

EXPONENTIAL ATTRACTORS FOR A CHEMOTAXIS GROWTH SYSTEM ON DOMAINS OF ARBITRARY DIMENSION

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Abstract. It is proven that there exist exponential attractors for dynamical systems defined by a general chemotaxis system defined on a domain of arbitrary dimension n .

1. Introduction. There has been great interest in studying strongly coupled parabolic systems. Fundamental issues, such as the question of local existence of solutions, were settled in [1] but global existence results and long time dynamics of solutions seem to be answered in only very few cases. The discussion in [1] also revealed that the most suitable functional spaces one should use to study the dynamics of solutions to strongly coupled parabolic systems, which are defined on a domain of dimension n , are the Sobolev spaces $W^{1,p}(\Omega)$ with $p > n$.

In this paper we will consider a class of triangular cross diffusion systems given on an open bounded domain Ω in \mathbb{R}^n with $n \geq 2$. Let us consider the following parabolic system

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{A}_u(u, v) + g(u, v), & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = \mathcal{A}_v(v) + f(u, v), & x \in \Omega, t > 0. \end{cases} \quad (1.1)$$

where $\mathcal{A}_u, \mathcal{A}_v$ are quasilinear/linear differential operators described in Section 3. This system comes with the following mixed boundary conditions for $x \in \partial\Omega$ and $t > 0$

$$\chi(x) \frac{\partial v}{\partial n}(x, t) + (1 - \chi(x))v(x, t) = 0, \quad \text{and} \quad \bar{\chi}(x) \frac{\partial u}{\partial n}(x, t) + (1 - \bar{\chi}(x))u(x, t) = 0, \quad (1.2)$$

where $\chi, \bar{\chi}$ are given functions on $\partial\Omega$ with values in $\{0, 1\}$. The initial conditions are described by

$$v(x, 0) = v^0(x), \quad u(x, 0) = u^0(x), \quad x \in \Omega \quad (1.3)$$

for nonnegative functions v^0, u^0 in $X = W^{1,p}(\Omega)$ for some $p > n$ (see [1]). In (1.1), \mathcal{A}_u includes the *self-diffusion* and *cross-diffusion* pressures, and \mathcal{A}_v presents only the *self-diffusion* pressure (see hypothesis (H.1) below).

Concerning long time behavior of the solutions, the notion of exponential attractors in Hilbert spaces was introduced by Eden et al. [2], and had been shown

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to be very important in the study of the dynamics of solutions to nonlinear diffusion equations [11]. Recently, an extension of this theory for dissipative dynamical systems in Banach spaces was done in [6] (see also [3]). Basically, we consider a dynamical system $\{S(t)\}_{t \geq 0}$ defined on a Banach space \mathcal{X} and suppose that the dynamical system $\{S(t)\}_{t \geq 0}$ possesses a compact absorbing set $\mathcal{B} \subset \mathcal{X}$. That is, for any bounded set $K \subset \mathcal{X}$, there is a finite T_K such that $S(t)x \in \mathcal{B}$ for any $x \in K$ and $t \geq T_K$. We established in [6] the following result.

Theorem 1. *Assume the following conditions for the map $G : [0, \infty) \times \mathcal{X} \rightarrow \mathcal{X}$, $G(t, x) = S(t)x$.*

(C): *For large $t > 0$, the map $x \rightarrow G(t, x)$ is differentiable on \mathcal{B} . Moreover, we have the decomposition $D_x G(t, x) = K_x + C_x$, where K_x is compact and C_x is a contraction.*

(L): *$G(t, x)$ is Lipschitz on $[0, T] \times \mathcal{B}$ for any finite $T > 0$.*

Then there exists an exponential attractor \mathcal{M} , which is a positively invariant compact subset of \mathcal{X} with finite fractal dimension to which every solution is attracted at an exponential rate.

Unlike the theory by [2] for Hilbert spaces, where its applications usually required some effort to verify the so called "squeezing property", the assumptions in the above theorem are often easy to be established. Indeed, (C) is almost immediate from the regularizing property of many parabolic systems. Likewise, (L) is often trivial.

For cross diffusion systems defined on 2 dimensional domains (and classical reaction diffusion systems defined on any dimensional domains), one can work in the Hilbert space $W^{1,2}$ (see [11, 12, 13]) so that the theory in [2] is applicable. However, if one wants to study cross diffusion systems on domains of dimension greater than 2, then dynamical systems in Banach spaces $W^{1,p}$ should be in question (see [1]). Theorem 1 could be used here.

We are interested not only in the question of global existence of solutions to (1.1) but also in long time dynamics of its solutions. The assumptions on the parameters defining (1.1) will be specified later in Section 3 and they are general enough to cover many interesting models investigated in literature. Furthermore, our conclusion is far more stronger, in some cases, than what have been known about these systems. To demonstrate this, in the next section, we will first discuss some well studied systems and state our findings, which are the consequences of our general Theorem 7. Roughly speaking, we establish the following.

A solution (u, v) of (1.1) exists globally in time if the norms $\|v(\bullet, t)\|_\infty$ and $\|u(\bullet, t)\|_1$ do not blow up in finite time. Moreover, if these norms of the solutions are ultimately uniformly bounded then there exists an exponential attractor, with finite fractal dimension, attracting all solutions at an exponential rate.

Our assumption on (1.1) and the main results, together with their proofs, will be given in Section 3 and Section 4. Finally, in Section 5, we provide the proof of the theorems for the examples in Section 2.

2. Applications. In this section, we will apply our theorems in Section 3 to several cross diffusion parabolic systems modeling biological and ecological phenomena. Most of them have been studied by many authors assuming that the domain Ω is of two dimensional. In very few cases, they can only establish global existence results. We impose here no restriction on the dimension of Ω , and further assert the existence of exponential attractors for these systems.

In order to understand theoretically certain chemotactic pattern formation of the *E. coli* bacteria, Mimura and Tsujikawa [8] proposed the following modified version of the famous Keller-Segel model

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot [(d_1 + \alpha_{11}u)\nabla u] + \alpha_{12}\nabla \cdot (u\nabla\chi(v)) + f(u), \\ \frac{\partial v}{\partial t} = d_2\Delta v + bu - cv, \end{cases} \quad (2.1)$$

with homogeneous Neumann boundary conditions. The constants b, c are taken to be positive. Furthermore, $\chi(v)$ is a smooth function in v with bounded derivatives, and $f(u)$ is a function behaves like $(-\mu u + \nu)u$ when u is large. The constant α_{11} was assumed to be zero in [8]. Here, we introduce the "crowding effect" by letting $\alpha_{11} > 0$.

In this form, (2.1) is a special case of the general model (1.1). Our Theorem 8 applies here to conclude that

Theorem 2. *Consider the modified Keller-Segel (2.1) on a bounded domain Ω of any dimension n . This system defines a smooth dynamical system on $W^{1,p}(\Omega)$, $p > n$ and admits an exponential attractor for the system.*

If $n = 2$ and $\alpha_{11} = 0$, the above theorem was established by Osaki et al. in [13] where they considered solutions in the phase space $H^1(\Omega)$ and employed the theory of exponential attractors in Hilbert spaces. Their method seems to be not applicable to our case (particularly, when $n \geq 3$).

Our next example is a cross diffusion model in population dynamics. Recently, Lou et al. ([7]) and Yagi [12] established global existence results for the Shigesada, Kawasaki, Teramoto system (see [10])

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1u - c_1v), \\ \frac{\partial v}{\partial t} = \Delta[(d_2 + \alpha_{22}v)v] + v(a_2 - b_2u - c_2v), \end{cases} \quad (2.2)$$

with Neumann boundary conditions. They assumed that Ω is a bounded domain in \mathbb{R}^2 . In [4], we discussed not only global existence but also long time dynamics of solutions to a class of cross diffusion systems which includes (2.2). Again, we had to assume that the dimension of the domain Ω is two. We have here a better result.

Theorem 3. *Consider (2.2) on a bounded domain Ω of any dimension n . If $\alpha_{22} = 0$ the dynamical system associated with (2.2) possesses an exponential attractor in $W^{1,p}(\Omega)$, $p > n$.*

If $\alpha_{22} = 0$ and $n = 2$, the method of this paper can combine with that of [4] to give the same conclusion.

Finally, we look at a system modeling bio-reactors with chemotactic and crowding effects.

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot [(d_1 + \alpha_{11}u)u\nabla u] + \nabla \cdot (u\Phi(S)\nabla S) + u(f(S) - k), \\ \frac{\partial S}{\partial t} = d_2\Delta S - \gamma u f(S), \\ \frac{\partial u}{\partial n} + \alpha u = 0, \quad \frac{\partial S}{\partial n} + \beta S = S_0, \quad x \in \partial\Omega, t > 0. \end{cases} \quad (2.3)$$

Here, $k, \alpha, \beta, \gamma, \alpha_{11}$ are positive constants and $\Phi(S), f(S)$ are continuous functions, and $f(S) > 0$. This system was studied in [5, 14] where Ω is assumed to be an interval in \mathbb{R} . Although the boundary condition in (2.3) is of Robin type, but our proof can be easily modified to cover this case. Our Theorem 8 applies again and asserts that

Theorem 4. *Consider (2.3) on a bounded domain Ω of any dimension n . The dynamical system associated with (2.3) possesses an exponential attractor in $W^{1,p}(\Omega)$, $p > n$.*

3. Main results. In this section, we will specify our assumptions on the general system (1.1) and state our main results. Let (v^0, u^0) be given functions in $X = W^{1,p_0}(\Omega)$, $p_0 > n$. Let (u, v) be the solution of system (1.1), and $I := I(u^0, v^0)$ be its maximal interval of existence (see [1]).

In order to simplify the statements of our theorems and proof, we will make use of the following terminology.

Definition 5. Consider the initial-boundary problem (1.1),(1.2) and (1.3) . Assume that there exists a solution (u, v) defined on a subinterval I of \mathbb{R}_+ . Let \mathcal{O} be the set of functions ω on I such that there exists a positive constant C_0 , which may generally depend on the parameters of the system and the W^{1,p_0} norm of the initial value (u^0, v^0) , such that

$$\omega(t) \leq C_0, \quad \forall t \in I. \tag{3.1}$$

Furthermore, if $I = (0, \infty)$, we say that ω is in \mathcal{P} if $\omega \in \mathcal{O}$ and there exists a positive constant C_∞ that depends only on the parameters of the system but does not depend on the initial value of (u^0, v^0) such that

$$\limsup_{t \rightarrow \infty} \omega(t) \leq C_\infty. \tag{3.2}$$

If $\omega \in \mathcal{P}$ and $I = (0, \infty)$, we will say that ω is *ultimately uniformly bounded*.

Examples of functions in \mathcal{P} include $\omega(t) = e^{-t} \|u^0\|_\infty$. On the other hand, if $\|u(\bullet, t)\|_\infty, \|v(\bullet, t)\|_\infty$, as functions in t , belong to \mathcal{O} then (3.1) says that the supremum norms of the solutions to (1.1) do not blow up in any finite time interval and are bounded by some constant that may depend on the initial conditions. This implies that the solution exists globally (see [1]). Moreover, if these norms are in \mathcal{P} , then (3.2) says that they can be majorized eventually by a universal constant independent of the initial data. This property implies that there is an absorbing ball for the solution and therefore shows the existence of the global attractor if certain compactness is proven (see [11]).

We will consider the following conditions on the parameters of the system.

(H.1): There are differentiable functions $P(u, v), R(u, v)$ such that \mathcal{A}_u is given by

$$\mathcal{A}_u(u, v) = \nabla \cdot (P(u, v)\nabla u + R(u, v)\nabla v).$$

There exist a continuous function Φ and positive constants C, d such that

$$P(u, v) \geq d(1 + u) > 0, \quad \forall u \geq 0, \tag{3.3}$$

$$|R(u, v)| \leq \Phi(v)u. \tag{3.4}$$

Moreover, the partial derivatives of P, R with respect to u, v can be majorized by some powers of u, v .

The operator \mathcal{A}_v is regular linear elliptic in divergence form. That is, for some $\alpha > 0$ and functions $Q(x, t) \in C^{1+\alpha}(\Omega \times (0, \infty))$ and $c(x, t) \in C^\alpha(\Omega \times (0, \infty))$

$$\mathcal{A}_v(v) = \nabla \cdot (Q(x, t)\nabla v) + c(x, t)v, \quad Q(x, t) \geq d > 0, \quad c(x, t) \leq 0. \quad (3.5)$$

We will be interested only in nonnegative solutions, which are relevant in many applications. Therefore, we will assume that the solution u, v stay nonnegative if the initial data u^0, v^0 are nonnegative functions. We will also impose the following assumption on the reaction terms.

(H.2): There exists a nonnegative continuous function $C(v)$ such that

$$|f(u, v)| \leq C(v)(1+u), \quad g(u, v)u^p \leq C(v)(1+u^{p+1}), \quad \forall u, v \geq 0 \text{ and } p > 0. \quad (3.6)$$

Remark 3.1. *The fact that u, v stay positive can be verified by further assumptions on P, Q, R and g, f using a maximum principle argument. For example, u, v will stay positive if $P(0, v), R_v(0, v), Q(u, 0)$ and $g(0, v), f(u, 0)$ are nonnegative. It is easy to see that our examples verify these conditions.*

Our first result is the following global existence result.

Theorem 6. *Assume (H.1) and (H.2). Let (u, v) be a nonnegative solution to (1.1), with its maximal existence interval I . If $\|v(\bullet, t)\|_\infty$ and $\|u(\bullet, t)\|_1$ are in \mathcal{O} then there exists $\nu > 1$ such that*

$$\|v(\bullet, t)\|_{C^\nu(\Omega)}, \quad \|u(\bullet, t)\|_{C^\nu(\Omega)} \in \mathcal{O}. \quad (3.7)$$

This also implies that the solution exists globally in time.

If we have better bounds on the norms of the solutions then a stronger conclusion follows.

Theorem 7. *Assume (H.1) and (H.2). Let (u, v) be a nonnegative solution to (1.1), with its maximal existence interval I . If $\|v(\bullet, t)\|_\infty$ and $\|u(\bullet, t)\|_1$ are in \mathcal{P} then there exists $\nu > 1$ such that*

$$\|v(\bullet, t)\|_{C^\nu(\Omega)}, \quad \|u(\bullet, t)\|_{C^\nu(\Omega)} \in \mathcal{P}. \quad (3.8)$$

Therefore, if $\|v(\bullet, t)\|_\infty$ and $\|u(\bullet, t)\|_1$ are in \mathcal{P} for every solution (u, v) of (1.1), then there exists an absorbing ball where all solutions will enter eventually. Thus, if the system (1.1) is autonomous then there is a compact global attractor with finite Hausdorff dimension which attracts all solutions.

Remark 3.2. *We can still establish (3.8) by allowing \mathcal{A}_v to be a quasilinear operator, and relaxing the crowding effect in (3.3). However, we must start with somewhat stronger a priori estimate on the L^p norm of u . In particular, the conclusions of Theorem 7 (respectively, Theorem 6) continue to hold if $\|v(\bullet, t)\|_\infty$ and $\|u(\bullet, t)\|_p$ are in \mathcal{P} (respectively \mathcal{O}) for some $p > n/2$. The proof of this fact is a little more involved and will appear elsewhere.*

Finally, we have the following result on the existence of exponential attractors for (1.1).

Theorem 8. Consider the autonomous system (1.1). Assume the conditions of Theorem 7. The system (1.1) defines a dynamical system $\{S(t)\}_{t \geq 0}$ on the Banach space $\mathcal{X} = W^{1,p_0}(\Omega) \times W^{1,p_0}(\Omega)$, $p_0 > n$. That is,

$$S(t)(u_0, v_0)(x) = (u(x, t), v(x, t)), \quad (u_0, v_0) \in \mathcal{X}, \quad t \geq 0,$$

where (u, v) is the solution to (1.1) with the initial data (u_0, v_0) . Moreover, this dynamical system possesses an exponential attractor in $C^{2+\alpha, 1+\alpha/2}(\Omega) \times C^{2+\alpha, 1+\alpha/2}(\Omega) \cap \mathcal{X}$ for some $\alpha > 0$.

4. Proof of the main results. In the proof we will use $\omega(t), \omega_1(t), \dots$ to denote various continuous functions in \mathcal{O} or \mathcal{P} . Our proof is based on the following crucial technical lemma.

Lemma 4.3. Given the conditions of Theorem 6 (respectively Theorem 7), for any finite $p \geq 1$, there exists a function $\omega_p \in \mathcal{O}$ (respectively \mathcal{P}) such that

$$\|u(\bullet, t)\|_p \leq \omega_p(t). \tag{4.1}$$

The proof of this is similar to that of [4, Lemma 2.6] if we can establish the following.

Lemma 4.4. Given the conditions of Theorem 6 (respectively Theorem 7). For any $p > \max\{n/2, 1\}$, we set $y(t) = \int_{\Omega} u^p dx$. We can find $\beta \in (0, 1)$ and positive constants A, B, C, δ and functions $\omega_i \in \mathcal{O}$ (respectively, \mathcal{P}) such that the following inequality holds

$$\begin{aligned} \frac{d}{dt} y &\leq -Ay^\eta + (\omega_0(t) + \|u(\bullet, t)\|_1)y + B\omega(t) + \\ &Cy^\theta \left\{ \omega_1(t) + \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega_2(s) \|u(\bullet, s)\|_1^\zeta y^\vartheta(s) ds \right\}^\Xi. \end{aligned} \tag{4.2}$$

Here, $\eta = \frac{p+1}{p}$, $\theta = \frac{p-1}{p}$, $\Xi = 2$, $\vartheta = \frac{(r-1)}{r(p-1)}$, and $\zeta = \frac{(p-r)}{r(p-1)}$. Moreover, $\eta > \theta + \Xi\vartheta$.

Proof. We assume the conditions of Theorem 7 as the proof for the other case is identical. We multiply the equation for u by u^{p-1} and integrate over Ω . Using integration by parts and noting that the boundary integrals are all zero thanks to the boundary condition on u , we see that

$$\begin{aligned} \int_{\Omega} u^{p-1} \frac{d}{dt} u dx + \int_{\Omega} P(u, v) \nabla u \nabla (u^{p-1}) dx &\leq \int_{\Omega} (-R(u, v) \nabla (u^{p-1}) \nabla v dx \\ &+ \int_{\Omega} g(u, v) u^{p-1} dx. \end{aligned}$$

Using the conditions (3.3),(3.4) and Young’s inequality for the integrals involving P, R , we deduce (for some positive constants $C(d, p), \varepsilon, C(\varepsilon, d, p)$)

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^p dx + C(d, p) \int_{\Omega} u^{p-1} |\nabla u|^2 dx &\leq C(\varepsilon, d, p) \int_{\Omega} (u^{p-1} \Phi^2(v) |\nabla v|^2 dx \\ &+ \int_{\Omega} C(v) (u^p + 1) dx. \end{aligned} \tag{4.3}$$

Here, we have also used (3.6). The second term on the left can be estimated by

$$\begin{aligned} \int_{\Omega} u^{p-1} |\nabla u|^2 dx &= C(p) \int_{\Omega} |\nabla(u^{(p+1)/2})|^2 dx \\ &\geq C \int_{\Omega} u^{p+1} dx - C \left(\int_{\Omega} u^{(p+1)/2} dx \right)^2 \\ &\geq C \left(\int_{\Omega} u^p dx \right)^{\frac{p+1}{p}} - C \|u\|_1 \int_{\Omega} u^p dx. \end{aligned}$$

Here, we have used the Hölder’s inequality

$$\left(\int_{\Omega} u^{(p+1)/2} dx \right)^2 = \left(\int_{\Omega} u^{\frac{1}{2}} u^{\frac{p}{2}} dx \right)^2 \leq \|u\|_1 \int_{\Omega} u^p dx.$$

We next consider the first integral on the right of (4.3). By our assumption on L^∞ norm of v , $\Phi(v) \leq \omega_1(t)$ for some $\omega_1 \in \mathcal{P}$. Using the Hölder inequality, we have

$$\begin{aligned} \int_{\Omega} u^{p-1} \Phi^2(v) |\nabla v|^2 dx &\leq \omega_1(t) \left(\int_{\Omega} u^p dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega} |\nabla v|^{2p} dx \right)^{\frac{1}{p}} \\ &= \omega_1(t) y^{\frac{p-1}{p}} \|\nabla v\|_{2p}^2. \end{aligned}$$

As $p > \max\{n/2, 1\}$, there exists $r \in (1, p)$ such that $\frac{1}{n} + \frac{1}{2p} > \frac{1}{r} > \frac{1}{p}$. This implies $2 > 1 - n/2p + n/r$. Hence, we can find $\beta \in (0, 1)$ such that $2\beta > 1 - n/2p + n/r$. Using the $W^{1,q}$ version of [4, (2.15)], with $q = 2p > r$, we have

$$\|\nabla v\|_{2p} \leq \omega_0(t) + \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) \|u(\bullet, s)\|_r ds. \tag{4.4}$$

Applying the above estimates in (4.3), we derive the following inequality for $y(t)$

$$\begin{aligned} \frac{d}{dt} y + y^{\frac{p+1}{p}} &\leq C y^{\frac{p-1}{p}} \omega_1(t) \left\{ \omega_0(t) + \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) \|u(\bullet, s)\|_r ds \right\}^2 \\ &\quad + C(\omega_2(t) + \|u\|_1) y + B\omega_2(t). \end{aligned} \tag{4.5}$$

As $1 < r < p$, we can use Hölder’s inequality

$$\|u\|_r \leq \|u\|_1^{1-\lambda} \|u\|_p^\lambda = \|u\|_1^{1-\lambda} y^{\frac{\lambda}{p}}$$

with $\lambda = \frac{1-1/r}{1-1/p} = \frac{p(r-1)}{r(p-1)}$. Applying this in (4.5) and re-indexing the functions ω_i , we prove (4.2). The last assertion of the lemma follows from the following equivalent inequalities

$$\eta > \theta + 2\vartheta \Leftrightarrow \frac{p+1}{p} > \frac{p-1}{p} + \frac{2(r-1)}{r(p-1)} \Leftrightarrow \frac{1}{p} > \frac{(r-1)}{r(p-1)} \Leftrightarrow rp-r > pr-p \Leftrightarrow p > r.$$

This completes the proof. □

Proof of Theorem 6 and Theorem 7. See the proof of [4, Theorem 2.2]. □

Proof of Theorem 8. Once (3.8) is established, using Schauder type estimates for parabolic equations, it is standard to check the conditions (C), (L) of Theorem 1. We leave the details to the reader. □

5. Proof of examples. We conclude our paper with the proof of our theorems stated in Section 2.

Proof of Theorem 2. For ii), by integrating the equation for u we easily see that the L^1 norm of u is in \mathcal{P} so that $\|u\|_1 \in \mathcal{P}$. However, it is not so easy to show that $\|v\|_\infty \in \mathcal{P}$ and therefore our theorems, as they were stated, are not immediately applicable here. However, a careful inspection of the proof of Lemma 4.4 reveals that the only places where we make use of the assumption $\|v\|_\infty \in \mathcal{P}$ to derive (4.2) are when we estimate $\Phi(v)$ and $\|\nabla v\|_{2p}$. In this case, by our assumption, $\Phi(v) = D_v \chi(v)$ is bounded. Moreover, if we define $\mathcal{A}_v(v) = d_2 \Delta v - cv$ and $f(u, v) = bu$, which is independent of v and thus $C(v) = \text{const}$ in (3.6), we then find that the proof of Lemma 4.4 is still in force to get (4.2). This shows that $\|u\|_p \in \mathcal{P}$ for all $p > 1$. Using this in the equation for v , we get $\|v\|_\infty \in \mathcal{P}$. Our Theorem 8 can apply here to conclude our proof. \square

Proof of Theorem 3. It is easy to see that (2.2) is a special case of (1.1) with $P(u, v) = d_1 + 2\alpha_{11}u + \alpha_{12}v$, $R(u, v) = \alpha_{12}u$ and $Q(v) = d_2 + 2\alpha_{22}$. The fact that $\|v(\bullet, t)\|_\infty$ and $\|u(\bullet, t)\|_1$ are in \mathcal{P} is easy to show (see [4]). If $\alpha_{22} = 0$ then the conditions of Theorem 7 are fulfilled and we can use Theorem 8 to assert the theorem. \square

Proof of Theorem 4. By comparison principles, one can show easily that $\|S\|_\infty \in \mathcal{P}$. Multiplying the equation of u by γ and adding the result to the equation of S , we can easily prove that $\|u\|_1 \in \mathcal{P}$ by integrating over Ω . This fact has been proven in [5] where we assumed that $\alpha_{11} = 0$ and $n = 1$. Applying our Theorem 8 to the case $\alpha_{11} > 0$, we obtain the theorem. \square

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