DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
SUPPLEMENT 2007

Website: www.AIMSciences.org

pp. **694–703**

POSITIVE SOLUTIONS FOR A NONLOCAL FOURTH ORDER EQUATION OF KIRCHHOFF TYPE

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Abstract. Existence and multiplicity of positive solutions for the fourth order equation

$$u'''' - M(\textstyle\int_0^1 |u'|^2\,dx)\,u'' = q(x)f(x,u,u'),$$

which models simply supported extensible beams, are considered using fixed point theorems in cones of ordered Banach spaces.

1. **Introduction.** In this paper we study the existence of positive solutions for the fourth order boundary value problem

$$u'''' - M(\textstyle\int_0^1 |u'|^2\,dx)\,u'' = q(x)f(x,u,u'), \quad 0 < x < 1, \tag{1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{2}$$

where $f \in C([0,1] \times [0,\infty) \times \mathbb{R})$ and $M \in C[0,\infty)$ are nonnegative real functions and $q \in C[0,1]$ is a positive real function.

This kind of problem models the bending equilibrium of simply supported extensible beams on nonlinear foundations. The function f represents the force that the foundation exerts on the beam and $M(\int u'^2)$ models the effects of the small changes in the length of the beam. The function M was introduced by Kirchhoff in [10] and it is typically a line with positive slope. For a discussion about modeling and justification of Kirchhoff type strings and beams, we refer the reader to the references [3, 10, 14, 16, 18].

When f does not depend on u', existence and multiplicity results for the problem (1)-(2) were considered in [14, 15], through variational methods. Indeed, in this case, the solutions are critical points of the C^1 functional $\Psi : H^2[0,1] \cap H_0^1[0,1] \to \mathbb{R}$, defined by

$$\Psi(u) = \frac{1}{2}\|u''\|_2^2 + \frac{1}{2}\hat{M}\left(\|u'\|_2^2\right) - \int_0^1 q(x)\hat{f}(x,u)\,dx,$$

where $\hat{M}$ and $\hat{f}$ are, respectively, the primitives of $M(\xi)$ and $f(x,\xi)$ with respect to ξ, and H_0^1, H^2 are standard Sobolev spaces. If $\hat{M}(u) \geq M(u)u$ for all $u \geq 0$, then we can easily prove that Ψ satisfies the Palais-Smale compactness condition for a large class of nonlinearities f. This would allow us to use several critical point theorems. However, in this case, M cannot be increasing, which is not usual in the

2000 *Mathematics Subject Classification.* Primary: 34B18; Secondary: 34B60.
Key words and phrases. beam equation, fixed point theorem, multiple solutions.
Supported by the Brazilian agencies CNPq (473316/2004-8) and FAPESP (2006/04523-4).

applications. Some arguments for controlling M are discussed in [2, 16] for second order elliptic equations.

Related fourth order problems involving nonlinear dependence on the second order derivative,

$$u'''' = f(x, u, u''), \quad 0 < x < 1, \tag{3}$$

have been considered by several authors, with many different boundary conditions. Some of them are based on the two-parameter eigenvalue problem

$$u'''' = \alpha u - \beta u'', \quad u(0) = u(1) = u''(0) = u''(1) = 0,$$

discussed in [6]. In that direction, existence results are obtained under nonresonance conditions. See for example [6, 7, 13] and the references therein. See also [1, 17] for some other related works. Our problem (1)-(2) is different because it involves a nonlocal term $M(\int u'^2)$ and has nonlinear dependence on the first derivative of u.

The objective of the present paper is to show some existence and multiplicity results for positive solutions of the problem (1)-(2), using fixed points theorems to cones in ordered Banach spaces. Following recent results from [4, 5, 9], our results are also stated assuming local conditions on f and on M. Roughly speaking, if there exist suitable constants $m, d, C_1, C_2 > 0$ such that

$$0 \le M(t) \le m \quad \text{if} \quad t \in [0, d^2]$$

and

$$C_1 d \le f(t, u, v) \le C_2 d \quad \text{if} \quad (t, u, v) \in [0, 1] \times [0, \tfrac{d}{2}] \times [-d, d],$$

then problem (1)-(2) has at least a positive solution (Theorem 3). The key argument is the Lemma 2 which provides sufficient conditions for positivity. Under some additional conditions, the problem may have three positive solutions (Theorem 4).

The paper is organized as follows. In Section 2 we establish the fixed point framework for the problem and recall the fixed point theorems of Krasnosel'skii [11, 8] and of Avery and Peterson [4]. In Section 3 we consider the existence of a positive solution. In Section 4 we consider the existence of three positive solutions.

2. **Fixed Point Setting.** We begin by reducing problem (1)-(2) to a system of second order equations

$$u'' = M(\|u'\|_2^2)u + v, \qquad u(0) = u(1) = 0, \tag{4}$$

and

$$v'' = qf(x, u, u'), \qquad v(0) = v(1) = 0, \tag{5}$$

where $\|u\|_p$ denotes $L^p[0, 1]$ norms. Let us denote by G the Green's function of the problem $w'' = 0$, $w(0) = w(1) = 0$, which is defined by

$$G(x, t) = \begin{cases} x(1-t), & x \le t, \\ t(1-x), & t \le x. \end{cases} \tag{6}$$

Then from the system (4)-(5) we see that

$$u(x) = \int_0^1 -G(x, t)\,(\,M(\|u'\|_2^2)u(t) + v(t)\,)\,dt,$$

where

$$\|u'\|_2^2 = \int_0^1 |u'(s)|^2\,ds$$

and

$$v(t) = \int_0^1 -G(t,s)q(s)f(s,u(s),u'(s))ds.$$

Therefore we infer that u is a solution of the problem (1)-(2) if and only if it is a fixed point of $T : C^1[0,1] \to C^1[0,1]$ defined by

$$Tu(x) = \int_0^1 G(x,t)\left(\int_0^1 G(t,s)q(s)f(s,u(s),u'(s))ds - M(\|u'\|_2^2)u(t)\right)dt. \tag{7}$$

Because the hypotheses of the continuity of q, f and M, the operator T is completely continuous in $C^1[0,1]$ by the Arzela-Ascoli's theorem.

The existence results are derived by applying the well-known Krasnosel'skii fixed point theorem of compression of a cone (see [11, 8]).

Theorem 1 (Krasnosel'skii)**.** *Let P be a cone in a Banach space E and let Ω_1 and Ω_2 be open subsets of E, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that*

$$T : P \cap (\Omega_2 \backslash \overline{\Omega}_1) \to P$$

is a completely continuous operator such that

$$\|Tu\| \geq \|u\| \ \text{ if } \ u \in P \cap \partial\Omega_1 \quad \text{and} \quad \|Tu\| \leq \|u\| \ \text{ if } \ u \in P \cap \partial\Omega_2. \tag{8}$$

Then T has a fixed point in $P \cap (\Omega_2 \backslash \overline{\Omega}_1)$.

In the present context we consider the cone

$$P = \{u \in C^1[0,1] \,|\, u(0) = u(1) = 0 \text{ and } u \text{ is concave}\}, \tag{9}$$

which is a closed set in the Banach space $C^1[0,1]$, equipped with the norm

$$\|u\| = \max\{\|u\|_\infty\,,\, \|u'\|_\infty\}.$$

Then if $u \in P$ is a nonzero fixed point of T (as defined in (7)), it follows that u is a positive solution of (1)-(2), that is, $u > 0$ in $(0,1)$.

We have the following useful lemma.

Lemma 1. *If $u \in P$ then*

$$\max_{t\in[0,1]} |u(t)| \leq \frac{1}{2} \max_{t\in[0,1]} |u'(t)|, \tag{10}$$

which implies that

$$\|u\| = \max_{t\in[0,1]} |u'(t)|. \tag{11}$$

In addition, if $p \in (0, \frac{1}{2})$, then

$$u(t) \geq p \max_{t\in[0,1]} |u(t)|, \quad \forall\, t \in [p, 1-p]. \tag{12}$$

Proof. The proof follows from the properties of concave functions. We give a proof of (12). Since u is concave and $u(0) = u(1) = 0$, one has $u(t)$ is below the line passing by $(p, u(p))$ and $(1,0)$, if $t \leq p$. Likewise, $u(t)$ is below the line passing by $(0,0)$ and $(p, u(p))$, if $t \geq p$. Then

$$u(t) \leq \frac{u(p)}{1-p}(1-t) \ \text{ if } \ 0 \leq t \leq p \quad \text{and} \quad u(t) \leq \frac{u(p)}{p}t \ \text{ if } \ p \leq t \leq 1.$$

Hence

$$\max_{t\in[0,1]} |u(t)| \leq \frac{u(p)}{p}.$$

Analogously we get

$$\max_{t\in[0,1]} |u(t)| \le \frac{u(1-p)}{p},$$

and then (12) follows. □

We end this section recalling a recent multiple fixed point theorem by Avery and Peterson [4], which is a generalization of the triple fixed point theorem of Leggett-Williams [12]. This will be used in Section 4. Given four functionals $\gamma, \alpha, \theta, \psi$ defined on the cone P, and positive numbers a, b, c, d, one define the following sets:

$$P(\gamma, d) = \{u \in P |\, \gamma(u) < d\},$$
$$P(\gamma, \alpha, b, d) = \{u \in P |\, b \le \alpha(u) \text{ and } \gamma(u) \le d\},$$
$$P(\gamma, \theta, \alpha, b, c, d) = \{u \in P |\, b \le \alpha(u),\ \theta(u) \le c \text{ and } \gamma(u) \le d\},$$

and

$$R(\gamma, \psi, a, d) = \{u \in P |\, a \le \psi(u) \text{ and } \gamma(u) \le d\}.$$

Theorem 2 (Avery-Peterson)**.** *Let P be a cone in a real Banach space E and let $\gamma, \theta, \alpha, \psi$ be nonnegative continuous functionals on P. Suppose that γ, θ are convex and α is concave. Suppose in addition that ψ satisfies $\psi(\lambda u) \le \lambda\psi(u)$ for $0 \le \lambda \le 1$, and that for some positive numbers μ and d,*

$$\alpha(u) \le \psi(u) \quad \text{and} \quad \|u\| \le \mu\gamma(u), \quad \forall\, u \in \overline{P(\gamma, d)}.$$

Let

$$T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$$

be a completely continuous operator and suppose there exist positive numbers a, b, c, with $a < b$, such that

$$\{u \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(u) > b\} \ne \emptyset \text{ and } \alpha(Tu) > b \text{ for } u \in P(\gamma, \theta, \alpha, b, c, d), \tag{13}$$

$$\alpha(Tu) > b \text{ for } u \in P(\gamma, \alpha, b, d) \text{ with } \theta(Tu) > c, \tag{14}$$

$$0 \notin R(\gamma, \psi, a, d) \text{ and } \psi(Tu) < a \text{ for } u \in R(\gamma, \psi, a, d) \text{ with } \psi(u) = a. \tag{15}$$

Then T has at least three distinct fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, d)}$, such that

$$\alpha(u_2) < b < \alpha(u_1) \quad \text{and} \quad \psi(u_3) < a < \psi(u_2).$$

3. **Positive Solutions.** Let us define

$$A = \min\left\{\int_0^1 G(p, s)q(s)\, ds\,,\ \int_0^1 G(1-p, s)q(s)\, ds\right\}, \tag{16}$$

where p is a given constant with $0 < p < 1/2$, and

$$B = \max\left\{\int_0^1\int_0^1 tG(t, s)q(s)\, dsdt\,,\ \int_0^1\int_0^1 (1-t)G(t, s)q(s)\, dsdt\right\}. \tag{17}$$

Theorem 3. *Suppose that there exist $m, d > 0$ such that*

$$M(t) \le m \quad \text{if} \quad t \in [0, d^2], \tag{18}$$

$$f(t, u, v) \ge \frac{md}{2A} \quad \text{if} \quad (t, u, v) \in [0, 1] \times [0, \tfrac{d}{2}] \times [-d, d], \tag{19}$$

$$f(t, u, v) \le \frac{d}{B} \quad \text{if} \quad (t, u, v) \in [0, 1] \times [0, \tfrac{d}{2}] \times [-d, d]. \tag{20}$$

Then problem (1)-(2) *has at least a positive solution.*

Remark 1. If $q(t) = 1$ is constant and $p = 1/4$, then $A = 3/32$ and $B = 1/24$. So hypotheses (19)-(20) become simply

$$\frac{16m}{3}\, d \le f(t,u,v) \le 24d \quad \text{if} \quad (t,u,v) \in [0,1] \times [0, \tfrac{d}{2}] \times [-d,d].$$

Of course, in this case m must satisfy $m < 9/2$.

To prove Theorem 3 we apply the Krasnosel'skii fixed point theorem to show that operator T, defined in (7), has a fixed in the cone P of positive concave functions defined in (9).

In order to guarantee that operator T is positive, it suffices to have

$$z(t) := \int_0^1 G(t,s)q(s)f(s,u,u')ds - M(\|u'\|_2^2)u(t) \ge 0, \quad t \in [0,1]. \tag{21}$$

This is given in the next lemma.

Lemma 2. *Suppose that* (18) *and* (19) *hold. Then* $Tu \ge 0$ *for all* $u \in P$ *satisfying* $\|u\| \le d$.

Proof. We show that

$$\begin{aligned} w(t) &:= \int_0^1 G(t,s)q(s)f(s,u(s),u'(s))\,ds \\ &\ge M(\|u'\|_2^2)u(t), \quad t \in [0,1], \end{aligned} \tag{22}$$

for any $u \in P$ with $\|u'\|_\infty \le d$. We have three cases. Firstly suppose that $t \in [p, 1-p]$. Then since (10) implies that $(u(s), u'(s)) \in [0, \frac{d}{2}] \times [-d,d]$ for all $s \in [0,1]$, one has from (19),

$$w(t) \ge \frac{md}{2A} \int_0^1 G(t,s)q(s)ds.$$

Since $y(t) = \int_0^1 G(t,s)q(s)ds$ is concave because $y'' = -q(t) \le 0$, we get from (16) that

$$\int_0^1 G(t,s)q(s)ds \ge A \quad \text{if} \quad t \in [p, 1-p].$$

Therefore

$$w(t) \ge md/2 \quad \text{if} \quad t \in [p, 1-p].$$

Besides, noting that $M(\|u'\|_2^2)u(t) \le md/2$, the inequality (22) holds for $t \in [p, 1-p]$.

Next we take $t \in [0,p]$. Since $w = w(t)$ is also concave and $w(0) = 0$, $w(t)$ lies above the line joining $(0,0)$ and $(p, w(p))$, that is, $w(t) \ge (w(p)/p)t$ if $t \in [0,p]$. Then because $w(p) \ge md/2$ and $2p < 1$, we conclude that

$$w(t) \ge \frac{md}{2p}\, t \ge md\, t \quad \text{if} \quad t \in [0,p]. \tag{23}$$

On the other hand, from $u(0) = 0$ and $u'(t) \le d$ for all $t \in [0,1]$, it follows that $u(t) \le d\, t$ in $[0,1]$. Therefore

$$M(\|u'\|_2^2)u(t) \le md\, t, \quad \forall\, t \in [0,1]. \tag{24}$$

Combining (23) and (24) we see that (22) holds for $t \in [0,p]$. Similar arguments show that (22) also holds for $t \in [1-p, 1]$. □

Proof of Theorem 3. From Lemma 2 we know that T applies $P \cap \overline{B}(0,d)$ into P. Suppose $\|u\| = d$. Since Tu is concave

$$\begin{aligned} \|Tu\| &= \max\{|(Tu)'(0)|\,,\,|(Tu)'(1)|\} \\ &= \max\left\{\int_0^1 (1-t)z(t)\,dt\,,\,\int_0^1 tz(t)\,dt\right\}, \end{aligned}$$

where $z(t)$ is defined in (21). Then noting that $M(\|u'\|_2^2)u(t) \geq 0$, we have

$$z(t) \leq \int_0^1 G(t,s)q(s)f(s,u(s),u'(s))ds,$$

and consequently from (17)

$$\|Tu\| \leq \max_{s\in[0,1]} f(s,u(s),u'(s)) \times B.$$

Since

$$(u(s),u'(s)) \in [0,\tfrac{d}{2}] \times [-d,d], \quad \forall\, s \in [0,1],$$

using (20) we conclude that

$$\|Tu\| \leq \|u\| \quad \text{if} \quad \|u\| = d.$$

So, the second part of the condition (8) holds with $\Omega_2 = B(0,d)$.

Now take $r > 0$ small and let $u \in P$ with $\|u\| = r$. As before

$$\begin{aligned} \|Tu\| &= \max\left\{\int_0^1 (1-t)z(t)dt, \int_0^1 tz(t)dt\right\} \\ &\geq \int_p^{1-p} t(w(t) - M(\|u'\|_2^2)u(t))dt, \end{aligned}$$

where $w(t)$ is defined in (22). We know from Lemma 2 that $w(t) \geq md/2$ for $t \in [p, 1-p]$, and that $\|u\| \leq r$ implies $u(t) \leq r/2$. Therefore

$$w(t) - M(\|u'\|_2^2)u(t) \geq \frac{m(d-r)}{2}.$$

If we take $r \leq md/(m + (4/(1-2p)))$, then

$$\|Tu\| \geq \int_p^{1-p} t\frac{m(d-r)}{2}dt = m(d-r)\frac{1-2p}{4} \geq r,$$

that is,

$$\|Tu\| \geq \|u\| \quad \text{if} \quad \|u\| = r.$$

Therefore the first part of condition (8) holds with $\Omega_1 = B(0,r)$. Hence from the Krasnosel'skii theorem, problem (1)-(2) has a positive solution. □

Example 1. Let us suppose

$$f(t,u,v) = 1 + u^2 + v^2, \quad M(t) = 1 + t, \quad q(t) = \cos t.$$

Then (18) holds for $d = 1/10$ and $m = 11/10$. Besides,

$$1 \leq f(t,u,v) \leq \frac{405}{400} \quad \text{if} \quad (t,u,v) \in [0,1] \times [0,\tfrac{1}{20}] \times [-\tfrac{1}{10},\tfrac{1}{10}].$$

Taking $p = 1/4$ we have $A \geq 7/100$. Since $B \leq 4/100$, we then have

$$\frac{md}{2A} \leq \frac{11}{14} \quad \text{and} \quad \frac{5}{2} \leq \frac{d}{B}.$$

This shows that (19) and (20) also hold. Hence from Theorem 3, problem (1)-(2) has a positive solution.

4. **Three Positive Solutions.** Our multiplicity result is motivated by early works from [4, 5, 9, 19] concerned with second order problems. For sake of readability, we assume $p = 1/4$. Let

$$C = \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)\,ds, \tag{25}$$

$$D = \min\left\{ \int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tfrac{1}{4},t)G(t,s)q(s)\,dsdt, \int_0^1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tfrac{3}{4},t)G(t,s)q(s)\,dsdt \right\}. \tag{26}$$

Theorem 4. *Suppose that hypotheses of Theorem 3 hold. Suppose in addition that there exist constants* $0 < a < b < 4b \le c < d/2$ *such that*

$$f(t,u,v) < \frac{8a}{C} \quad \textit{if} \quad (t,u,v) \in [0,1]\times[0,a]\times[-d,d], \tag{27}$$

$$f(t,u,v) \ge \frac{7b}{4D} \quad \textit{if} \quad (t,u,v) \in [\tfrac{1}{4},\tfrac{3}{4}]\times[b,c]\times[-d,d]. \tag{28}$$

Then problem (1)-(2) *has at least three positive solutions.*

Proof. We apply the Avery-Peterson theorem with T and P defined as before. Let us define the following functionals:

$$\gamma(u) = \|u\|,$$

$$\psi(u) = \theta(u) = \max_{t\in[0,1]} |u(t)|,$$

$$\alpha(u) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} |u(t)|.$$

Since the hypotheses of Theorem 3 holds, we known that T is a completely continuous operator defined in $P \cap \overline{B}(0,d)$. Therefore

$$T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}.$$

Let us check condition (13). It is easy to see that the function

$$u(t) = \begin{cases} 8bt & \text{if} \quad t \le \frac{1}{4}, \\ 16b\left(-t^2+t-\frac{1}{16}\right) & \text{if} \quad \frac{1}{4} \le t \le \frac{3}{4}, \\ 8b(1-t) & \text{if} \quad t \ge \frac{3}{4}, \end{cases}$$

belongs to P and satisfies $\alpha(u) = 2b$, $\theta(u) = 3b < c$ and $\gamma(u) = 8b < d$. Therefore

$$\{u \in P(\gamma,\theta,\alpha,b,c,d) |\, \alpha(u) > b\} \ne \emptyset.$$

Let us show that $u \in P(\gamma,\theta,\alpha,b,c,d)$ then $\alpha(Tu) > b$. By convexity,

$$\begin{aligned} \alpha(Tu) &= \min\left\{(Tu)(\tfrac{1}{4}),(Tu)(\tfrac{3}{4})\right\} \\ &= \min\left\{\int_0^1 G(\tfrac{1}{4},t)z(t)dt, \int_0^1 G(\tfrac{3}{4},t)z(t)dt\right\}, \end{aligned}$$

where as before,

$$z(t) = \int_0^1 G(t,s)q(s)f(s,u(s),u'(s))ds - M(\|u'\|_2^2)u(t).$$

Then from (18) we have

$$\begin{aligned}(Tu)(\tfrac{1}{4}) &\geq \int_0^1 G(\tfrac{1}{4},t)\left(\int_0^1 G(t,s)q(s)f(s,u(s),u'(s))ds - mc\right)dt \\ &\geq \int_0^1 G(\tfrac{1}{4},t)\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)q(s)f(s,u,u')dsdt - mc\int_0^1 G(\tfrac{1}{4},t)dt.\end{aligned}$$

On the other hand, from (19) and (27) we see that

$$\frac{md}{2A} < \frac{8a}{C},$$

and since $a < b$, $c < d/2$ and $A \leq C$, we infer that

$$mc < 8b.$$

Therefore

$$(Tu)(\tfrac{1}{4}) > \min_{s\in[\frac{1}{4},\frac{3}{4}]} |f(s,u(s),u'(s))| \int_0^1 G(\tfrac{1}{4},t)\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)q(s)dsdt - \frac{3b}{4}.$$

Using (26) and (28) we get

$$(Tu)(\tfrac{1}{4}) > \frac{7b}{4D}\times D - \frac{3b}{4} = b.$$

Analogously we get

$$(Tu)(\tfrac{3}{4}) > b.$$

Then $\alpha(Tu) > b$ and the condition (13) holds.

In order to verify (14) let $u \in P(\gamma,\alpha,b,d)$ with $\theta(Tu) > c$. Using (12) with $p = 1/4$ and since $c \geq 4b$,

$$\alpha(Tu) \geq \frac{1}{4}\theta(Tu) > \frac{c}{4} \geq b.$$

Therefore condition (14) holds.

Finally we check condition (15). Since $\psi(0) = 0 < a$, it follows that

$$0 \notin R(\gamma,\psi,a,d).$$

Let $u \in R(\gamma,\psi,a,d)$. Then noting that $\int_0^1 |G(x,t)|\,dt \leq \frac{1}{8}$ for all $x \in [0,1]$ and using (25) and (27),

$$\begin{aligned}\psi(Tu) &= \max_{x\in[0,1]} |Tu(x)| \\ &\leq \frac{1}{8}\max_{t\in[0,1]}\int_0^1 G(t,s)q(s)f(s,u(s),u'(s))ds \\ &< \frac{1}{8}\frac{8a}{C}\times C = a.\end{aligned}$$

Thus (15) also holds. Hence from the Avery-Peterson theorem, problem (1)-(2) has three positive solutions. □

Example 2. We give an example satisfying all the assumptions of Theorem 4. Suppose that

$$M(t) = \frac{1}{10} + \frac{t}{9000}, \quad q(t) = 1, \quad p = \frac{1}{4}.$$

Then (18) holds for $m = 1/5$ and $d = 30$. Besides,

$$A = \frac{3}{32},\ B = \frac{1}{24},\ C = \frac{1}{8},\ D = \frac{5}{768}.$$

Taking $a = 1$, $b = 2$ and $c = 8$, it follows that

$$\frac{md}{2A} = 32,\ \frac{d}{B} = 720,\ \frac{8a}{C} = 64,\ \frac{7b}{4D} = 537.6.$$

Now we define f as

$$f(t,u,v) = \begin{cases} 34 + 24u^4 + (\frac{v}{30})^3 & \text{if} \quad 0 \le u \le 1, \\ -424 + 482u + (\frac{v}{30})^3 & \text{if} \quad 1 < u \le 2, \\ 540 + (\frac{v}{30})^3 & \text{if} \quad 2 < u \le 15. \end{cases}$$

Then (19)-(20) and (27)-(28) hold because

$$\begin{aligned} f(t,u,v) &\ge 33 \quad (t,u,v) \in [0,1] \times [0,15] \times [-30,30], \\ f(t,u,v) &\le 541 \quad (t,u,v) \in [0,1] \times [0,15] \times [-30,30], \\ f(t,u,v) &\le 59 \quad (t,u,v) \in [0,1] \times [0,1] \times [-30,30], \\ f(t,u,v) &\ge 539 \quad (t,u,v) \in [\tfrac{1}{4}, \tfrac{3}{4}] \times [2,8] \times [-30,30]. \end{aligned}$$

Therefore problem (1)-(2) has at least three positive solutions.

Acknowledgements. The author is indebted to Professor K. Q. Lan and to the anonymous referee for their valuable comments and suggestions.

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Received September, 2006; revised March 2007.

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