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PARABOLIC SYSTEMS WITH NON CONTINUOUS COEFFICIENTS

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Abstract. In this note we are interested in the local regularity of the highest order derivatives of the solutions of the system

$$\mathcal{T}u = f_i(y) \qquad i = 1, \dots, N$$

where the known terms f_i are in Lebesgue spaces and the differential the parabolic operator $\mathcal{T}$ has the form

$$u_t - \sum_{j=1}^{N} \sum_{|\alpha|=2s} a_{ij}^{(\alpha)}(y) D^\alpha u_j(y) + \sum_{j=1}^{N} \sum_{|\alpha|\le 2s-1} b_{ij}^{(\alpha)}(y) D^\alpha u_j(y).$$

have discontinuous coefficients.

1. **Introduction and Some Definitions.** In this paper we study the local regularity of the solutions of the parabolic system:

$$\mathcal{T}u \equiv u_t - \sum_{j=1}^{N} \sum_{|\alpha|=2s} a_{ij}^{(\alpha)}(y) D^\alpha u_j(y) + \sum_{j=1}^{N} \sum_{|\alpha|\le 2s-1} b_{ij}^{(\alpha)}(y) D^\alpha u_j(y) = f_i(y)$$

for $i = 1, \dots, N$. In the study made by the author in the note [21] the case of elliptic systems is taken into account.

In the last thirty years parabolic systems have been studied by many authors, for istance we remember the papers of Gugliemino [15], Campanato [4], [5] and the note of Cannarsa [6]; useful contributions to this study has been given by the results contained in [16], [1], [2], [18], [19]. But at our knowledge the L^p estimates we have obtained are the first available for systems with discontinuous coefficients, for every $1 < p < \infty$.

More precisely the coefficients $a_{ij}^{(\alpha)}$ belong to the class of vanishing mean oscillation functions we will introduce in the following.

The method used by the author has been inspired from the papers written by Chiarenza, Frasca and Longo [7] and [8], where the authors consider a linear differential equations of elliptic type in nondivergence form.

The type of regularity results we obtain is not close to that one of the famous paper [14] where Giaquinta study elliptic systems in nonlinear form, in this last note the coefficient p belong to a suitable neighbourhood of 2 and our regularity is true for all p between 1 and ∞.

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Let us consider in $\mathbb{R}^{n+1}$ the *parabolic metric* $d(y,z) = \rho(y-z)$, where

$$\rho(y) = \sqrt{\frac{|x|^2 + \sqrt{|x|^4 + 4t^2}}{2}}.$$

Let us define *parabolic cubes* of center $y = (x,t) = (x_1, x_2, \dots, x_n, t) \in \mathbb{R}^{n+1}$ and radius r

$$I = I_r(y) = \left\{\zeta = (x', s) \in \mathbb{R}^{n+1} : |x - x'| < r, |t - s| < r^2\right\}.$$

The above parabolic metric was first considered by Fabes and Riviere in the paper [12].

We are now ready to define the class of bounded mean oscillation functions (BMO) first introduced by John and Nirenberg in [17].

Definition 1. Let $a \in L^1_{loc}(\mathbb{R}^{n+1})$. We say that a belongs to the parabolic BMO space ($BMO(\mathbb{R}^{n+1})$) if is finite the seminorm

$$\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where Q belongs to the class of the parabolic cubes in $\mathbb{R}^{n+1}$ and a_Q is the integral average

$$\frac{1}{|Q|} \int_Q a(y) dy.$$

To define the vanishing mean oscillation class VMO we consider $a \in BMO$ and set

$$\eta_f(r) = \sup_{\rho \le r} \frac{1}{|Q_\rho|} \int_{Q_\rho} |a(y) - a_{Q_\rho}| dy$$

where Q_ρ ranges over the class of parabolic cubes in $\mathbb{R}^{n+1}$ of radius ρ.

Definition 2. (see e.g. [20]). We say that the function $f \in BMO$ is in the Sarason class $VMO(\mathbb{R}^{n+1})$ if

$$\lim_{r \to 0^+} \eta(r) = 0$$

In the sequel we denote with η_{ij} the VMO modulus of a_{ij} with $i, j = 1, \dots, n$ and

$$\eta(r) = (\Sigma_{i,j=1}^n \eta_{ij}^2(r))^{\frac{1}{2}}.$$

We suppose that the known terms f_i, $i = 1, \dots, N$ of the above system belong to the space $L^p(Q_T)$ $1 < p < \infty$, we now define.

Definition 3. (see e.g. [22]). Let us set $L^p(Q_T), 1 < p < \infty$ the class of measurable functions $f(x,t)$ such that

$$\|f\|_{L^p(Q_T)} \equiv \left(\int_0^T \left(\int_\Omega |f(x,t)|^p \, dx\right) dt\right)^{\frac{1}{p}} < \infty.$$

Elliptic systems of arbitrary order have been studied in [9] and the technique of our proofs to estimate the highest order derivatives of the solutions is mainly along the lines we hinted at the elliptic case.

In particular we estimate the derivatives of the solutions considering the L^p estimates of the parabolic singular integrals and commutators.

2. **Preliminary tools and main results.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $Q_T = \Omega \times (0,T)$, $0 < T < +\infty$, and let $n \geq 3$ be an integer.

As usual D_t and D_i $(i = 1, \ldots, n)$ stands, respectively, for the derivative respect to the variable t and the spatial derivatives.

If $u : Q_t \to \mathbb{R}^{n+1}$, then we consider the parabolic system

$$\mathcal{T}u \equiv u_t - \sum_{j=1}^{N} \sum_{|\alpha|=2s} a_{ij}^{(\alpha)}(y) D^\alpha u_j(y) + \sum_{j=1}^{N} \sum_{|\alpha|\leq 2s-1} b_{ij}^{(\alpha)}(y) D^\alpha u_j(y) = f_i(y) \quad (1)$$

where $i = 1, \ldots, N$; α is a multi-index and $|\alpha| = \alpha_1 + \ldots + \alpha_n$,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$

$f_i \in L^p_{loc}(Q_T)$, $i = 1, \ldots, N$, $p \in]1, +\infty[$ and the coefficients $a_{ij}^{(\alpha)}$ and $b_{ij}^{(\alpha)}$ satisfy the following conditions:

$$(H1)\ a_{ij}^{(\alpha)}(y) \in VMO(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}^{n+1})$$

for every $i, j = 1, \ldots, N,$ where $|\alpha| = 2s$;

$$(H2)\ (\textit{strong ellipticity}) \quad \exists K > 0 \text{ such that}$$

$$det \left(\tau \sum_{i,j=1}^{N} \sum_{|\alpha|=2s} a_{ij}^{(\alpha)}(y) \tau^\alpha \right) \geq K|\tau|^{2sN} \quad \forall \tau \in \mathbb{R}^n,$$

for almost every $y \in Q_T$; where

$$\sum_{|\alpha|=2s} a_{ij}^{(\alpha)} \tau^\alpha,$$

for all $i, j = 1, \ldots, N$, and almost every $\tau \in \mathbb{R}^n$, is the homogeneous polynomial of degree $2s$ in τ obtained substituting D^α with

$$\tau^\alpha = \tau_1^{\alpha_1} \ldots \tau_n^{\alpha_n}.$$

Let us set n odd as in [10] and the coefficients of the lower terms such that:

$$(H3)\ b_{ij}^{(\alpha)} \in L^{\tilde{p}}(Q_T) \text{ where } \tilde{p} \text{ is defined as follows}$$

$$\tilde{p} \begin{cases} = \frac{n+2}{m} \text{ if } 1 < p < \frac{n+2}{m} \\ > \frac{n+2}{m} \text{ if } p = \frac{n+2}{m} \\ = p \text{ if } p > \frac{n+2}{m}. \end{cases}$$

Let us also set:

$$\bar{M} = \max_{\substack{i,j=1,\ldots,N \\ |\alpha|=2s}} \sup_{\Omega} |\, a_{ij}^{(\alpha)} \,|.$$

In the present paper we study the local regularity of the solutions of the above system, the main results are the following.

Theorem 1. *Let us suppose that are true the hypothesis on the coefficients* $a_{ij}^{(\alpha)}$ $(H1)$, $(H2)$; $b_{ij}^{(\alpha)}$ *verify* $(H3)$ *and* $f_i \in L^p_{loc}(Q_T)$, $\forall i = 1, \ldots, N$, $1 < p < \infty$.

If $u \in W^{2s,q}_{loc}(Q_T)$, $1 < q \le p < \infty$, is solution of (1) *then, for any $Q_\sigma \subset\subset Q_T$, there exists a constant $c = c(n, p, K, \eta, dist(Q_\sigma, Q_T))$ such that:*

$$\sum_{\substack{j=1\\|\alpha|=2s}}^{N} \|D^\alpha u_j\|_{L^p(Q_{\frac{\sigma}{2}})} \le c\left(\sum_{j=1}^{N} \|u_j\|_{L^p(Q_\sigma)} + \|f_j\|_{L^p(Q_\sigma)}\right), \quad \forall Q_\sigma \subset\subset Q_T \tag{2}$$

and

$$\|u_t\|_{L^p(Q_{\frac{\sigma}{2}})} \le c\left(\sum_{j=1}^{N} \|u_j\|_{L^p(Q_\sigma)} + \|f_j\|_{L^p(Q_\sigma)}\right), \quad \forall Q_\sigma \subset\subset Q_T. \tag{3}$$

Theorem 2. *Let $a^{(\alpha)}_{ij}$ be such that $(H1)$, $(H2)$, are satisfied $b^{(\alpha)}_{ij}$ verifying $(H3)$ and also $f_i \in L^p_{loc}(Q_T)$, $\forall i = 1, \dots, N$, $1 < p < \infty$.*

Let us suppose $u \in W^{2s,q}_{loc}(Q_T)$, for $1 < q \le p < \infty$, where u is a solution of (1).

Then, for a suitable constant C independent on u_i and f_i, and every $Q_\sigma \subset\subset Q_T$:

$$\sum_{\substack{j=1\\|\alpha|=2s-1}}^{N} \|D^\alpha u_j\|_{C^{0,\gamma}(\bar{Q}_{\frac{\sigma}{2}})} \le C\left(\sum_{j=1}^{N} \|u_j\|_{L^p(Q_\sigma)} + \|f_j\|_{L^p(Q_\sigma)}\right), \quad \forall Q_\sigma \subset\subset Q_T. \tag{4}$$

where $\gamma = 1 - \frac{(n+1)}{p}$.

In the sequel we will use the integral operators and results about them.

Definition 4. (see [3]). We say that a function k is a Parabolic Calderón–Zygmund kernel (PCZ kernel) on $\mathbb{R}^{n+1}$ endowed with the metric ρ above introduced if

1. k is smooth on $\mathbb{R}^{n+1}\backslash\{0\}$;
2. $k(rx', r^2t) = r^{-(n+1)}k(x', t)$, $\forall r > 0$ (homogeneity condition);
3. $\displaystyle\int_{\rho(y)=r} k(y)d\sigma(y) = 0$, $\forall r > 0$ (vanishing property on ellipsoids).

Definition 5. (see [3]). Let us define a function $K(y, y')$, $y \in \mathbb{R}^{n+1}$, $y' \in \mathbb{R}^{n+1}\backslash\{0\}$, a *variable parabolic Calderón–Zygmund kernel* if are satisfied the following conditions:

1. $K(y, \cdot)$ is a parabolic Calderón–Zygmund kernel, for a. e. $y \in \mathbb{R}^{n+1}$;
2. $\sup_{y' \in \Sigma_{n+1}} \| \left(\frac{\partial}{\partial y'}\right)^\beta K(y, y')\| \le C(\beta)$ independent of y.

Let us point out that are finite the following quantities :

$$\sup_y \sup_{y' \in \Sigma_{n+1}} |\nabla_{y'} K(y, y')|$$

and

$$\sup_y \int_{\Sigma_{n+1}} |K(y, y')| d\sigma(y').$$

The following three propositions are proved in [3].

Proposition 1. *Let $K(y, y')$ be a variable PCZ kernel.*

Then for any $f \in L^p$, $1 < p < \infty$, Kf exists in L^psense and

$$\|Kf\|_p \le c\,\|f\|_p$$

for some $c = c(p, K)$.

Proposition 2. *Let $K(y,y')$ be a variable PCZ kernel and a a function belonging to the space L^∞.*

Let us also consider $C[a,f]$ the commutator with kernel K.

Then, for any $1<p<\infty$ there exists $C=C(p,K)$ such that, for $f\in L^p$

$$\|C[a,f]\|_p \le C\|a\|_* \cdot \|f\|_p.$$

From the above proposition it follows the next result.

Proposition 3. *Let $K(y,y')$ be a variable PCZ kernel, $a\in L^\infty$. Let us also observe that if the function a belong to the space $VMO\cap L^\infty(\mathbb{R}^{n+1})$ then, for every $\epsilon>0$, there exists r_0 depending on ϵ and the VMO modulus η_a of a, such that for every $f\in L^p(Q_r)$ with $r\le r_0$ we have*

$$\|C[a,f]\|_{L^p(Q_r)} \le C\cdot\epsilon\cdot \|f\|_{L^p(Q_r)}.$$

3. **Proof of the theorems.**

Proof of Theorem 1 . Let $Q_\sigma\subset Q_T$ be a cylinder, $\tilde{Q}$ the subset of Q_T where the coefficients $a_{ij}^{(\alpha)}$ are defined and satisfy the strongly ellipticity hypotheses.

For fixed $y_0\in\tilde{Q}$, let us consider (as in [19] pg. 619) the constant coefficient operator obtained by $\mathcal{T}$ freezing the coefficient y_0 :

$$\tilde{\mathcal{T}} \equiv u_t \,-\, det\left(\Sigma_{|\alpha|=2s}a_{ij}^{(\alpha)}(y_0)D^\alpha\right), i,j=1,\dots,N.$$

Let us consider $\Gamma(y_0,y)$ its fundamentl solution in $\mathbb{R}^{n+1}$,

it is zero if $t\le 0$ and if $t>0$ is express in terms of an opportune analytic function.

For every $v\in C_0^\infty(Q_\sigma)$ we have

$$v(y) \,=\, \int_{\mathbb{R}^{n+1}} \Gamma(y_0,y-y')\tilde{\mathcal{T}}_0 v(y')dy', \quad \forall y\in\mathbb{R}^{n+1}.$$

Because of y_0 range over $\tilde{Q}$ we can consider Γ as a function of two variables and we will write $\Gamma(y,y')$ where y is the frozen variable considered above.

This function $\Gamma\in C^\infty(\mathbb{R}^{n+1}\backslash\{0\})$, is such that $D^\alpha\Gamma(y_0,y')$, for $|\alpha| = 2sN$ is homogeneous of degree $-n$ and have zero mean value on the sphere $|y'|=1$.

Following the method used in [9] for any vector $u=(u_1,\dots,u_n)\in C_0^\infty(\tilde{Q})$ the derivatives of $D^\alpha u_i$ for $|\alpha|=2s$ can be expressed a. e. in $\tilde{Q}$ as a linear combination of the terms

$$\text{P.V.}\int_B D^{\gamma+\alpha}\Gamma(y,y-y')\sum_{\substack{j=1\\|\alpha|=2s}}^N \left(a_{kj}^{(\alpha)}(y)-a_{kj}^{(\alpha)}(y')\right)D^\alpha u_j(y')dy',$$

$$\text{P.V.}\int_B D^{\gamma+\alpha}\Gamma(y,y-y')\sum_{j=1}^N\left(\sum_{|\alpha|=2s}a_{ij}^{(\alpha)}(y')D^\alpha u_j(y')\right)dy',$$

$$\sum_{j=1}^N\left(\sum_{|\alpha|=2s}a_{ij}^{(\alpha)}(y)D^\alpha u_j(y)\right)$$

and let us point out that, because of a density argument, if $u\in W_0^{2s,p}(\tilde{Q})$ the same representation formula of the case $u\in C_0^\infty(\tilde{Q})$ can be obtained.

In order to write the interior representation formula we can estimate the singular integral operators and commutators with Calderón–Zygmund kernels using the estimates of the derivatives of Γ contained [19] formula (10.9) (see also (2.8) in [13] or [11]), Propositions 1 and 3 and Theorem 3.3 in [22]. .

Then the L^p estimate for $D^\alpha u_j$, $|\alpha| = 2s$, $j = 1, \dots, N$, has been obtained.

Let us now write

$$u_t \,=\, \mathcal{T}u \,+\, \sum_{j=1}^{N} \sum_{|\alpha|=2s} a_{ij}^{(\alpha)}(y) D^\alpha u_j(y) + \sum_{j=1}^{N} \sum_{|\alpha|\le 2s-1} b_{ij}^{(\alpha)}(y) D^\alpha u_j(y).$$

Let us set

$$\Sigma = \{y \in \mathbb{R}^{n+1} : |y| = 1\}.$$

Then using the last equality we can obtain the final estimate.

We finally observe that the constant C is dependent on

$$\tau \,=\, \max_{i,j=1,\dots,N} \max_{|\alpha|\le 2s} \| \frac{\partial^\alpha \Gamma_j(y,y')}{\partial^\alpha y'} \|_{L^\infty(\Omega\times\Sigma)}$$

but it is independent of $u_i$and f_i. □

Proof of Theorem 2. The estimate is a consequence of the Theorem 1 and the known imbedding properties concerning functions belong to L^p spaces and Hölder functions. □

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