DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **7**, Number **2**, March **2007**

Website: http://AIMsciences.org

pp. **425–440**

EXPONENTIAL APPROXIMATIONS FOR THE PRIMITIVE EQUATIONS OF THE OCEAN

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Abstract. We show that in the limit of small Rossby number ε, the primitive equations of the ocean (OPEs) can be approximated by "higher-order quasi-geostrophic equations" up to an exponential accuracy in ε. This approximation assumes well-prepared initial data and is valid for a timescale of order one (independent of ε). Our construction uses Gevrey regularity of the OPEs and a classical method to bound errors in higher-order perturbation theory.

1. **Introduction.** We consider the primitive equations for the ocean (henceforth OPEs), scaled as in [13]

$$\begin{aligned}
&\partial_t \boldsymbol{v} + \frac{1}{\varepsilon}\big[\boldsymbol{v}^\perp + \boldsymbol{\nabla} p\big] + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{v} = \mu\Delta_3\boldsymbol{v} + f_{\boldsymbol{v}},\\
&\partial_t \rho - \frac{1}{\varepsilon} w + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho = \mu\Delta_3\rho + f_\rho,\\
&\boldsymbol{\nabla}\cdot\boldsymbol{u} = \boldsymbol{\nabla}\cdot\boldsymbol{v} + w_z = 0,\\
&\rho = -p_z.
\end{aligned} \tag{1}$$

Here $\boldsymbol{u} = (u, v, w)$ is the three-dimensional fluid velocity, with $\boldsymbol{v} = (u, v)$ its horizontal component and $\boldsymbol{v}^\perp = (-v, u)$; p is the pressure; ρ is the perturbation density (not including the mean stable stratification which figures into the ε in the equation for ρ). We write $\boldsymbol{\nabla}_2 := (\partial_x, \partial_y)$ and $\boldsymbol{\nabla}_3 := (\partial_x, \partial_y, \partial_z)$; when no ambiguity may arise, we simply write $\boldsymbol{\nabla}$. Similarly, we write $\Delta_2 := \partial_x^2 + \partial_y^2$ and $\Delta_3 := \partial_x^2 + \partial_y^2 + \partial_z^2$. The parameter ε is related to the Rossby and Froude numbers; in this article we shall be concerned with the limit $\varepsilon \to 0$, and for convenience we assume that $\varepsilon \le 1$ (further restrictions on ε will be stated below). In general the viscosity coefficients for $\boldsymbol{v}$ and ρ are different; we have set them all to μ for clarity of presentation (the general case does not introduce any more essential difficulty). The forcings $f_{\boldsymbol{v}}$ and f_ρ are assumed to be independent of time.

We work in three spatial dimensions, $\boldsymbol{x} = (x, y, z) \in [0, L_1]\times[0, L_2]\times[-L_3/2, L_3/2] =: \mathcal{M}$, with periodic boundary conditions assumed. Following common practice in

2000 *Mathematics Subject Classification.* Primary: 35B25, 76U05.

Key words and phrases. Singular perturbation, exponential asymptotics, Gevrey regularity, primitive equations.

This research was supported by grants NSF 0305110, DOE DE–FG02–01ER63251:A000 and the Research Fund of Indiana University.