PROCEEDINGS OF THE FOURTH INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
May 24 – 27, 2002, Wilmington, NC, USA pp. **798–808**

ASYMPTOTIC STABILITY FOR DYNAMICAL SYSTEMS ASSOCIATED WITH THE ONE-DIMENSIONAL FRÉMOND MODEL OF SHAPE MEMORY ALLOYS

Ken Shirakawa

Department of Information Environment Design and Integration
School of Information Environment, Tokyo Denki University
2-1200 Muzai Gakuendai, Inzai, Chiba, 270-1382, JAPAN

Abstract. In this paper, an evolution dynamical system, which is generated by one-dimensional Frémond models of shape memory alloys, is considered. Assuming that forcing terms converge to some time-independent terms in appropriate senses as time goes to infinity, we shall characterize the asymptotic stability for our dynamical system by the global attractor for the limiting autonomous dynamical system.

1. **Introduction.** Let us put $\Omega := (0,1)$. In this paper, we consider the following system, denoted by (SMA):

$$\begin{cases} (c_0\vartheta - L\chi_1)_t - h\vartheta_{xx} = f(t,x), \text{ a.e. } (t,x) \in Q := (0,+\infty)\times\Omega, \\ \mu\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_t - \lambda\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_{xx} + \partial I_K(\chi_1,\chi_2) \ni \begin{pmatrix} l(\vartheta^*-\vartheta) \\ -\beta g(t)\alpha(\vartheta) + \beta\alpha(\vartheta)^2\chi_2\end{pmatrix}, \text{ a.e. } (t,x)\in Q. \end{cases}$$

The system (SMA) is a simplified version of the mathematical model of shape memory alloys, which was proposed by Frémond (cf. [3]). The first equation is the (simplified) heat equation, and ϑ is the absolute temperature with the critical temperature ϑ^* (positive constant). The second equation is the kinetic equation of the microstructures in shape memory alloys (e.g. austenite and martensites). In this model, it is assumed that there are only two kinds of martensites and two parameters χ_i $(i=1,2)$ are order parameters related to local proportions of martensites. The system (SMA) is considered under the following initial-boundary conditions:

$$\begin{cases} -h\vartheta_x(t,0) + k(\vartheta(t,0)-\gamma_0(t)) = h\vartheta_x(t,1) + k(\vartheta(t,1)-\gamma_1(t)) = 0, \text{ a.e. } t>0, \\ (\chi_i)_x(t,0) = (\chi_i)_x(t,1) = 0, \ i=1,2, \text{ a.e. } t>0, \\ \vartheta(0,x) = \vartheta_0(x), \ \chi_i(0,x) = \chi_{i,0}, \ i=1,2, \text{ a.e. } x\in\Omega. \end{cases}$$

Here, we assume that:

(a1) $f \in L^2_{loc}(0,+\infty;L^2(\Omega))$, $g\in W^{1,2}_{loc}(0,+\infty)$, $\gamma_j \in W^{1,2}_{loc}(0,+\infty)$, $j=0,1$, and a function $f^*\in L^2_{loc}[0,+\infty)$, defined as

$$f^*(t) := \left\{ \ 1 + |f(t)|^2_{L^2(\Omega)} + |g(t)|^2 + |g'(t)|^2 + \sum_{j=0}^{1}(|\gamma_j(t)|^2 + |\gamma_j'(t)|^2) \ \right\}^{\frac{1}{2}}, \ t\ge 0,$$

1991 *Mathematics Subject Classification.* Primary: 35B41, 93D20; Secondary: 82B26.

Key words and phrases. Global attractor, one-dimensional Frémond model, shape memory alloys.

satisfies $S(f^*) := \sup_{s\geq 0} |f^*(\cdot + s)|_{L^2(0,1)} < +\infty$, where g' and γ_j' are derivatives of functions g and γ_j $(j = 0, 1)$, respectively;

(a2) $\alpha \in W^{1,\infty}(0, +\infty)$;

(a3) c_0, L, h, k, μ, λ, l, β and ϑ^* are all positive constants;

(a4) I_K is the indicator function of the following two-dimensional triangle

$$K := \left\{ \ (\xi, \eta) \in \mathbb{R}^2 \ \middle| \ 0 \leq \xi \leq 1, \ |\eta| \leq \xi \ \right\},$$

and ∂I_K is the subdifferential of I_K.

The subdifferential ∂I_K is a maximal monotone (set-valued mapping) from $\mathbb{R}^2$ into itself, and it stands on an important position to describe the mechanical constitutive law by set-valued functions.

In recent years, the mathematical theory for Frémond models of shape memory alloys has been studied by a lot of mathematicians (for example, please see [2, 3, 4]). Especially, on the large time behavior, the characterization from the viewpoint of steady-state solutions has been discussed in [4]. In this paper, we shall try to characterize the large time behavior from the another angle with that in [4]. Namely, our main objective is to investigate the large time behavior of solutions of (SMA) from the viewpoint of global attractors.

Definition 1. (Global attractor) Let H be a real Hilbert space, and let $D_0 \subset H$ be a closed and convex subset in H. Let $\{T(t) : D_0 \longrightarrow D_0 \mid t \geq 0\}$ be a semigroup on D_0. Then, a set $\mathcal{A} \subset D_0$ is called a global attractor for $\{T(t)\}$, if:

(A1) $\mathcal{A}$ is nonempty, connected and compact in H;

(A2) (invariance) $T(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$;

(A3) (attractiveness) for any bounded set $B \subset D_0$, $\mathrm{dist}_H(T(t)B, \mathcal{A}) \to 0$ as $t \to +\infty$, where $\mathrm{dist}_H(\cdot, \cdot)$ is the Hausdorff semi-distance between two subsets in H, defined as $\mathrm{dist}_H(A, B) := \sup_{a\in A} \inf_{b\in B} |a-b|_H$ for every subsets A and $B \subset H$.

If the system is autonomous, then the dynamical system, associated with solution operators, forms a semigroup. But, as is easily seen, the system (SMA) is nonautonomous, so we need some devices for the characterization of the asymptotic stability. As one of the method of the characterization, the general theory established by Chepyzhov-Vishik (cf. [6]) has been well known.

In this paper some results, which is obtained partly in the paper [5], will be reported. The main point of the result is to characterize the asymptotic stability for (SMA) in the framework of the general theory obtained in [9]. In [9], an abstract evolution equation governed by time-dependent subdifferentials and demi-closed perturbations is considered. Then, the dynamical system forms an evolution system (not semigroup) such that the domain of initial values is also time-dependent. Authors in [9] first set a limiting autonomous equation for their original nonautonomous equation, and secondly characterized the asymptotic stability for the evolution system by the global attractor for the limiting semigroup system. Unfortunately, the general theory as in [9] is not applicable for our system (SMA), because the perturbation does not show the demi-closedness in the abstract setting. But, in this paper, it will be seen that a similar characterization as in [9] also holds

for (SMA). Roughly speaking, it implies that the global attractor for the limiting autonomous system may be regarded as an attractor for the nonautonomous system.

2. **Statement of main results.** Let $(\cdot,\cdot)$ be the usual inner product in $L^2(\Omega)$. Let V be the Sobolev space $H^1(\Omega)$ with the following inner product

$$(u,v)_V := (u_x, v_x) + \frac{k}{h}(u(0)v(0) + u(1)v(1)) \quad \text{for any } u, v \in V,$$

and let V^* be the dual space of V. As is well known, $V \subset L^2(\Omega) \subset V^*$ with compact injections.

For functions γ_j, $j = 0, 1$, as in the assumption (a1), let us define

$$\gamma(t,x) := \frac{k(\gamma_1(t) - \gamma_0(t))}{2h+k}x + \frac{h\gamma_1(t) + (h+k)\gamma_0(t)}{2h+k}, \ (t,x) \in Q.$$

Clearly, $\gamma \in W^{1,2}_{loc}(0,+\infty; H^2(\Omega))$, $\gamma_{xx}(t,x) = 0$ for any $(t,x) \in Q$, and $-h\gamma_x(t,0) + k(\gamma(t,0) - \gamma_0(t)) = h\gamma_x(t,1) + k(\gamma(t,1) - \gamma_1(t)) = 0$.

Definition 2. (Definition of solutions of (SMA)) A triple $[\vartheta, \chi_1, \chi_2]$ of functions $\vartheta : [0,+\infty) \longrightarrow L^2(\Omega)$ and $\chi_i : [0,+\infty) \longrightarrow L^2(\Omega)$, $i = 1, 2$, is called a solution of (SMA), if:

(s1) $\vartheta \in C([0,+\infty); L^2(\Omega)) \cap L^2_{loc}(0,+\infty; V) \cap W^{1,2}_{loc}((0,+\infty); L^2(\Omega))$,
$\chi_i \in W^{1,2}_{loc}(0,+\infty; L^2(\Omega)) \cap L^2_{loc}(0,+\infty; H^1(\Omega))$, $i = 1, 2$;

(s2) $\vartheta(0,\cdot) = \vartheta_0$ in $L^2(\Omega)$, and

$$\begin{gathered}((c_0\vartheta - L\chi_1)_t(t), z) + h(\vartheta(t) - \gamma(t), z)_V = (f(t), z) \\ \text{for any } z \in V \text{ and a.e. } t > 0;\end{gathered} \tag{1}$$

(s3) $\chi_i(0,\cdot) = \chi_{i,0}$ in $H^1(\Omega)$, and there exists a pair $[\xi_1, \xi_2]$ of two functions $\xi_i \in L^2_{loc}(0,+\infty;\, L^2(\Omega))$, $i = 1, 2$, such that

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \partial I_K(\chi_1, \chi_2), \text{ a.e. in } Q, \text{ and} \tag{2}$$

$$\begin{gathered}\sum_{i=1}^{2}\Big\{\mu((\chi_i)_t(t), z_i) + \lambda((\chi_i)_x(t), (z_i)_x) + (\xi_i(t), z_i)\Big\} = l(\vartheta^* - \vartheta(t), z_1) \\ +(-\beta g(t)\alpha(\vartheta(t)) + \beta\alpha(\vartheta(t))^2\chi_2(t), z_2) \\ \text{for any } z_i \in H^1(\Omega),\ i = 1, 2, \text{ and a.e. } t > 0.\end{gathered} \tag{3}$$

Now, for a simplicity, we define a Hilbert space W by putting

$$W := L^2(\Omega) \times H^1(\Omega) \times H^1(\Omega)$$

$$\text{with the norm: } |[u, v_1, v_2]|_W := \Big(|u|^2_{L^2(\Omega)} + \sum_{i=1}^{2}|v_i|^2_{H^1(\Omega)}\Big)^{\frac{1}{2}},\ [u, v_1, v_2] \in W,$$

and put

$$D := \left\{ [u, v_1, v_2] \in W \;\middle|\; \begin{array}{l} u \in L^2(\Omega), \text{ and } v_i \in H^1(\Omega),\ i = 1, 2, \\ \text{such that } [v_1, v_2] \in K, \text{ a.e. in } \Omega \end{array} \right\}.$$

As is easily seen, W and D indicate the range of solutions and the domain of initial values, respectively.

Our first theorem is concerned with the existence and uniqueness of solutions of (SMA).

Theorem 1. *(Existence and uniqueness) Let us assume conditions (a1)∼(a4). Then, for any $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D$, (SMA) admits a unique solution $[\vartheta, \chi_1, \chi_2]$.*

Our second theorem is concerned with the boundedness of solutions.

Theorem 2. *(Boundedness) Under conditions (a1)∼(a4), let $[\vartheta, \chi_1, \chi_2]$ be the solution of (SMA) with the initial value $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D$. Then the following estimates hold for the solution $[\vartheta, \chi_1, \chi_2]$:*

(i) *there exists a positive constant N_0, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, such that*

$$\begin{aligned} &|[\vartheta(t), \chi_1(t), \chi_2(t)]|_W^2 + |\vartheta|^2_{L^2(t,t+1;V)} + \sum_{i=1}^{2} |\chi_i|^2_{W^{1,2}(t,t+1;L^2(\Omega))} \\ &\quad \le N_0 \left\{ |[\vartheta_0, \chi_{1,0}, \chi_{2,0}]|_W^2 + S(f^*)^2 \right\} \text{ for any } t \ge 0; \end{aligned}$$

(ii) *for any (small) $\delta > 0$, there exists a positive constant N_δ, depending only on δ, such that*

$$\begin{aligned} &|\vartheta|^2_{W^{1,2}(t,t+1;L^2(\Omega))} + |\vartheta(t)|_V^2 + \sum_{i=1}^{2} \left\{ |\chi_i|^2_{W^{1,2}(t,t+1;H^1(\Omega))} + |(\chi_i)_t(t)|^2_{L^2(\Omega)} \right\} \\ &\quad \le N_\delta (1 + |g|^2_{L^\infty(0,+\infty)}) \left\{ |[\vartheta_0, \chi_{1,0}, \chi_{2,0}]|_W^2 + S(f^*)^2 \right\} \text{ for any } t \ge \delta. \end{aligned}$$

Remark 1. Under the same conditions as in Theorem 2, for any pair $[\xi_1, \xi_2]$ of functions satisfying (2) and (3) we can derive the following estimates:

(iii) there is a positive constant M_0, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, such that

$$\begin{aligned} &\sum_{i=1}^{2} \left\{ |\chi_i|^2_{L^2(t,t+1;H^2(\Omega))} + |\xi_i|^2_{L^2(t,t+1;L^2(\Omega))} \right\} \\ &\quad \le M_0 \left\{ |[\vartheta_0, \chi_{1,0}, \chi_{2,0}]|_W^2 + S(f^*)^2 \right\} \quad \text{for any } t \ge 0; \end{aligned}$$

(iv) for any $\delta > 0$, there is a positive constant M_δ, depending only on δ, such that

$$\begin{aligned} &\sum_{i=1}^{2} \left\{ |\chi_i(t)|^2_{H^2(\Omega)} + |\xi_i(t)|^2_{L^2(\Omega)} \right\} \\ &\le M_\delta (1 + |g|^2_{L^\infty(0,+\infty)}) \left\{ |[\vartheta_0, \chi_{1,0}, \chi_{2,0}]|_W^2 + S(f^*)^2 \right\} \quad \text{for any } t \ge \delta. \end{aligned}$$

The above remark is checked as follows. Since (3) is equivalent to

$$\begin{cases} -\lambda(\chi_1)_{xx}(t) + \xi_1(t) = l(\vartheta^* - \vartheta(t)) - \mu(\chi_1)_t(t) \text{ in } L^2(\Omega), \\ -\lambda(\chi_2)_{xx}(t) + \xi_2(t) = -\beta g(t)\alpha(\vartheta(t)) + \beta\alpha(\vartheta(t))^2\chi_2(t) - \mu(\chi_2)_t(t) \text{ in } L^2(\Omega), \end{cases}$$

for a.e. $t > 0$, we see from the condition (a2) and the regularity result as in [1] that there is a positive constant M_1, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, satisfying

$$\sum_{i=1}^{2} \left\{ |\chi_i(t)|^2_{H^2(\Omega)} + |\xi_i(t)|^2_{L^2(\Omega)} \right\} \le M_1 \left\{ 1 + |g(t)|^2 + |\vartheta(t)|^2_{L^2(\Omega)} + \sum_{i=1}^{2} |(\chi_i)_t(t)|^2_{L^2(\Omega)} \right.$$

for a.e. $t > 0$. Thus, by (a1), (i) and (ii) of Theorem 2, we obtain assertions (iii) and (iv), respectively. □

Now, for any $s \geq 0$, we denote by $f^s \in L^2_{loc}(0,+\infty;L^2(\Omega))$, $g^s \in W^{1,2}_{loc}(0,+\infty)$ and $\gamma^s_j \in W^{1,2}_{loc}(0,+\infty)$, $j=0,1$, forcing terms translated by s, more precisely

$$f^s(t) := f(t+s) \text{ in } L^2(\Omega),$$

$$g^s(t) := g(t+s) \text{ and } \gamma^s_j(t) := \gamma_j(t+s),\ j=0,1,\ \text{for } t>0.$$

Here, we denote by $(SMA)^s$ $(s \geq 0)$ the system (SMA) with translated forcing terms f^s, g^s and γ^s_j, $j=0,1$. Then, on account of Theorem 1, we can define a solution operator $E(t,s) : D \longrightarrow D$ $(0 \leq s \leq t < +\infty)$, which assigns to any initial value $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$ the value $[\vartheta(t-s), \chi_1(t-s), \chi_2(t-s)]$ of the solution of $(SMA)^s$ at time $t-s$. Then, the family $\{E(t,s) \mid 0 \leq s \leq t < +\infty\}$ satisfies the following evolution property:

(E1) $E(s,s) = I$ (identity) for any $s \geq 0$;

(E2) $E(t,s) = E(t,t_0) \circ E(t_0,s)$ for any $0 \leq s \leq t_0 \leq t < +\infty$.

Next, we assume that

$$\begin{aligned} &f^s \to f^\infty \text{ in } L^2_{loc}(0,+\infty;L^2(\Omega)),\ g^s \to g^\infty \text{ in } W^{1,2}_{loc}(0,+\infty) \\ &\quad \text{and } \gamma^s_j \to \gamma^\infty_j \text{ in } W^{1,2}_{loc}(0,+\infty),\ j=0,1, \text{ as } s \to +\infty. \end{aligned} \tag{4}$$

Then, we can consider the following system, denoted by $(SMA)^\infty$, as the limiting system:

$$\begin{cases} (c_0\vartheta - L\chi_1)_t - h\vartheta_{xx} = f^\infty(x),\ \text{a.e. } (t,x) \in Q, \\ \mu \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_t - \lambda \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_{xx} + \partial I_K(\chi_1,\chi_2) \ni \begin{pmatrix} l(\vartheta^* - \vartheta) \\ -\beta g^\infty \alpha(\vartheta) + \beta\alpha(\vartheta)^2\chi_2 \end{pmatrix},\ \text{a.e. } (t,x) \in Q, \\ -h\vartheta_x(t,0) + k(\vartheta(t,0) - \gamma^\infty_0) = h\vartheta_x(t,1) + k(\vartheta(t,1) - \gamma^\infty_1) = 0,\ \text{a.e. } t>0, \\ (\chi_i)_x(t,0) = (\chi_i)_x(t,1) = 0,\ i=1,2,\ \text{a.e. } t>0, \\ \vartheta(0,x) = \vartheta_0(x),\ \chi_i(0,x) = \chi_{i,0},\ i=1,2,\ \text{a.e. } x \in \Omega. \end{cases}$$

The well-posedness of solutions of $(SMA)^\infty$ can be shown as the special case of that of (SMA). In this case, we can also define a solution operator $S(t) : D \longrightarrow D$ $(t \geq 0)$, which assigns to any initial value $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$ the value $[\vartheta(t), \chi_1(t), \chi_2(t)]$ of the solution of $(SMA)^\infty$ at time t. Here, we note that the family $\{S(t) \mid t \geq 0\}$ forms a semigroup on D, since $(SMA)^\infty$ is an autonomous system.

Definition 3. (ω-limit sets) Let $\{E(t,s)\}$ and $\{S(t)\}$ be dynamical systems as the above. For any subset B of D,

(i) a set $\omega_E(B)$ defined by $\omega_E(B) := \bigcap_{\tau \geq 0} \overline{\bigcup_{\substack{t \geq \tau \\ s \geq 0}} E(t+s,s)B}^W$ is called an ω-limit set of B for $\{E(t,s)\}$;

(ii) a set $\omega_S(B)$ defined by $\omega_S(B) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t)B}^W$ is called an ω-limit set of B for $\{S(t)\}$.

Our final result is concerned with a characterization of the asymptotic stability for $\{E(t,s)\}$ by the global attractor for the limiting semigroup system $\{S(t)\}$.

Theorem 3. *Assume all conditions (a1)∼(a4) and (4). Let $\{E(t,s)\}$ and $\{S(t)\}$ be dynamical systems as the above. Then:*

(i) *(cf.* [7, 10]*) for the semigroup $\{S(t)\}$, there exists a global attractor $\mathcal{A}_\infty$;*

(ii) $\omega_E(B) \subset \mathcal{A}_\infty$ *for any bounded subset* B*;*
(iii) *there exists a bounded set* $B_E \subset D$ *such that* $\mathcal{A}_\infty = \omega_E(B_E)$*.*

The assertions (ii) and (iii) in the above theorem implies that the compact set $\mathcal{A}_\infty$ may be called as an attractor for the evolution process $\{E(t,s)\}$ in some sense. Actually, it will be shown (in (14)) that $\mathcal{A}_\infty$ satisfy an attracting property for $\{E(t,s)\}$ like the condition (A3) in Definition 2.

3. **Proofs.** In this section, we give proofs of our main theorems. Since proofs of the well-posedness and the boundedness are also proved in [5], we will show only the rough sketch of them, and refer to [5] for details.

Let us put $D_0 := \left\{ \ [u, v_1, v_2] \in D \ \middle| \ u \in V \ \right\}$, and first show the existence in the case of $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D_0$.

Proof of the existence in the case of $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D_0$*:* We first note that the system (SMA) can be reformulated to a system of two kinds of abstract evolution equations governed by subdifferentials of appropriate convex functions. This kind of system was treated in [8] to show the local existence result for some phase transition models. Indeed, by a similar argument as in [8, section 3], we can also prove the local existence of solutions of (SMA) (see [5] for details).

The global existence is shown by a contradiction. Let us assume that

$$T^* := \sup \left\{ \ T > 0 \ \middle| \ (SMA) \text{ has a solution on } [0,T] \ \right\} < +\infty. \tag{5}$$

Then, by the local existence result, we immediately have $T^* > 0$. Here, it follows from the smoothing effect of solutions that $[\vartheta(T^*), \chi_1(T^*), \chi_2(T^*)] \in D_0$. Hence, taking $[\vartheta(T^*), \chi_1(T^*), \chi_2(T^*)]$ as the initial value, we can extend the solution $[\vartheta, \chi_1, \chi_2]$ in time beyond T^*. It contradicts (5). □

Next, we consider the case of $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D$. In this case, the existence is shown on the basis of the continuous dependence of solutions with respect to the initial values. So, we first prove the uniqueness and the boundedness, because we need them in the proof of the continuous dependence.

Proof of the uniqueness: Let $[\vartheta, \chi_1, \chi_2]$ and $[\tilde{\vartheta}, \tilde{\chi}_1, \tilde{\chi}_2]$ be two solutions of (SMA) with two initial values $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$ and $[\tilde{\vartheta}_0, \tilde{\chi}_{1,0}, \tilde{\chi}_{2,0}]$, respectively. Here, let us subtract (1) for $[\tilde{\vartheta}, \tilde{\chi}_1, \tilde{\chi}_2]$ from the one for $[\vartheta, \chi_1, \chi_2]$, as well as (3), and take the sum of results. Then, by the monotonicity of ∂I_K, we find positive constants ν_0 and ν_1, independent of forcing terms and initial values, such that

$$\frac{d}{dt} J_0(t) \le \nu_0 \left\{ J_0(t) + \left(|\vartheta_0 - \tilde{\vartheta}_0|^2_{L^2(\Omega)} + \sum_{i=1}^{2} |\chi_{i,0} - \tilde{\chi}_{i,0}|^2_{L^2(\Omega)} \right) \right\}, \text{ a.e. } t > 0,$$

for a function

$$J_0(t) := \int_0^t |(\vartheta - \tilde{\vartheta})(\tau)|^2_{L^2(\Omega)} \, d\tau + \sum_{i=1}^{2} \int_0^t |(\chi_i - \tilde{\chi}_i)(\tau)|^2_{L^2(\Omega)} \, d\tau$$
$$+ \nu_1 \left| \int_0^t (\vartheta - \tilde{\vartheta})(\tau) \, d\tau \right|^2_V, \ t \ge 0.$$

Now, applying Gronwall's lemma to J_0, we conclude the uniqueness. □

Proof of (i) of Theorem 2: For any solution $[\vartheta, \chi_1, \chi_2]$ of (SMA) with the initial value $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, we define

$$J(t) := \frac{c_0}{2L}|\vartheta(t) - \gamma(t)|^2_{L^2(\Omega)} + \frac{\lambda}{2l}\sum_{i=1}^{2}|\chi_i(t)|^2_{H^1(\Omega)},\ t \geq 0. \tag{6}$$

Let us put $z = \frac{1}{L}(\vartheta(t) - \gamma(t))$ in (1), $z_i = \frac{\lambda}{l\mu}\chi_i(t)$ in (3), $z_i = \frac{1}{l}(\chi_i)_t(t)$ in (3), $i = 1, 2$, and take the sum of results. Then, applying the condition (a1) and Young's inequality, we find positive constants ν_2, ν_3, ν_4 and N_1, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, such that

$$\frac{d}{dt}J(t) + \nu_2 J(t) \leq \nu_3|f^*(t)|^2, \text{ and} \tag{7}$$

$$\begin{aligned}\frac{d}{dt}J(t) + \nu_4\Big\{|\vartheta(t) - \gamma(t)|^2_V + \sum_{i=1}^{2}\Big(|(\chi_i)_x(t)|^2_{L^2(\Omega)} + |(\chi_i)_t(t)|^2_{L^2(\Omega)}\Big)\Big\} \\ \leq N_1|f^*(t)|^2 \text{ for a.e. } t \geq 0.\end{aligned} \tag{8}$$

Now, applying Gronwall's lemma to (7) and integrating the both sides of (8) over $[t, t+1]$, we obtain (i) of Theorem 2. □

Proof of (ii) of Theorem 2: Let us put $z = \frac{1}{L}(\vartheta_t(t) - \gamma_t(t))$ in (1), $z_i = \frac{1}{l}(\chi_i)_t(t)$ in $\frac{d}{dt}(3)$, $i = 1, 2$, and take the sum of results. Then, applying the condition (a1) and Young's inequality, we find a positive constant N_2, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, such that

$$\begin{aligned}\frac{c_0}{4L}|\vartheta_t(t)|^2_{L^2(\Omega)} + \frac{\lambda}{l}\sum_{i=1}^{2}|(\chi_i)_{xt}|^2_{L^2(\Omega)} + \frac{d}{dt}\Big\{\frac{h}{2L}|\vartheta(t) - \gamma(t)|^2_V + \frac{\mu}{2l}\sum_{i=1}^{2}|(\chi_i)_t(t)|^2_{L^2(\Omega)}\Big\} \\ \leq N_2(1 + |g|^2_{L^\infty(0,+\infty)})\Big\{\sum_{i=1}^{2}|\chi_i(t)|^2_{L^2(\Omega)} + |f^*(t)|^2\Big\} \text{ for any } t \geq 0.\end{aligned}$$

Here for any $0 \leq s \leq t \leq s+2$, let us multiply the both sides by $(t-s)$ and integrate them over $[s, t]$. Then, by (8), we find positive constant N_3, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, such that

$$\begin{aligned}&\int_s^t (\tau - s)\Big(|\vartheta_t(\tau)|^2_{L^2(\Omega)} + \sum_{i=1}^{2}|(\chi_i)_{xt}(\tau)|^2_{L^2(\Omega)}\Big)\ d\tau \\ &+(t-s)\Big(|\vartheta(t) - \gamma(t)|^2_V + \sum_{i=1}^{2}|(\chi_i)_t(t)|^2_{L^2(\Omega)}\Big) \\ &\leq N_3(1 + |g|^2_{L^\infty(0,+\infty)})\Big\{J(s) + \int_s^t |f^*(\tau)|^2\ d\tau\Big\}\end{aligned} \tag{9}$$

for any $0 \leq s \leq t \leq s+2$. Now, combining the condition (a1), (6), (9) and (i) of Theorem 2, we conclude (ii) of Theorem 2. □

Remark 2. For solutions of systems $(SMA)^s$ with $s > 0$, we also obtain the same estimates as in Theorem 2 just as in the above argument. Then, we notice that constants N_0 and N_δ can be chosen independent of s, since forcing terms f^s, g^s and γ^s_j $(j = 0, 1)$ are estimated by $S(f^*)$ uniformly with respect to s.

Now, we are on the stage to prove the existence of solutions in the case of $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D$. The keypoint of the proof is to show the continuous dependence of solutions, stated as follows.

Proposition 1. *Let* $\{f_n\} \subset L^2_{loc}(0,+\infty;L^2(\Omega))$, $\{g_n\} \subset W^{1,2}_{loc}(0,+\infty)$, $\{\gamma_{j,n}\} \subset W^{1,2}_{loc}(0,+\infty)$, $j=0,1$. *For any* $n \in \mathbb{N}$, *let* $[\vartheta_{0,n},\chi_{1,0,n},\chi_{2,0,n}] \in D$, *and let* $[\vartheta_n,\chi_{1,n},\chi_{2,n}]$ *be the solution of* (SMA) *corresponding to forcing terms* f_n, g_n, $\gamma_{j,n}$, $j=0,1$ *and the initial value* $[\vartheta_{0,n},\chi_{1,0,n},\chi_{2,0,n}]$. *If there exist functions* $f \in L^2_{loc}(0,+\infty;L^2(\Omega))$, $g \in W^{1,2}_{loc}(0,+\infty)$, $\gamma_j \in W^{1,2}_{loc}(0,+\infty)$, $j=0,1$, *and* $[\vartheta_0,\chi_{1,0},\chi_{2,0}] \in D$ *such that*

$$\begin{cases} f_n \to f \text{ in } L^2_{loc}(0,+\infty;L^2(\Omega)), \\ g_n \to g,\ \gamma_{j,n} \to \gamma_j \text{ in } W^{1,2}_{loc}(0,+\infty),\ j=0,1, \\ [\vartheta_{0,n},\chi_{1,0,n},\chi_{2,0,n}] \to [\vartheta_0,\chi_{1,0},\chi_{2,0}] \text{ in } W \text{ as } n \to +\infty, \end{cases} \tag{10}$$

then there exists a triple $[\vartheta,\chi_1,\chi_2] \in L^2_{loc}(0,+\infty;W)$ *such that*

$$[\vartheta_n,\chi_{1,n},\chi_{2,n}] \to [\vartheta,\chi_1,\chi_2] \text{ in } C(J_0,W) \quad \text{as } n \to +\infty, \tag{11}$$

for any compact interval $J_0 \subset (0,+\infty)$. *Moreover, the triple* $[\vartheta,\chi_1,\chi_2]$ *is a solution of* (SMA).

Proof. For any $n \in \mathbb{N}$, let $[\xi_{1,n},\xi_{2,n}]$ be the pair of functions $\xi_{i,n} \in L^2_{loc}(0,+\infty;L^2(\Omega))$, $i=1,2$, as in (2) and (3) with $[\vartheta,\chi_1,\chi_2]=[\vartheta_n,\chi_{1,n},\chi_{2,n}]$. Then, by (10), (i) of Theorem 2 and (iii) of Remark 1, we find subsequences $\{\vartheta_{n_k}\} \subset \{\vartheta_n\}$, $\{\chi_{i,n_k}\} \subset \{\chi_{i,n}\}$, $\{\xi_{i,n_k}\} \subset \{\xi_{i,n}\}$ and functions $\vartheta,\chi_i,\xi_i \in L^2_{loc}(0,+\infty;L^2(\Omega))$, $i=1,2$, such that $\vartheta_{n_k} \to \vartheta$ in $L^2(J,L^2(\Omega))$, weakly in $L^2(J,V)$,

$$\begin{cases} \chi_{i,n_k} \to \chi_i & \text{in } C(J;L^2(\Omega)), \text{ weakly in } W^{1,2}(J;L^2(\Omega)), \\ & \text{in } L^2(J;H^1(\Omega)), \text{ weakly in } L^2(J;H^2(\Omega)), \\ \xi_{i,n_k} \to \xi_i & \text{weakly in } L^2(J;L^2(\Omega)),\ i=1,2, \end{cases}$$

$$g_{n_k}\alpha(\vartheta_{n_k}) \to g\alpha(\vartheta) \text{ in } L^2(J;L^2(\Omega)), \text{ and}$$
$$\alpha(\vartheta_{n_k})^2\chi_{2,n_k} \to \alpha(\vartheta)^2\chi_2 \text{ in } L^2(J,L^2(\Omega)), \text{ as } k \to +\infty,$$

for any compact interval $J \subset [0,+\infty)$. Now, by the demi-closedness of ∂I_K, $[\vartheta,\chi_1,\chi_2]$ and $[\xi_1,\xi_2]$ satisfy (2) and (3). Since ϑ_n is a solution of the heat equation with the forcing term $\frac{1}{c_0}(f+L(\chi_{1,n})_t)$ for any $n \in \mathbb{N}$, we easily check that $\vartheta_{n_k} \to \vartheta$ in $C(J;L^2(\Omega))$ as $k \to +\infty$, and $[\vartheta,\chi_1,\chi_2]$ satisfies (1).

Hence, the triple $[\vartheta,\chi_1,\chi_2]$ is a solution of (SMA). On account of the uniqueness of solutions and (ii) of Theorem 2, we conclude (11). □

Proof of the existence in the case of $[\vartheta_0,\chi_{1,0},\chi_{2,0}] \in D$*:* For any $[\vartheta_0,\chi_{1,0},\chi_{2,0}] \in D$, let us choose a sequence $\{[\vartheta_{0,n},\chi_{1,0,n},\chi_{2,0,n}]\} \subset D_0$ such that

$$[\vartheta_{0,n},\chi_{1,0,n},\chi_{2,0,n}] \to [\vartheta_0,\chi_{1,0},\chi_{2,0}] \text{ in } W \text{ as } n \to +\infty.$$

For any $n \in \mathbb{N}$, let $[\vartheta_n,\chi_{1,n},\chi_{2,n}]$ be the solution of (SMA) with the initial value $[\vartheta_{0,n},\chi_{1,0,n},\chi_{2,0,n}] \in D_0$. Then, by Proposition 1, we find a solution $\{\vartheta,\chi_1,\chi_2\}$ of (SMA) with the initial value $[\vartheta_0,\chi_{1,0},\chi_{2,0}] \in D$ as the limit of the sequence $\{\vartheta_n,\chi_{1,n},\chi_{2,n}\}$ of solutions in the sense of (11). □

Finally, we show our third theorem. The keypoint of the proof is to find so-called compact (uniform) absorbing sets, stated in the following lemma.

Lemma 1. *We find a compact subset* $B^* \subset D$ *which satisfies the following absorbing property* $(*)$*:*

- $(*)$ *for any bounded subset* $B \subset D$, *there exists a finite time* t_B, *depending only on* B, *such that* $E(t+s,s)B \cup S(t)B \subset B^*$ *for any* $s \geq 0$ *and* $t \geq t_B$.

Proof. For any $0 \le s \le +\infty$, let us put

$$\gamma^s(t,x) := \frac{k(\gamma_1^s(t) - \gamma_0^s(t))}{2h+k}x + \frac{h\gamma_1^s(t) + (h+k)\gamma_0^s(t)}{2h+k} \quad \text{for any } (t,x) \in Q.$$

Here, we note that γ^∞ is time-independent, since γ_j^∞ $(j = 0, 1)$ are constants. Also, let us denote by $[\vartheta^s, \chi_1^s, \chi_2^s]$ the solution of $(SMA)^s$, and put

$$J^s(t) := \frac{c_0}{2L}|\vartheta^s(t) - \gamma^s(t)|^2_{L^2(\Omega)} + \frac{\lambda}{2l}\sum_{i=1}^{2}|\chi_i^s(t)|^2_{H^1(\Omega)}$$
$$\text{for any } 0 \le s \le +\infty \text{ and } 0 \le t < +\infty.$$

Let S^∞ be a positive constant defined as

$$S^\infty := \Big(1 + |f^\infty|^2_{L^2(\Omega)} + |g^\infty|^2 + \sum_{j=0}^{1}|\gamma_j^\infty|^2\Big)^{\frac{1}{2}},$$

Then, by a similar way to obtain (7), we find two positive constants ν_0^* and ν_1^*, independent of s, such that

$$\frac{d}{dt}J^s(t) + \nu_0^* J^s(t) \le \begin{cases} \nu_1^*|f^*(t)|^2, \text{ if } 0 \le s < +\infty, \\ \nu_1^*(S^\infty)^2, \text{ if } s = +\infty, \end{cases} \quad \text{for a.e. } t > 0.$$

So, applying conditions (a1), (4) and Gronwall's lemma to the above inequality, we also find a positive constant ν_2^*, independent of s, such that

$$|[\vartheta^s(t), \chi_1^s(t), \chi_2^s(t)]|^2_W \le \nu_2^* e^{-\nu_0^* t}|[\vartheta_0, \chi_{1,0}, \chi_{2,0}]|^2_W + \nu_0^*\nu_1^*\nu_2^*(S^\infty)^2$$
$$\text{for any } 0 \le s \le +\infty \text{ and } 0 \le t < +\infty. \tag{12}$$

Let us put $B_0^* := \left\{ [u, v_1, v_2] \in D \;\middle|\; |[u, v_1, v_2]|_W \le 1 + S^\infty\sqrt{\nu_0^*\nu_1^*\nu_2^*} \right\}$. Then, it is not so difficult to check that the set given by

$$B^* := \overline{\text{conv}}\Big(\bigcup_{s\ge 0} E(s+1,s)B_0^* \cup S(1)B_0^*\Big)$$

is the required compact set, where $\overline{\text{conv}}(\,\cdot\,)$ is the closed convex hull of any subset in W. □

Proof of (i) of Theorem 3: By Lemma 1, the construction of the global attractor $\mathcal{A}_\infty$ for $\{S(t)\}$ is a direct application of the general theory in [7, 10]. Accordingly, the global attractor $\mathcal{A}_\infty$ is given as the ω-limit set of the absorbing set B^*, namely $\mathcal{A}_\infty := \omega_S(B^*) = \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau} S(t)B^*}^{W}$. □

In the rest, let B be any bounded subset in D, let ε be any positive number, and let T be any finite time. Then, by (4) and Proposition 1, there exists a positive number $t_0 = t_0(B, \varepsilon, T) \ge \frac{1}{2\varepsilon}$, depending only on B and ε, such that

$$\sup_{\tau\in[\frac{T}{2},T]} |E(\tau + t + s, s)z - S(\tau) \circ E(t+s,s)z|_W \le \frac{\varepsilon}{2}$$
$$\text{for any } s \ge 0,\ z \in B \text{ and } t \ge t_0(B, \varepsilon, T). \tag{13}$$

Let $\mathcal{A}_\infty$ be the global attractor for $\{S(t)\}$. Then, by the attractiveness of $\mathcal{A}_\infty$, we find a finite time $\tau_0 = \tau_0(B, \varepsilon) \ge \frac{1}{2\varepsilon}$, depending only on B and ε, such that

$$\text{dist}_W(S(\tau) \circ E(t+s,s)B, \mathcal{A}_\infty) \le \frac{\varepsilon}{2} \text{ for any } s \ge 0,\ t \ge 0 \text{ and } \tau \ge \tau_0.$$

So, by (13),

$$\sup_{\tau\in[\frac{\tau_0}{2},\tau_0]} |E(\tau+t+s,s)z - S(\tau)\circ E(t+s,s)z|_W \le \frac{\varepsilon}{2}$$
$$\text{for any } s\ge 0,\ z\in B \text{ and } t\ge t_0(B,\varepsilon,\tau_0).$$

Thus, putting $t_1(B,\varepsilon) := t_0(B,\varepsilon,\tau_0) = t_0(B,\varepsilon,\tau_0(B,\varepsilon))$, we obtain that

$$\begin{aligned} & \mathrm{dist}_W(E(\tau_0+t+s,s)z,\mathcal{A}_\infty) \\ \le\ & |E(\tau_0+t+s,s)z - S(\tau_0)\circ E(t+s,s)z|_W + \mathrm{dist}_W(S(\tau_0)\circ E(t+s,s)z,\mathcal{A}_\infty) \\ \le\ & \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \ \text{ for any } s\ge 0,\ z\in B \text{ and } t\ge t_1(B,\varepsilon). \end{aligned} \tag{14}$$

Proof of (ii) of Theorem 3: Let us assume that $z_\infty \in \omega_E(B)$. Then, we find sequences $\{t_n\},\{s_n\}\subset[0,+\infty)$ and $\{z_n\}\subset B$ such that $t_n \ge \tau_0\left(B,\frac{1}{n}\right)+t_1\left(B,\frac{1}{n}\right)\ge n$ for any $n\in\mathbb{N}$, and $E(t_n+s_n,s_n)z_n \to z_\infty$ in W as $n\to+\infty$. Here, by (14),

$$\mathrm{dist}_W(E(t_n+s_n,s_n)z_n,\mathcal{A}_\infty) \le \frac{1}{n} \ \text{ for any } n\in\mathbb{N}, \text{ so that}$$

$$\begin{aligned} \mathrm{dist}_W(z_\infty,\mathcal{A}_\infty) &\le |E(t_n+s_n,s_n)z_n - z_\infty|_W + \mathrm{dist}_W(E(t_n+s_n,s_n)z_n,\mathcal{A}_\infty) \\ &\le |E(t_n+s_n,s_n)z_n - z_\infty|_W + \frac{1}{n} \to 0 \ \text{ as } n\to+\infty. \end{aligned}$$

It implies that $z_\infty\in\mathcal{A}_\infty$, since $\mathcal{A}_\infty$ is compact in W. □

Proof of (iii) of Theorem 3: We shall show that putting $B_E := B^*$ gives a set with the required property, namely $\omega_E(B^*) = \mathcal{A}_\infty$. By the assertion (ii) of Theorem 3, we immediately have $\omega_E(B^*)\subset\mathcal{A}_\infty$. So, it is enough to show the converse inclusion. By Lemma 1, we find a finite time $t_{B^*}\ge 0$, depending only on B^*, such that

$$E(t+s,s)B^* \subset B^* \text{ for any } s\ge 0 \text{ and } t\ge t_{B^*}. \tag{15}$$

Let us assume that $z_\infty\in\mathcal{A}_\infty = \omega_S(B^*)$, equivalently there are sequences $\{t_n\}\subset(0,+\infty)$, $\{z_n\}\subset B^*$ such that

$$t_n \ge t_{B^*}+n \text{ for any } n\in\mathbb{N}, \text{ and } S(t_n)z_n\to z_\infty \text{ in } W \text{ as } n\to+\infty.$$

Now, it follows from Proposition 1 that for any $n\in\mathbb{N}$,

$$\sup_{\tau\in[t_n-t_{B^*},t_n]} |E(\tau+s,s)z_n - S(\tau)z_n|_W \to 0 \ \text{ as } s\to+\infty.$$

So, for any $n\in\mathbb{N}$, we find $s_n\ge n$ such that

$$|E(t_n+s_n,s_n)z_n - S(t_n)z_n|_W \le \frac{1}{n}. \tag{16}$$

On account of (15) and (16),

$$E(t_{B^*}+s_n,s_n)z_n \in B^* \ \text{ for any } n\in\mathbb{N}, \text{ and}$$

$$E(t_n+s_n,t_{B^*}+s_n)\circ E(t_{B^*}+s_n,s_n)z_n = E(t_n+s_n,s_n)z_n \to z_\infty$$
$$\text{in } W \text{ as } n\to+\infty,$$

which implies $z_\infty\in\omega_E(B^*)$. □

REFERENCES

[1] H. Brézis, M. G. Crandall and A. Pazy, Perturbations of nonlinear maximal monotone sets, Comm. Pure Appl. Math., **23** (1970), pp. 123-144.

[2] P. Colli, Mathematical study of an evolution problem describing the thermo-mechanical process in shape memory alloys, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. (9) Mat. Appl. **2** (1991), pp. 55-64.

[3] P. Colli, M. Frémond and A. Visintin, Thermo-mechanical evolution of shape memory alloys, Quart. Appl. Math., **48** (1990), pp. 31-47.

[4] P. Colli, Ph. Laurençot and U. Stefanelli, Long-time behavior for the full one-dimensional Frémond model for shape memory alloys, Contin. Mech. Thermodyn. **12** (2000), pp. 423-433.

[5] P. Colli and K. Shirakawa, Attractors for one-dimensional Frémond models of shape memory alloys, Preprint IAN-CNR, Pavia (2002).

[6] V. V. Chepyzhov and M. I. Vishik, Attractors of non-autonomous dynamical systems and their dimension, J. Math. Pures Appl. **73** (1994), pp. 279-333.

[7] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs **25**, Amer. Math. Soc., Providence, R. I. (1988).

[8] N. Kenmochi and M. Niezgódka, Evolution systems of nonlinear variational inequalities arising from phase change problems, Nonlinear Analysis **22**, No. 9 (1994), pp. 1163-1180.

[9] K. Shirakawa, A. Ito, N. Yamazaki and N. Kenmochi, Asymptotic stability for evolution equations governed by subdifferentials, *Recent Developments in Domain Decomposition Methods and Flow Problems,* GAKUTO Intern. Ser. Math. Sci. Appl. **11**, Gakkōtosho, Tokyo (1998), pp. 287-310.

[10] R. Temam, *Infinite Dimensional Systems in Mechanics and Physics*, Springer-Verlag, Berlin (1988).

Received September 2002; in revised March 2003.

E-mail address: `Kenboich@aol.com`