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STRONGLY DAMPED WAVE EQUATIONS IN $W_0^{1,p}(\Omega) \times L^p(\Omega)$

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Abstract. This paper is devoted to well posedness and regularity of the solutions of

$$u_{tt} + 2\eta A^{\frac{1}{2}} u_t + Au = f(u)$$

in $W_0^{1,p}(\Omega) \times L^p(\Omega)$, $p \in (1, \infty)$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\eta > 0$ and $-A$ is the Dirichlet Laplacian in $L^p(\Omega)$. We prove local well posedness result for nonlinearities $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(s) - f(t)| \leq C|s - t|(1 + |s|^{\rho-1} + |t|^{\rho-1})$ with $\rho < \frac{N+p}{N-p}$ $(N > p)$, and show that the solutions are classical. If f is dissipative and $\rho < \frac{N+2}{N-2}$ $(N \geq 3)$, we show that the associated semigroup has a global attractor $\mathcal{A}_{\eta,p}$ in $W_0^{1,p}(\Omega) \times L^p(\Omega)$, $p \in [2, \infty)$, which coincides with the attractor $\boldsymbol{A}_{\eta,2} =: \boldsymbol{A}_\eta$. We also obtain that $\boldsymbol{A}_\eta$ is compact in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ and attracts bounded subsets of $H_0^1(\Omega) \times L^2(\Omega)$ in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for each $\mu \in (0, 1)$.

1. **Introduction.** We discuss well posedness and regularity of solutions of semilinear strongly damped wave problems of the form

$$\begin{cases} u_{tt} + 2\eta A^{\frac{1}{2}} u_t + Au = f(u), \ t > 0, \ x \in \Omega, \\ u(0, x) = u_0(x), \ u_t(0, x) = v_0(x), \ x \in \Omega, \\ u(t, x) = 0, \ t \geq 0, \ x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $-A$ is the Dirichlet Laplacian in $L^p(\Omega)$, $p \in (1, \infty)$, with domain $D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $\eta > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying appropriate growth restrictions.

The strongly damped wave equations have been extensively studied in the Hilbert setting and much progress has been achieved. In [7, 8, 9] analyticity of a semigroup generated by the strongly damped wave operator $-\mathcal{A}_\eta = \begin{bmatrix} 0 & I \\ -A & -2\eta A^{1/2} \end{bmatrix}$ in $H_0^1(\Omega) \times L^2(\Omega)$ was proved and the corresponding fractional power spaces were characterized with the aid of the fractional power spaces X_2^α associated to A in $L^2(\Omega)$.

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It was proved in [5] that the problem (1) with (critical) nonlinearity $f(u) \sim |u|^{\frac{N+2}{N-2}}$ $(N \geq 3)$ is locally well posed in $H_0^1(\Omega) \times L^2(\Omega)$. In these results, which included also equations with the damping terms $A^\theta u_t$ $(\theta \in [\frac{1}{2}, 1])$ as well as nonlinearities depending on (u, u_t), the notion of an ε-regular solution (see [3, Definition 1]) was essentially exploited.

In [6] it was proved that, for dissipative nonlinearities $f \sim |u|^{\frac{N+2}{N-2}^-}$, the nonlinear semigroup associated to (1) has a global attractor in $H_0^1(\Omega) \times L^2(\Omega)$.

In this paper we consider the problem (1) in $W_0^{1,p}(\Omega) \times L^p(\Omega)$, $p \in (1, \infty)$. We show that the local solutions exist for nonlinearities satisfying

$$|f(s_1) - f(s_2)| \leq c|s_1 - s_2|(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}), \ s_1, s_2 \in \mathbb{R}^1, \tag{2}$$

with $1 < \rho < \frac{N+p}{N-p}$ $(N > p > 1)$ and we prove that they are classical solutions. When f satisfies (2) with $1 < \rho < \frac{N+2}{N-2}$ $(N \geq 3)$, we show that the attractors $\boldsymbol{A}_{\eta,p}$ for the semigroups $\{T_\eta(t)\}$ associated with (1) in $W_0^{1,p}(\Omega) \times L^p(\Omega)$, $p \in [2, \infty)$, coincide with the attractor $\boldsymbol{A}_{\eta,2} =: \boldsymbol{A}_\eta$ obtained for the semigroup associated with (1) in $H_0^1(\Omega) \times L^2(\Omega)$. We also prove that $\boldsymbol{A}_\eta$ is compact in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ and attracts bounded subsets of $H_0^1(\Omega) \times L^2(\Omega)$ in the norm of $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for every $\mu \in (0, 1)$.

Related to the previous works [5, 6, 7], the above mentioned results greatly extend the knowledge about existence, regularity, and asymptotic behavior of the solutions to (1). The fact that $\mathcal{A}_\eta$ is sectorial in $W_0^{1,p}(\Omega) \times L^p(\Omega)$ for each $p \in (1, \infty)$ is crucial here. This fact cannot be located in the existing literature for $p \neq 2$.

The article is organized in a following way. In Section 2 we prove that $\mathcal{A}_\eta$ is sectorial in $W_0^{1,p}(\Omega) \times L^p(\Omega)$ for each $p \in (1, \infty)$. In Section 3 we obtain suitable embedding properties of the fractional power spaces associated to $\mathcal{A}_\eta$ and show that (1) is locally well posed in $W_0^{1,p}(\Omega) \times L^p(\Omega)$, $p \in (1, \infty)$. In Section 4 we present a 'bootstrapping' procedure for (1). In Section 5 we show global well posedness of (1) in $W_0^{1,p}(\Omega) \times L^p(\Omega)$, $p \in [2, \infty)$, as well as the existence and regularity of the global attractors. In the Appendix we prove a result on regularization of solutions of abstract parabolic equations uniform with respect to initial data in bounded sets. This result is essentially used in Section 4.

We remark that if instead of $2\eta A^{\frac{1}{2}} u_t$ a more general damping term $Bu_t := 2\eta A^\theta u_t + a u_t$ with $\theta \in [\frac{1}{2}, 1]$, $\eta > 0$ and $a > 0$ is considered in (1), the results on generation of analytic semigroup, local well posedness and bootstrapping will still hold but the analysis presented here needs to be adapted.

2. Linear strongly damped wave equations and analytic C^0 semigroups in $\mathbf{X}_\mathbf{p}^{\frac{1}{2}} \times \mathbf{X}_\mathbf{p}$, $\mathbf{p} \in (\mathbf{1}, \infty)$.

Let $\{X_p^\alpha, \alpha \in \mathbb{R}\}$ denote the fractional power scale generated by (A, X_p) where $X_p = L^p(\Omega)$ and $p \in (1, \infty)$. Recall that A has bounded imaginary powers (see [17]) and that spaces X_p^α, $\alpha \in (0, 1)$, $p > 1$, are characterized with the aid of complex interpolation as $[L^p(\Omega), H_{p,\{I\}}^2(\Omega)]_\alpha = H_{p,\{I\}}^{2\alpha}(\Omega) = X_p^\alpha$ (see [19]).

For $\eta > 0$ consider the strongly damped wave operator,

$$\mathcal{A}_\eta : D_p(A_\eta) \subset X_p^{\frac{1}{2}} \times X_p \to X_p^{\frac{1}{2}} \times X_p, \quad \mathcal{A}_\eta = \begin{bmatrix} 0 & -I \\ A & 2\eta A^{\frac{1}{2}} \end{bmatrix}, \quad D_p(\mathcal{A}_\eta) = X_p^1 \times X_p^{\frac{1}{2}},$$

where $p \in (1,\infty)$ and $\left\| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\|_{X_p^{\frac{1}{2}} \times X_p} = \|\phi\|_{X_p^{\frac{1}{2}}} + \|\psi\|_{X_p}$ is the norm in $X_p^{\frac{1}{2}} \times X_p$ equivalent to the 'usual' $W_0^{1,p}(\Omega) \times L^p(\Omega)-$norm.

Lemma 1. *The following conditions hold.*

i) $0 \in \rho(\mathcal{A}_\eta)$ *and* $\mathcal{A}_\eta$ *in* $X_p^{\frac{1}{2}} \times X_p$ *has compact resolvent.*

ii) *The spectrum of* $\mathcal{A}_\eta$ *in* $X_p^{\frac{1}{2}} \times X_p$ *consists of isolated eigenvalues* $\lambda_n^{\pm}$ *given by*

$$\lambda_n^{\pm} = (\eta \pm \sqrt{\eta^2 - 1})\sqrt{\mu_n}, \ \mu_n \in \sigma(A). \tag{3}$$

iii) *The operator* $-\mathcal{A}_\eta$ *generates in* $X_p^{\frac{1}{2}} \times X_p$ *a* C^0 *analytic semigroup* $\{e^{-\mathcal{A}_\eta t}\}$.

iv) *All the semigroups* $\{e^{-\mathcal{A}_\eta t}\}$ *in* $X_p^{\frac{1}{2}} \times X_p$ *are compact and asymptotically decaying.*

v) *For each* $\eta > 0$ *and* $\begin{bmatrix} \phi \\ \psi \end{bmatrix} \in X_p^1 \times X_p^{\frac{1}{2}}$

$$(1+2\eta)^{-1} \left\| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\|_{X_p^1 \times X_p^{\frac{1}{2}}} \le \left\| \mathcal{A}_\eta \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\|_{X_p^{\frac{1}{2}} \times X_p} \le (1+2\eta) \left\| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\|_{X_p^1 \times X_p^{\frac{1}{2}}}. \tag{4}$$

Proof. For the proof of i) simply note that

$$\mathcal{A}_\eta^{-1} : X_p^{\frac{1}{2}} \times X_p \to X_p^{\frac{1}{2}} \times X_p, \quad \mathcal{A}_\eta^{-1} = \begin{bmatrix} 2\eta A^{-\frac{1}{2}} & A^{-1} \\ -I & 0 \end{bmatrix}. \tag{5}$$

Part ii) is a consequence of i) and [8, Lemma A.1].

In part iii) it is sufficient to show sectorial property of $\mathcal{A}_\eta = \begin{bmatrix} 0 & -I \\ A & 2\eta A^{\frac{1}{2}} \end{bmatrix}$, $\eta > 0$. Following ii), every complex number λ with $Re\lambda \le 0$ is in the resolvent set of $\mathcal{A}_\eta$, $\eta > 0$. We then have

$$\begin{aligned} &(\lambda I - \mathcal{A}_\eta)^{-1} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \lambda I & I \\ -A & \lambda I - 2\eta A^{\frac{1}{2}} \end{bmatrix}^{-1} \begin{bmatrix} \phi \\ \psi \end{bmatrix} \\ &= \begin{bmatrix} \lambda I - 2\eta A^{\frac{1}{2}} & -I \\ A & \lambda I \end{bmatrix} (\lambda^2 I - 2\eta\lambda A^{\frac{1}{2}} + A)^{-1} \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \quad Re(\lambda) \le 0, \ \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in X_p^{\frac{1}{2}} \times X_p. \end{aligned} \tag{6}$$

Focusing first on the case $\eta \in (0,1)$ we remark that $\eta \pm i\sqrt{1-\eta^2} = e^{\pm i\theta_\eta}$, $\theta_\eta \in (0, \frac{\pi}{2})$,

$$\lambda^2 I - 2\eta\lambda A^{\frac{1}{2}} + A = (\lambda e^{i\theta_\eta} I - A^{\frac{1}{2}})(\lambda e^{-i\theta_\eta} I - A^{\frac{1}{2}}), \tag{7}$$

and that $\lambda e^{\pm i\theta_\eta} \in \rho(A^{\frac{1}{2}})$ whenever $Re(\lambda) \le 0$ because $\sigma(A^{\frac{1}{2}})$ consists of strictly positive eigenvalues separated from zero. Thus for each $Re(\lambda) \le 0$ we get

$$(\lambda^2 I - 2\eta\lambda A^{\frac{1}{2}} + A)^{-1} = (\lambda e^{i\theta_\eta} I - A^{\frac{1}{2}})^{-1}(\lambda e^{-i\theta_\eta} I - A^{\frac{1}{2}})^{-1}. \tag{8}$$

and $\frac{\pi}{2} - \theta_\eta \le |\arg\big(\lambda e^{\pm i\theta_\eta}\big)| \le \pi$, where $\theta_\eta \to \frac{\pi}{2}^-$ as $\eta \to 0^+$.

Consider a sector

$$\mathcal{S}_{\theta_\eta} = \{\tilde{\lambda} \in \mathbb{C} : \frac{\pi}{2} - \theta_\eta \le |\arg(\tilde{\lambda})| \le \pi, \tilde{\lambda} \ne 0\}$$

contained in the resolvent set of $A^{\frac{1}{2}}$ in X_p, for which[3]

$$\|(\tilde{\lambda} I - A^{\frac{1}{2}})^{-1}\|_{L(X_p)} \le \frac{M_{\theta_\eta}}{|\tilde{\lambda}|}, \quad \tilde{\lambda} \in \mathcal{S}_{\theta_\eta}. \tag{9}$$

[3]We remark that condition (9) is true according to [18, Theorem 2] and [16].

First note with the aid of (8), (9) that

$$\begin{aligned}\|(\lambda^2 I - 2\eta\lambda A^{\frac{1}{2}} + A)^{-1}\phi\|_{X_p^{\frac{1}{2}}} =& \|(\lambda e^{i\theta_\eta} I - A^{\frac{1}{2}})^{-1}(\lambda e^{-i\theta_\eta} I - A^{\frac{1}{2}})^{-1} A^{\frac{1}{2}}\phi\|_{X_p} \\ \leq& \frac{M_{\theta_\eta}^2}{|\lambda|^2}\|A^{\frac{1}{2}}\phi\|_{X_p} = \frac{M_{\theta_\eta}^2}{|\lambda|^2}\|\phi\|_{X_p^{\frac{1}{2}}}, \quad Re(\lambda) < 0,\end{aligned} \tag{10}$$

and

$$\begin{aligned}&\|A^{\frac{1}{2}}(\lambda^2 I - 2\eta\lambda A^{\frac{1}{2}} + A)^{-1}\phi\|_{X_p^{\frac{1}{2}}} \\ =&\|A^{\frac{1}{2}}(\lambda e^{i\theta_\eta} I - A^{\frac{1}{2}})^{-1}(\lambda e^{-i\theta_\eta} I - A^{\frac{1}{2}})^{-1}\phi\|_{X_p^{\frac{1}{2}}} \\ =&\|(I + \lambda e^{i\theta_\eta}(A^{\frac{1}{2}} - \lambda e^{i\theta_\eta} I)^{-1})(A^{\frac{1}{2}} - \lambda e^{-i\theta_\eta} I)^{-1}\phi\|_{X_p^{\frac{1}{2}}} \\ \leq& \frac{M_{\theta_\eta} + M_{\theta_\eta}^2}{|\lambda|}\|\phi\|_{X_p^{\frac{1}{2}}}, \quad Re(\lambda) < 0.\end{aligned} \tag{11}$$

This gives the desired estimate for the upper left corner of the matrix in (6). The remaining three components can be estimated with the aid of (9), (10) and (11).

This leads to

$$\|(\lambda I - \mathcal{A}_\eta)^{-1}\|_{L(X_p^{\frac{1}{2}} \times X_p)} \leq \frac{\hat{\mathcal{M}}_{\theta_\eta}}{|\lambda|} \quad \text{for} \quad Re(\lambda) < 0, \tag{12}$$

which proves iii) in the case $\eta \in (0, 1)$.

When $\eta \geq 1$ instead of (8) and (9) we use the relations

$$(\lambda^2 I - 2\eta\lambda A^{\frac{1}{2}} + A)^{-1} = (\lambda(\eta + \sqrt{\eta^2 - 1})I - A^{\frac{1}{2}})^{-1}(\lambda(\eta - \sqrt{\eta^2 - 1})I - A^{\frac{1}{2}})^{-1},$$

and

$$\|(\tilde{\lambda} I - A^{\frac{1}{2}})^{-1}\|_{L(X_p)} \leq \frac{M_0}{|\tilde{\lambda}|}, \ Re(\tilde{\lambda}) < 0.$$

The required estimates are derived in a similar manner as in the formulas (10)-(12).

Part iv) follows from i), ii) and iii).

Concerning part v) note that

$$\begin{aligned}\left\|\mathcal{A}_\eta^{-1}\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right\|_{X_p^1 \times X_p^{\frac{1}{2}}} =& \left\|\begin{bmatrix}2\eta A^{-\frac{1}{2}}\phi + A^{-1}\psi\\ -\phi\end{bmatrix}\right\|_{X_p^1 \times X_p^{\frac{1}{2}}} \\ \leq& 2\eta\|A^{\frac{1}{2}}\phi\|_{X_p} + \|\psi\|_{X_p} + \|A^{\frac{1}{2}}\phi\|_{X_p} \\ \leq& (1 + 2\eta)\left\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right\|_{X_p^{\frac{1}{2}} \times X_p},\end{aligned} \tag{13}$$

which leads to the estimate

$$\left\|\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right\|_{X_p^1 \times X_p^{\frac{1}{2}}} \leq (1 + 2\eta)\left\|\mathcal{A}_\eta\begin{bmatrix}\phi\\ \psi\end{bmatrix}\right\|_{X_p^{\frac{1}{2}} \times X_p}, \ \begin{bmatrix}\phi\\ \psi\end{bmatrix} \in X_p^1 \times X_p^{\frac{1}{2}}. \tag{14}$$

This proves the left hand side inequality in (4). The proof of the right hand side inequality is similar (see [4, Proposition 2.1 iv)] for details). □

Remark 1. We note that for $p \neq 2$ the estimates in part iii) degenerate as $\eta \to 0$. This agrees with the fact that the operator $\mathcal{A}_0$ does not generate a C^0 semigroup in $W_0^{1,p}(\Omega) \times L^p(\Omega)$ unless $p = 2$ (see [15]).

3. **Local well posedness of** (15) **in** $X_p^{\frac{1}{2}} \times X_p$**.** With the set-up as described above we will consider further (1) in the form

$$\frac{d}{dt}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right),\ t > 0, \quad \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in X_p^{\frac{1}{2}} \times X_p, \tag{15}$$

where $\eta > 0$ and $F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 \\ f(u) \end{smallmatrix}\right]$.

Lemma 2. *Let $p \in (1, \infty)$ and let Y_p denote the extrapolation space of $X_p^{\frac{1}{2}} \times X_p$ generated by $\mathcal{A}_\eta$, $\eta > 0$. Then,*

$$Y_p = X_p \times X_p^{-\frac{1}{2}}. \tag{16}$$

Proof. Following [2] we recall that the extrapolation space of $X_p^{\frac{1}{2}} \times X_p$ is the completion of the normed space $(X_p^{\frac{1}{2}} \times X_p, \|\mathcal{A}_\eta^{-1} \cdot\|_{X_p^{\frac{1}{2}} \times X_p})$. Since for $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in X_p^{\frac{1}{2}} \times X_p$

$$\left\|\mathcal{A}_\eta^{-1}\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{X_p^{\frac{1}{2}} \times X_p} \le (1+2\eta)\left\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{X_p \times X_p^{-\frac{1}{2}}} \le (1+2\eta)^2 \left\|\mathcal{A}_\eta^{-1}\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{X_p^{\frac{1}{2}} \times X_p},$$

completions of $(X_p^{\frac{1}{2}} \times X_p, \|\mathcal{A}_\eta^{-1} \cdot\|_{X_p^{\frac{1}{2}} \times X_p})$ and $(X_p^{\frac{1}{2}} \times X_p, \|\cdot\|_{X_p \times X_p^{-\frac{1}{2}}})$ coincide. □

Consider the closed extension of $\mathcal{A}_\eta$ to Y_p (see [2, p. 262]) and still denote it by $\mathcal{A}_\eta$. Then $\mathcal{A}_\eta$ is a sectorial and positive operator in Y_p with the domain $Y_p^1 = X_p^{\frac{1}{2}} \times X_p$ (see [5]).

Lemma 3. *Fix $\eta > 0$, $p \in (1, \infty)$ and denote by $\{Y_p^\alpha, \alpha \ge 0\}$ the fractional power scale generated by $(\mathcal{A}_\eta, Y_p)$. Then*

$$Y_p{}^\theta \hookleftarrow X_p^{\frac{\sigma}{2}} \times X_p^{\frac{\sigma-1}{2}},\ p \in (1, \infty),\ 0 \le \theta < \sigma \le 1, \tag{17}$$

and

$$Y_p{}^{1+\gamma} \hookrightarrow X_p^{\frac{1+\alpha}{2}} \times X_p^{\frac{\alpha}{2}},\ p \in (1, \infty),\ 0 \le \alpha < \gamma \le 1. \tag{18}$$

Proof. Note that $\left[\begin{smallmatrix} A^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{smallmatrix}\right] : Y_p^1 \subset Y_p \to Y_p$ is sectorial and, for $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in X_p^1 \times X_p^{\frac{1}{2}}$, $\theta \in (0, 1)$, we have

$$\begin{aligned}
\left\|\mathcal{A}_\eta{}^\theta \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{Y_p} &\le c\left\|\mathcal{A}_\eta \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{Y_p}^\theta \left\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{Y_p}^{1-\theta} = c\left\|\left[\begin{smallmatrix} -\psi \\ A\phi + 2\eta A^{\frac{1}{2}}\psi \end{smallmatrix}\right]\right\|_{Y_p}^\theta \left\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{Y_p}^{1-\theta} \\
&\le \hat{c}(\|A^{\frac{1}{2}}\phi\|_{X^p} + \|\psi\|_{X^p})^\theta \left\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{Y_p}^{1-\theta} = \hat{c}\left\|\left[\begin{smallmatrix} A^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{smallmatrix}\right]\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{Y_p}^\theta \left\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{Y_p}^{1-\theta}.
\end{aligned}$$

Hence, using [14, Exercise 1.4.11] (see also [11, Lemma 17.1]), we obtain (17) from

$$\left\|\mathcal{A}_\eta{}^\theta \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{Y_p} \le C\left\|\left[\begin{smallmatrix} A^{\frac{\sigma}{2}} & 0 \\ 0 & A^{\frac{\sigma}{2}} \end{smallmatrix}\right]\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{Y_p} = C\left\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{X_p^{\frac{\sigma}{2}} \times X_p^{\frac{\sigma-1}{2}}},\ \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in X_p^1 \times X_p^{\frac{1}{2}}.$$

To prove (18) note that, for $\alpha \in (0, 1)$,

$$\begin{aligned}
\left\|\left[\begin{smallmatrix} A^{\frac{\alpha}{2}} & 0 \\ 0 & A^{\frac{\alpha}{2}} \end{smallmatrix}\right]\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{X_p^{\frac{1}{2}} \times X_p} &\le \|A^{\frac{1}{2}}\phi + 2\eta\psi\|_{X_p^{\frac{\alpha}{2}}} + \tilde{c}\|\psi\|_{X_p^{\frac{\alpha}{2}}} \\
&\le \tilde{C}\|A^{\frac{1}{2}}\phi + 2\eta\psi\|_{X_p^{\frac{1}{2}}}^\alpha \|A^{\frac{1}{2}}\phi + 2\eta\psi\|_{X_p}^{1-\alpha} + \tilde{C}\|\psi\|_{X_p^{\frac{1}{2}}}^\alpha \|\psi\|_{X_p}^{1-\alpha} \\
&\le \bar{C}\left\|\mathcal{A}_\eta \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{X_p^{\frac{1}{2}} \times X_p}^\alpha \left\|\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\right\|_{X_p^{\frac{1}{2}} \times X_p}^{1-\alpha}, \quad \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in X_p^1 \times X_p^{\frac{1}{2}}.
\end{aligned}$$

From this and [14, Exercise 1.4.11] (see also [11, Lemma 17.1]) we have

$$\left\| \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\|_{X_p^{\frac{\alpha+1}{2}} \times X_p^{\frac{\alpha}{2}}} = \left\| \begin{bmatrix} A^{\frac{\alpha}{2}} & 0 \\ 0 & A^{\frac{\alpha}{2}} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\|_{X_p^{\frac{1}{2}} \times X_p} \leq \hat{C} \left\| \mathcal{A}_\eta^\gamma \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right\|_{X_p^{\frac{1}{2}} \times X_p}, \ \gamma \in (\alpha, 1),$$

which proves (18). This concludes the proof. □

Theorem 1. *The problem* (15) *is locally well posed in* Y_p^1 *if*

i) $p > N$,
ii) $1 < p < N$ *and* f *satisfies* (2) *for some* $1 < \rho < \frac{N+p}{N-p}$, $c > 0$,
iii) $1 < p = N$ *and* f *satisfies* (2) *for some* $\rho \in (1, \infty)$.

Proof. If condition i) is satisfied, then the Nemytskiĭ map F is Lipschitz continuous on bounded sets from Y_p^1 into Y_p^1 as a consequence of the embedding $X_p^{\frac{1}{2}} \hookrightarrow L^\infty(\Omega)$.

When ii) holds, F is Lipschitz continuous on bounded sets from Y_p^1 into $X_p^{\frac{\sigma}{2}} \times X_p^{\frac{\sigma-1}{2}} \hookrightarrow Y_p^\theta$ whenever $0 < \theta < \sigma \leq \tilde{\sigma}_p$ and $\tilde{\sigma}_p = \min\{1, \frac{N}{p} + 1 - \rho(\frac{N}{p} - 1)\}$. Thus, if B is a bounded subset of Y_p^1 and $\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} \in B$, we have

$$\begin{aligned}
&\left\| F\left(\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}\right) - F\left(\begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix}\right) \right\|_{Y_p^\theta} \leq c_1 \left\| F\left(\begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}\right) - F\left(\begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix}\right) \right\|_{X_p^{\frac{\sigma}{2}} \times X_p^{\frac{\sigma-1}{2}}} \\
&= c_1 \|f(\phi_1) - f(\phi_2)\|_{X_p^{\frac{\sigma-1}{2}}} \leq c_2 \|f(\phi_1) - f(\phi_2)\|_{L^{\frac{Np}{N+(1-\sigma)p}}(\Omega)} \\
&\leq c_3 \|\phi_1 - \phi_2\|_{X_p^{\frac{1}{2}}} \left(1 + \|\phi_1\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2-\sigma}}(\Omega)} + \|\phi_2\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2-\sigma}}(\Omega)}\right) \leq c_4 \left\| \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} - \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} \right\|_{Y_p^1}.
\end{aligned}$$

In the case iii), $X_p^{\frac{1}{2}} \hookrightarrow L^q(\Omega)$ for each $q \in [1, \infty)$ and $F : Y_p^1 \to Y_p^1$ is Lipschitz continuous on bounded sets.

The proof now follows from Theorem 3.3.3 in [14]. □

4. Uniform bootstrapping.

In what follows, the regularizing properties of the local solutions will be discussed.

Theorem 2. *Under the hypothesis of Theorem 1, for each* $B \subset Y_p^1$ *bounded, there exists* $\tau = \tau(B) > 0$ *such that the solution* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])$ *of* (15) *exists until the time* τ *for all* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B$. *In addition, for each* $\mu \in (0, 1)$, $t \in (0, \tau)$, *the set* $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]) ; \ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B\}$ *is bounded in* $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$.

Proof. Suppose that $N > p$. Following the proof of Theorem 1 observe that $F : Y_p^1 \to Y_p^\alpha$ is Lipschitz continuous on bounded sets whenever $0 < \alpha < \tilde{\sigma}_p$. From Theorem 5 there exists $\tau_0 = \tau_0(B) > 0$ such that the sets

$$B_\epsilon = \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]) ; \ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B, \ t \in [\epsilon, \tau_0]\}, \ \dot{B}_\epsilon = \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_t (t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]) ; \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B, \ t \in [\epsilon, \tau_0]\}$$

are bounded in $Y_p^{1+\alpha^-}$. From (18) these sets are bounded in $X_p^{\frac{1+\beta}{2}} \times X_p^{\frac{\beta}{2}}$, $\beta \in (0, \alpha)$.

If $N > \alpha p$, then B_ϵ, $\dot{B}_\epsilon$ are bounded in $Y_{p_1}^1$ where $p_0^* := p \leq p_1 < \frac{Np}{N-\alpha p} =: p_1^*$. Note that $F : Y_{p_1}^1 \to Y_{p_1}^\alpha$ is Lipschitz continuous on bounded sets, where α is the same as before since $\tilde{\sigma}_p \leq \tilde{\sigma}_{p_1}$. The problem (15) can be considered in $Y_{p_1}^1$ and from Theorem 5 the sets B_ϵ, $\dot{B}_\epsilon$ are bounded in $Y_{p_1}^{1+\alpha^-}$. From (18), these sets are bounded in $X_{p_1}^{\frac{1+\beta}{2}} \times X_{p_1}^{\frac{\beta}{2}}$ with $\beta \in (0, \alpha)$ and $p_1 \in [p, p_1^*)$.

If $N > \alpha p_1^*$, then $N > 2\alpha p$ and B_ϵ, $\dot{B}_\epsilon$ are bounded in $Y_{p_2}^1$ with $p_1^* \leq p_2 < \frac{Np}{N-2\alpha p} =: p_2^*$.

Certainly, there is an integer k such that $N \leq \alpha p_k^*$ and B_ϵ, $\dot{B}_\epsilon$ are bounded in $Y^1_{\tilde{q}}$ for arbitrarily large $\tilde{q} \in (N, \infty)$. Also, for each choice of $\tilde{q} \in (N, \infty)$, map $F : Y^1_{\tilde{q}} \to Y^1_{\tilde{q}}$ is Lipschitz continuous on bounded sets. Applying Theorem 5 in $Y^1_{\tilde{q}}$, we conclude that B_ϵ and $\dot{B}_\epsilon$ are bounded in $Y^{2^-}_{\tilde{q}}$. According to (18) B_ϵ and $\dot{B}_\epsilon$ are bounded in $X^{1^-}_{\tilde{q}} \times X^{\frac{1}{2}^-}_{\tilde{q}} \hookrightarrow C^{1+\mu}(\bar{\Omega}) \times C^{\mu}(\bar{\Omega})$ with certain $\mu \in (0,1)$. Since $\Delta u = u_{tt} + 2\eta A^{\frac{1}{2}} u_t - f(u)$, Schauder type estimate (see [12, Theorems 9.19, 6.6]) implies boundedness of $\left\{\phi : \left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] \in B_\epsilon\right\}$ in $C^{2+\mu}(\bar{\Omega})$. Note that μ can be chosen arbitrarily close to 1 since $\tilde{q}$ can be taken as large as required. □

Corollary 1. *Under the assumptions of Theorem 1 the solutions of* (15) *are classical and* $\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t, \left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]) \in C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$, $\mu \in (0,1)$, *as long as it exists.*

5. **Global solutions to** (15) **and a family of global attractors.** In this section we prove global well posedness for (15) and show the existence of a global attractor (see [13, p. 39]). To that end we assume the dissipativeness condition

$$\limsup_{|s|\to\infty} \tfrac{f(s)}{s} < \lambda_1, \tag{19}$$

where λ_1 is the first positive eigenvalue of A.

Theorem 3. *Assume that $N = 1$ or that $N = 2$ and* (2) *is satisfied for some $\rho \in (1, \infty)$ or that $N \geq 3$ and* (2) *holds with $1 < \rho < \frac{N+2}{N-2}$. If* (19) *is satisfied then,*

i) *the problem* (15) *is globally well posed in Y^1_p, $p \in [2, \infty)$,*
ii) *positive orbits of bounded subsets of Y^1_p, $p \in [2, \infty)$, are eventually bounded in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for each $\mu \in (0,1)$.*

Proof. The energy functional $\mathcal{L} : Y^1_2 \to \mathbb{R}$ defined by

$$\mathcal{L}(\left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right]) = \frac{1}{2}\|w_1\|^2_{X_2^{\frac{1}{2}}} + \frac{1}{2}\|w_2\|^2_{X_2} - \int_\Omega \int_0^{w_1} f(s)dsdx, \ \left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right] \in Y^1_p, \tag{20}$$

satisfies

$$\mathcal{L}(\left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right]) \geq \frac{\varepsilon}{2\lambda_1}\|w_1\|^2_{X_2^{\frac{1}{2}}} + \frac{1}{2}\|w_2\|^2_{X_2} - C_\varepsilon$$

for some $\varepsilon > 0$, $C_\varepsilon > 0$, decreases along $\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t, \left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right])$ and

$$\| \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t, \left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]) \|_{Y^1_2} \leq c_{\mathcal{L}} \sqrt{1 + \mathcal{L}(\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right])}. \tag{21}$$

Consequently, the solution is bounded in Y^1_2 as long as it exists.

From Theorem 1 the problem (15) is locally well posed in Y^1_2 and (21) implies that all the solutions (in the Y^1_2-setting) are defined on $[0, \infty)$. From (21), orbits of bounded subsets of Y^1_2 are bounded in Y^1_2 and the proof follows from Theorem 2. □

Corollary 2. *If $\mathcal{E}$ is the set of stationary solutions for* (15), $\mathcal{E}_1 = \{u_0 : \left[\begin{smallmatrix}u_0\\ 0\end{smallmatrix}\right] \in \mathcal{E}\}$ *and $\mu \in (0,1)$, then the set $\mathcal{E}_1$ (which is independent of η) is bounded in $C^{2+\mu}(\bar{\Omega})$.*

Theorem 4. *Under the assumptions of Theorem 3, for $p \geq 2$, we have that:*

i) *problem* (15) *defines in Y^1_p compact C^0 semigroups $\{T_\eta(t)\}$, $\eta > 0$, of the global solutions with bounded orbits of bounded sets,*
ii) *for each $\eta > 0$ the semigroup $\{T_\eta(t)\}$ in Y^1_p possesses a global attractor $\boldsymbol{A}_{\eta,p}$,*
iii) *the attractors $\boldsymbol{A}_{\eta,p}$, $\eta > 0$, $p \in [2, \infty)$ are independent of p; namely,*

$$\boldsymbol{A}_{\eta,p} = \boldsymbol{A}_{\eta,2} =: \boldsymbol{A}_\eta \quad \text{for each} \quad p \in [2, \infty),$$

iv) *$\boldsymbol{A}_\eta$ is compact in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for every $\mu \in (0,1)$, $\eta > 0$,*

v) *if B is a bounded subset of Y_2^1 and $\mu \in (0,1)$, then B is attracted by $\boldsymbol{A}_\eta$, $\eta > 0$, in the sense of the Hausdorff semidistance in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$.*

Proof. Part i) follows from Theorem 3 and Lemma 1.

Part ii) comes back to the theory of dissipative system [13] and is a consequence of i), Corollary 2 and the properties of the functional $\mathcal{L}$ defined in (20).

In the light of Theorem 2 and the invariance property of the attractors, each set $\boldsymbol{A}_{\eta,p}$ is bounded in $C^{2+\tilde{\mu}}(\bar{\Omega}) \times C^{1+\tilde{\mu}}(\bar{\Omega}) \hookrightarrow C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$, $0 < \mu < \tilde{\mu} < 1$. Since $\boldsymbol{A}_{\eta,p}$ and $\boldsymbol{A}_{\eta,2}$ attracts one another, they in fact coincide, which proves iii). Since the above mentioned embedding among spaces of Hölder functions is compact (see [1, Theorem 1.31]), condition iv) is true as well.

If B is bounded in Y_2^1 and $\mu \in (0,1)$, then for $\tilde{\mu} \in (\mu, 1)$ the orbit of B is eventually bounded in $C^{2+\tilde{\mu}}(\bar{\Omega}) \times C^{1+\tilde{\mu}}(\bar{\Omega})$ and therefore eventually precompact in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$. Thus, $\boldsymbol{A}_\eta$ attracts B in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})-$norm. □

6. Appendix: Uniformly regularizing properties of the local solutions of abstract parabolic equations.

In this section we derive estimates of the solutions of abstract parabolic equations uniform with respect to initial conditions varying in bounded sets. These estimates complement the results of [14].

Theorem 5. *Let Y be a Banach space, $\mathcal{A} : D(\mathcal{A}) \subset Y \to Y$ be a sectorial operator with* Re $\sigma(\mathcal{A}) > 0$, $0 \le \alpha < 1$, *and $f : Y^\alpha \to Y$ be Lipschitz continuous in bounded sets. Then, for each bounded set $B \subset Y^\alpha$ there exists $\tau_B > 0$ such that the solution of*

$$\begin{aligned} \dot{y}(t) &= \mathcal{A}y + f(y) \\ y(0) &= y_0 \in B \end{aligned}$$

exists for $t \in [0, \tau_B]$ and

$$\sup_{t\in[0,\tau_B]} \sup_{y_0\in B} \|y(t,y_0)\|_{Y^\alpha} < \infty. \tag{22}$$

Furthermore, for any $0 \le \theta < 1$ and $0 < t_0 < \tau_B$,

$$\sup_{t\in[t_0,\tau_B]} \sup_{y_0\in B} \|\dot{y}(t,y_0)\|_{Y^\theta} < \infty \tag{23}$$

and

$$\sup_{t\in[t_0,\tau_B]} \sup_{y_0\in B} \|y(t,y_0)\|_{Y^1} < \infty. \tag{24}$$

Additionaly, if f takes bounded subsets of Y^1 into bounded subsets of Y^β for some $\beta > 0$, then for any $0 \le \gamma \le \beta$ and $0 < t_0 < \tau_B$,

$$\sup_{t\in[t_0,\tau_B]} \sup_{y_0\in B} \|y(t,y_0)\|_{Y^{1+\gamma}} < \infty. \tag{25}$$

Proof. Let $M \ge 1$ be such that

$$\|e^{-\mathcal{A}t}\|_{L(Y^{\beta_1},Y^{\beta_2})} \le M t^{\beta_1-\beta_2} \quad \text{for } t > 0 \text{ and } 0 \le \beta_1 \le \beta_2 \le 1. \tag{26}$$

Proceeding as in [10] we define $\rho = \sup_{y_0\in B} \|u_0\|_{Y^\alpha}$ and consider a ball $B(M\rho+1)$ in Y^α centered at zero and of radius $M\rho+1$. Note that the solutions originating in points of a bounded set $B \subset Y^\alpha$ fulfil the estimate

$$\|y(t)\|_{Y^\alpha} \le M\rho + M(1-\alpha)^{-1} \sup_{z\in B(M\rho+1)} \|f(z)\|_Y \; t^{1-\alpha}$$

as long as they stay in the mentioned ball. Therefore they will not leave this ball at least until the second term above becomes equal to 1, which proves (22).

Let L be the Lipschitz constant of f in the set $B^* = \{y(t,y_0) : 0 \le t \le \tau_B,\ y_0 \in B\}$ and $N = \sup_{y\in B^*} \|f(y)\|_Y$. For $0 \le t \le t+h \le \tau_B$ we have

$$\begin{aligned} y(t+h) - y(t) =& (e^{-\mathcal{A}h} - I)e^{-\mathcal{A}t}y_0 + \int_0^t e^{-\mathcal{A}(t-s)}(f(y(s+h)) - f(y(s)))ds \\ &+ \int_0^h e^{-\mathcal{A}(t+h-s)}f(y(s))ds \end{aligned}$$

and from [14, Theorem 1.4.3] and (26)

$$\begin{aligned} &\|y(t+h) - y(t)\|_\alpha \le \frac{M^2}{\gamma}h^\gamma t^{-\gamma}\|y_0\|_{Y^\alpha} \\ &+ ML\int_0^t (t-s)^{-\alpha}\|y(s+h) - y(s)\|_{Y^\alpha}ds + MN\int_0^h (t+h-s)^{-\alpha}ds \\ \le& \frac{M^2}{\gamma}h^\gamma t^{-\gamma}\|y_0\|_{Y^\alpha} + ML\int_0^t (t-s)^{-\alpha}\|y(s+h) - y(s)\|_{Y^\alpha}ds + hMNt^{-\alpha}. \end{aligned}$$

If $\gamma \in [\alpha, 1)$ and $0 < t \le t+h \le \tau_B < 1$, then from Gronwall's inequality we have

$$\|y(t+h) - y(t)\|_\alpha \le C(M, N, \sup_{y_0\in B}\|y_0\|_{Y^\alpha}, \alpha, \gamma)\, h^\gamma\, t^{-\gamma} =: C_0 h^\gamma\, t^{-\gamma}$$

and therefore we get

$$\|f(y(t+h)) - f(y(t))\|_Y \le C_0\, L\, h^\gamma\, t^{-\gamma},\ 0 < t \le t+h \le \tau_B < 1. \tag{27}$$

Since

$$\mathcal{A}\int_0^t e^{-\mathcal{A}(t-s)}f(y(s))ds = \int_0^t \mathcal{A}e^{-\mathcal{A}(t-s)}(f(y(s)) - f(y(t)))ds + f(y(t)) - e^{-\mathcal{A}t}f(y(t))$$

we can rewrite a time derivative of y in the form

$$\begin{aligned} \dot y(t) &= -\mathcal{A}y(t) + f(y(t)) \\ &= -\mathcal{A}e^{-\mathcal{A}t}y_0 + \int_0^t \mathcal{A}e^{-\mathcal{A}(t-s)}(f(y(t)) - f(y(s)))ds + e^{-\mathcal{A}t}f(y(t)). \end{aligned} \tag{28}$$

We apply (27) with $t := s$, $h := t - s$ to estimate the integral in (28) and get

$$\begin{aligned} &\left\| \int_0^t \mathcal{A}e^{-\mathcal{A}(t-s)}(f(y(t)) - f(y(s)))ds \right\|_{Y^\theta} \\ &\le \int_0^t M(t-s)^{-1-\theta}\|f(y(t)) - f(y(s))\|_Y ds \\ &\le C_0 LM\int_0^t (t-s)^{-1-\theta+\gamma}s^{-\gamma}ds = C_0 LM\mathcal{B}(1-\gamma, \gamma-\theta)t^{-\theta}, \end{aligned} \tag{29}$$

where $\alpha < \theta < \gamma < 1$ and $\mathcal{B}(a,b) = \int_0^1 r^{a-1}(1-r)^{b-1}dr$ for $a, b > 0$. Hence, it follows from (28), (29) that

$$\begin{aligned} &\sup_{t\in[t_0,\tau_B]}\sup_{y_0\in B}\|\dot y(t,y_0)\|_{Y^\theta} \le Mt_0^{-1-\theta+\alpha}\sup_{y_0\in B}\|y_0\|_{Y^\alpha} \\ &+ C_0 LM\mathcal{B}(1-\gamma,\gamma-\theta)t_0^{-\theta} + MNt_0^{-\theta},\ \alpha < \theta < \gamma < 1, \end{aligned}$$

which shows that the estimate of the derivative in (23) holds with $\theta > \alpha$ and with a bound depending only on B, α, θ, M, f, t_0 and a choice of $\gamma \in (\theta, 1)$. Furthermore, recalling that $\mathcal{A}y = -\dot y + f(y)$, we also observe that

$$\sup_{t\in[t_0,\tau_B]}\sup_{y_0\in B}\|y(t,y_0)\|_{Y^1} \le c\sup_{t\in[t_0,\tau_B]}\sup_{y_0\in B}\|\dot y(t,y_0)\|_{Y^\theta} + N.$$

Finally, to prove (25), we proceed as above writing $\mathcal{A}y = -\dot{y} + f(y)$ and using (23) and (24). □

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REFERENCES

[1] R. A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.
[2] H. Amann, "Linear and Quasilinear Parabolic Problems," Birkhäuser, Basel, 1995.
[3] J. M. Arrieta and A. N. Carvalho, *Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations*, Trans. Amer. Math. Soc., **352** (2000), 285–310.
[4] S. M. Bruschi, A. N. Carvalho, J. W. Cholewa and T. Dlotko, *Uniform exponential dichotomy and continuity of attractors for singularly perturbed damped wave equations*, J. Dynam. Differential Equations, **18** (2006), 767–814.
[5] A. N. Carvalho and J. W. Cholewa, *Local well posedness for strongly damped wave equations with critical nonlinearities*, Bull. Austral. Math. Soc., **66** (2002), 443–463.
[6] A. N. Carvalho and J. W. Cholewa, *Attractors for strongly damped wave equations with critical nonlinearities*, Pacific J. Math., **207** (2002), 287–310.
[7] S. Chen, R. Triggiani, "Proof of two conjectures of G. Chen and D. L. Russell on structural damping for elastic systems: The case $\alpha = \frac{1}{2}$," in Notes in Mathematic 1354, Springer, 1988, 234–256.
[8] S. Chen and R. Triggiani, *Proof of extension of two conjectures on structural damping for elastic systems*, Pacific J. Math., **136** (1989), 15–55.
[9] S. Chen and R. Triggiani, *Characterization of domains of fractional powers of certain operators arising in elastic systems, and applications*, J. Differential Equations, **88** (1990), 279–293.
[10] J. W. Cholewa and T. Dlotko, "Global Attractors in Abstract Parabolic Problems," Cambridge University Press, Cambridge, 2000.
[11] A. Friedman, "Partial Differential Equations," Holt, Rinehart and Winston, New York, 1969.
[12] D. Gilbarg and N. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer, Berlin, 1983.
[13] J. K. Hale, "Asymptotic Behavior of Dissipative Systems," AMS, Providence, 1988.
[14] D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Springer, Berlin, 1981.
[15] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta. Math., **104** (1960), 93–140.
[16] T. Kato, *Fractional powers of dissipative operators*, J. Math. Soc. Japan, **13** (1961), 246–274.
[17] J. Prüss, H. Sohr, *Imaginary powers of elliptic second order differential operators in L^p-spaces*, Hiroshima Math. J., **23** (1993), 161–192.
[18] K. Taira, "Analytic Semigroups and Semilinear Initial Boundary Value Problems," Cambridge University Press, Cambridge, 1995.
[19] H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," North-Holland, Amsterdam, 1978.

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