; in revised March 2003.

E-mail address: lazzo@dm.uniba.it