

## A CHARACTERIZATION OF THE CONCEPT OF DUALITY

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ABSTRACT. In this paper we discuss the abstract concept of duality. We consider various classes: convex functions,  $s$ -concave functions, convex bodies and log-concave functions. We demonstrate how very little a-priori information is needed in order to arrive at the concrete formulas for duality in each of the classes. Some of these transforms, such as the Legendre transform or polarity in convexity, are classical and well-known, and some more recently discovered and used. In most cases the information we require is only, essentially, that inequalities between functions are reversed. In another case the information is that the transform exchanges summation with the operation of inf-convolution.

### 1. INTRODUCTION

The notion of duality is one of the central concepts both in geometry and in analysis. At the same time, it is usually defined in a very concrete way, using very concrete structures. In this paper we investigate the notion of duality, and show how the standard definitions arise, explicitly, from two very simple and natural properties which we call “abstract duality” properties, and which are, as is clear to us now, the essence of the concept of duality. Here is a sample of our main results.

Denote the class of lower semi-continuous convex functions  $\phi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$  by  $Cvx(\mathbb{R}^n)$ . Denote by  $\langle \cdot, \cdot \rangle$  the standard scalar product on  $\mathbb{R}^n$ . Recall the definition of the classical Legendre transform  $\mathcal{L}$  defined for functions  $\phi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$  by

$$(1) \quad (\mathcal{L}\phi)(x) = \sup_y (\langle x, y \rangle - \phi(y)).$$

For background on the Legendre transform and its many applications, see, e.g., [1], [11] and [14]. Let us list some remarkable properties of this transform.

- (1)  $\mathcal{L}\phi \in Cvx(\mathbb{R}^n)$  for all  $\phi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$ ,
- (2)  $\mathcal{L} : Cvx(\mathbb{R}^n) \rightarrow Cvx(\mathbb{R}^n)$  is 1-1 and onto,
- (3)  $\mathcal{L}\mathcal{L}\phi = \phi$  for  $\phi \in Cvx(\mathbb{R}^n)$
- (4)  $\phi \leq \psi$  implies  $\mathcal{L}\phi \geq \mathcal{L}\psi$ ,
- (5)  $\mathcal{L}(\max(\phi, \psi)) = \hat{\min}(\mathcal{L}\phi, \mathcal{L}\psi)$  where  $\hat{\min}$  denotes “regularized minimum”, that is, the largest l.s.c. convex function below all functions participating in the minimum,
- (6)  $\mathcal{L}\phi + \mathcal{L}\psi = \mathcal{L}(\phi \square \psi)$  where  $(\phi \square \psi)(x) := \inf\{\phi(y) + \psi(z) : x = y + z\}$ .

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It turns out that all these remarkable properties arise from very simple restrictions on the transform, which we will later call “concept of duality”. For example, we proved the following theorem which shows why it is, in some sense, the only natural transform to associate with duality of convex functions.

**Theorem 1.** *Let  $\mathcal{T} : \text{Cvx}(\mathbb{R}^n) \rightarrow \text{Cvx}(\mathbb{R}^n)$  be a transform (defined on the whole domain  $\text{Cvx}(\mathbb{R}^n)$ ) satisfying*

- (1)  $\mathcal{T}\mathcal{T}\phi = \phi$ ,
- (2)  $\phi \leq \psi$  implies  $\mathcal{T}\phi \geq \mathcal{T}\psi$ .

*Then  $\mathcal{T}$  is essentially the classical Legendre transform, namely there exists a constant  $C_0 \in \mathbb{R}$ , a vector  $v_0 \in \mathbb{R}^n$ , and a symmetric transformation  $B \in GL_n$  such that*

$$(\mathcal{T}\phi)(x) = (\mathcal{L}\phi)(Bx + v_0) + \langle x, v_0 \rangle + C_0.$$

The difference between the usual Legendre transform and the formula in Theorem 1 is in the additional fixed  $B, v_0$  and  $C_0$ . However, not only is it easy to check that this modified transform satisfies the two conditions, but it is also very much expected that we will arrive at these extra factors. Indeed, first note that the choice of 0 plays no special role for convex functions, which remain convex also when translated. The choice of 0 in  $\mathbb{R}^n$  accounts for the translation vector  $v_0$  (which appears also as a linear factor to promise an involution). However, functions can be thought of as graphs in  $\mathbb{R}^{n+1}$ , and the constant  $C_0$  accounts for the choice of 0 in the additional direction  $\mathbb{R}$  of the image of the functions. Finally, to account for  $B$ , note that we fixed, arbitrarily, a scalar product on  $\mathbb{R}^n$  with which we have defined the Legendre transform  $\mathcal{L}$  above. Since it was arbitrary, any other scalar product could be used instead. Note that  $B$  is assumed symmetric but not necessarily positive definite. This is because we are free to choose the coordinates both in the space  $\mathbb{R}^n$  and in its dual space (which is also  $\mathbb{R}^n$ ) as we please. Of course, to arrive at the involution condition (1) the bases should be mutually orthogonal, but the directions of the basic vectors can be opposite for some of the vectors, and this produces the signature of  $B$ .

For the details of the proof of Theorem 1, as well as those of the next one and a few consequences, see [2]. Let us consider a few more examples, the first of which is a direct consequence of the previous theorem. Denote by  $LC(\mathbb{R}^n)$  (for “log-concave”) the class of upper semi continuous non-negative functions with convex *non-empty* support, such that on their support, their logarithm is concave.

**Theorem 2.** *Let  $\mathcal{T} : LC(\mathbb{R}^n) \rightarrow LC(\mathbb{R}^n)$  be a transform (defined on the whole domain  $LC(\mathbb{R}^n)$ ) satisfying*

- (1)  $\mathcal{T}\mathcal{T}f = f$ ,
- (2)  $f \leq g$  implies  $\mathcal{T}f \geq \mathcal{T}g$ .

*Then there exists a constant  $0 < C_0 \in \mathbb{R}$ , a vector  $v_0 \in \mathbb{R}^n$ , and an invertible symmetric linear transformation  $B \in GL_n$  such that  $\mathcal{T}$  is defined as follows:*

$$(\mathcal{T}f)(x) = C_0 e^{\langle v_0, x \rangle} \inf_y \frac{e^{-\langle Bx + v_0, y \rangle}}{f(y)}.$$

(In fact,  $(\mathcal{T}f)(x) = C_0 e^{\langle v_0, x \rangle} f^\circ(Bx + v_0)$  for the functional duality  $f^\circ$  defined in [4] and which we will recall below in the equation (2).)

Define for  $s > 0$  the set  $\text{Conc}_s(\mathbb{R}^n)$  to be the set of all upper semi-continuous non-negative functions on  $\mathbb{R}^n$  which are  $s$ -concave, namely, have convex support that includes 0, and  $f^{1/s}$  is concave on the support.

**Theorem 3.** *Let  $\mathcal{T} : \text{Conc}_s(\mathbb{R}^n) \rightarrow \text{Conc}_s(\mathbb{R}^n)$  be a transform (defined on the whole domain) satisfying*

- (1)  $\mathcal{T}\mathcal{T}f = f$ ,
- (2)  $f \leq g$  implies  $\mathcal{T}f \geq \mathcal{T}g$ .

*Then there exists a constant  $C_0 \in \mathbb{R}$  and a symmetric  $B \in GL_n$  such that*

$$(\mathcal{T}f)(x) = C_0 \inf_{\{y: f(By) > 0\}} \frac{(1 - \langle x, y \rangle)_+^s}{f(By)}.$$

(In fact,  $(\mathcal{T}f)(x) = C_0 \mathcal{L}_s(B^{-1}x)$  for the  $s$ -duality transform  $\mathcal{L}_s$  defined in [4] and which we will recall below in the equation (3).)

For the details of the proof of this theorem, some other close relatives of it, and consequences, see [3].

These theorems suggest that the concrete formulae which are used as “duality formulae” in several important examples are a direct consequence of two very natural conditions, which we call “concept of duality”, and a certain class of functions on which we require the operation to be defined.

**Definition 4** (Concept of Duality). *We will say that a transform  $\mathcal{T}$  generates a duality transform on a set of functions  $\mathcal{S}$  on  $\mathbb{R}^n$  if the following two properties are satisfied:*

- (1) *For any  $f \in \mathcal{S}$  we have  $\mathcal{T}\mathcal{T}f = f$ ,*
- (2) *For any two functions in  $\mathcal{S}$  satisfying  $f \leq g$  we have that  $\mathcal{T}f \geq \mathcal{T}g$ .*

Let us describe in detail various examples where duality notions are frequently used.

1. **(a) Duality for Normed Spaces:** For a Banach space  $X$ , one considers the space  $X^*$  of all bounded linear functionals on  $X$ . The pairing of an element  $x \in X$  and  $y \in X^*$  is simply the action of  $y$  on  $x$ ,  $y(x) \in \mathbb{R}$ . When the space is finite dimensional,  $X = (\mathbb{R}^n, \|\cdot\|)$ , it is natural to identify the dual space, which is also of dimension  $n$ , with the same  $\mathbb{R}^n$ . To this end one introduces a positive bilinear form (inner product)  $\langle \cdot, \cdot \rangle$  with  $y(x) = \langle y, x \rangle$ , and then one may identify the dual space with  $\mathbb{R}^n$ , endowed with the dual norm, namely  $\|y\|^* = \sup_{x \neq 0} \frac{\langle y, x \rangle}{\|x\|}$ . Of course, one may identify a normed space with its closed unit ball, which is a compact, convex, centrally symmetric set in  $\mathbb{R}^n$ , with 0 in its interior. This one-to-one correspondence transfers the notion of duality to the domain of centrally symmetric convex sets in  $\mathbb{R}^n$ .

**(b) Polarity in Convexity:** In convexity theory, the basic element is a closed compact convex set in  $\mathbb{R}^n$  (convex body). Once a scalar product is fixed, polarity is defined for any set which contains zero in its interior. For a convex body  $K \subset \mathbb{R}^n$  the polar is defined by

$$K^\circ = \{y \in \mathbb{R}^n : \sup_{x \in K} \langle y, x \rangle \leq 1\}.$$

We remark that one may also consider the class where the origin is allowed to be on the boundary of the set  $K$ , and then for duality to be well-defined

one also includes non-compact convex sets. For a later reference it will be useful to consider instead of a set  $K$  its indicator function  $\chi_K$  defined to be 1 on the body and 0 outside, so that duality (or polarity) is a transformation defined on this class of functions. Of course, for centrally symmetric sets in  $\mathbb{R}^n$ , which are unit balls of finite dimensional normed spaces, this is simply the definition given above in the corresponding normed-space formulation.

2. **Lower Semi-Continuous Convex Functions:** For the class of lower semi-continuous convex functions with values in  $\mathbb{R} \cup \{\pm\infty\}$ , which we denote by  $Cvx(\mathbb{R}^n)$ , one usually interprets the standard Legendre transform (1) as a form of duality.
3. **Log-Concave Functions:** For the class of upper semi-continuous log-concave functions on  $\mathbb{R}^n$  a form of duality was defined and investigated in [4]. It is defined (again, after fixing a scalar product on  $\mathbb{R}^n$ ) for a function  $f = e^{-\phi}$  by the formula  $f^\circ = e^{-\mathcal{L}\phi}$ , or, more explicitly, by

$$(2) \quad f^\circ(x) = \inf_{y \in \mathbb{R}^n} \frac{e^{-\langle x, y \rangle}}{f(y)}.$$

Treating this operation as a form of “duality” was partially justified in the paper [4] (one may also add to this the results from [12] and [13]). It was checked, for example, that several facts, which we are used to associate with “duality”, hold, such as the Santaló-type inequalities (a fact that for the case of even functions, it turned out, was proven years before in [5]).

4. **Non-negative  $s$ -Concave Function, non-zero at the origin:** Another meaningful class is, for a fixed  $s > 0$ , the class of upper semi-continuous non-negative functions on  $\mathbb{R}^n$  which are  $s$ -concave, namely,  $f^{1/s}$  is concave on its convex support that includes 0. (Another option is to consider only bounded functions, and require  $f(0) > 0$ , or to ask 0 to be in the interior of the support and the support to be compact.) The duality transform that was defined and applied in [4] for this class of functions is as follows

$$(3) \quad (\mathcal{L}_s f)(x) = \inf_{\{y: f(y) > 0\}} \frac{\left(1 - \frac{\langle x, y \rangle}{s}\right)_+^s}{f(y)}.$$

It is clear that all the above examples satisfy abstract duality, namely, they are involutive and order-reversing. Another noticeable property is that the infimum of two functions, or, when it is not in the class, its regularization, is transformed to the supremum (or, if it is not in the class, to its regularization) and vice versa. This property is, in fact, *implied* by the two aforementioned properties, as presented in Proposition 2.1. There are also some special properties of each of the specific transforms, such as were listed above for the Legendre transform.

The second named author has recently brought up the question regarding an abstract characterization of duality. The original question, which he posed on several occasions, was regarding the usual polarity of convex bodies in Example 2. This was prompted by the joint work in [4], where the definition for functional duality from Example 3 above was suggested, which seemed “right” in the sense that the inequalities it anticipated turned out to be either known, or provable. However, many more inequalities of the “same nature” were found and proven, see [8], [13] and also the older results of [7], which disturbed the picture. It became clear that

a more basic understanding of what should characterize duality was necessary. The question whether there are different possible “duality” notions, became intriguing and even bothering. We had then realized that even in the very well-known and commonly used notion of polarity for convex bodies, no axiomatic definition was known, and the concrete definition was simply used as “god-given”.

For this reason, it was important to understand whether the concrete form follows from several abstract conditions. The conditions, which were first suggested, were not exactly the “abstract duality” defined above, which seemed too optimistic, but rather the involutive property, and the exchange of the operations of convex hull and intersection. This was, very recently, answered by Böröczky and Schneider in the paper [6], where it was shown that, indeed, for polarity of convex bodies, these conditions alone imply already that for some choice of an Euclidean structure (and signature), duality is defined in the standard way. Thus, both Examples 1 and 2 above were settled, abstractly, up to some difference in the conditions. For the case of symmetric convex bodies, namely Example 1 above, they show that it follows quite easily, using one additional trick, from a Theorem of Gruber [9] [10] (although he did not regard this specific type of question), and they give a full proof of the non-symmetric case. We formulate their result in Section 4.

The fact that such an elegant theorem holds in the case of convex bodies, increased our belief that it may also be true for other forms of duality, in particular, the various functional dualities that were presented in the examples above. It turned out that the formally weaker “abstract duality” defined above is enough to characterize duality, and to characterize it in **all** the above examples simultaneously.

Theorems 1, 2 and 3 show how the condition of “abstract duality” uniquely determines the form of the transform, up to linear transforms and constants, in several important classes  $\mathcal{S}$ , corresponding to Examples 2, 3 and 4 above. As explained above, examples 1(a) and 1(b) were settled in [6], see also [9] and [10].

Another point, which is interesting, is that, in our understanding, there is a clear division of these type of results into two classes: where 0 plays a special role, and where the choice of 0 is of no intrinsic importance. The family of convex bodies, of normed spaces, and the family of  $s$ -concave functions and its relatives corresponds to the first class. The family of convex functions and its various relatives, such as log-concave functions, corresponds to the second class. The proofs for the two types of classes are quite different, and as a rule seem to be slightly more difficult for the first type.

We would like to remark that, as may be anticipated, the proofs of the various theorems are quite technical, and a detailed inspection of many cases, and the division into many steps, is often required. However, the end results are a clean characterization of the duality concept for classes of functions, and we feel it is well worth the effort. For these technical details, see [2] and [3], as here we only sketch the proofs.

We chose to include a detailed proof for two new theorems, which were not presented elsewhere. The first is a theorem of a somewhat different flavor than what we presented above, concerning the Legendre transform. We show that the condition of involution, together with the fact that sum is exchanged with inf-convolution, is enough to imply the special form of the transform. The proof is somewhat simpler than the proofs of the theorems mentioned above. Second, we are able to essentially weaken the condition of Theorem 1 and Corollary 12 below,

and arrive at a slightly more general conclusion. More precisely, we show that if a 1-1 and onto transformation  $\mathcal{T}$  from  $Cvx(\mathbb{R}^n)$  to itself satisfies that a comparable pair of functions (meaning that one is greater than or equal to the other) is always mapped to a comparable pair of functions, and the same is true for  $\mathcal{T}^{-1}$ , then it must be the case that either the order is reversed for *all* pairs of comparable functions, which, in turn, will mean that up to linear terms  $\mathcal{T}$  is the Legendre transform, or the order is preserved for *all* pairs of comparable functions, which means that  $\mathcal{T}$  is, up to linear terms, identity of  $Cvx(\mathbb{R}^n)$ .

The paper is organized as follows: In Section 2 we explain one of the simple tools used in the proofs of several of the theorems. In Section 3 we discuss the main ideas in the proof of Theorem 1. In Section 4 we discuss the corresponding theorems for convex bodies. In Section 5 we present some elements of the proofs for Theorem 3 regarding  $s$ -concavity. In Section 6 we present the new theorem characterizing transformations that exchange inf-convolution with summation, with a complete proof. In Section 7 we present a corollary regarding order-preserving transformations, and another regarding the class of Log-concave functions. Finally, in Section 8 we characterize transformations that map comparable pairs to comparable pairs. We then remark on variants for some other classes.

## 2. INTERCHANGING MAX AND MIN

Let  $\mathcal{S}$  be some abstract class of functions. It need not be closed under the operations of min and max (for example,  $Cvx(\mathbb{R}^n)$  is not closed under min). However, in many cases, one can define for any two functions in the class their regularized minimum and maximum. Namely, for two functions  $f, g \in \mathcal{S}$ , one may define the function  $\hat{\max}(f, g) := \inf\{h(x) : h \in \mathcal{S}, h \geq f \text{ and } h \geq g\}$  and  $\hat{\min}(f, g) := \sup\{h(x) : h \in \mathcal{S}, h \leq f \text{ and } h \leq g\}$ . These may or may not belong to  $\mathcal{S}$ , and notice that in all the aforementioned classes, they do belong to the class. (For example, in the class  $Cvx(\mathbb{R}^n)$ ,  $\hat{\max}$  is the usual maximum, and  $\hat{\min}$  is the supremum of all l.s.c. convex functions that lie below  $\min(f, g)$  and thus is also convex and lower semi-continuous.) The following propositions are very useful in the proofs of the theorems mentioned in the introduction.

**Proposition 2.1.** *Let  $\mathcal{T} : \mathcal{S} \rightarrow \mathcal{S}$  be a 1-1 and onto transform satisfying for all  $\phi$  and  $\psi$*

- (1)  $\phi \leq \psi$  implies  $\mathcal{T}\phi \geq \mathcal{T}\psi$ ,
- (2)  $\mathcal{T}\phi \leq \mathcal{T}\psi$  implies  $\phi \geq \psi$ .

*Then for any  $f, g$  such that  $\hat{\min}(f, g) \in \mathcal{S}$  and  $\hat{\max}(\mathcal{T}f, \mathcal{T}g) \in \mathcal{S}$ , we have*

$$\mathcal{T}(\hat{\min}(f, g)) = \hat{\max}(\mathcal{T}f, \mathcal{T}g),$$

*and vice versa, if  $\hat{\max}(f, g) \in \mathcal{S}$  and  $\hat{\min}(\mathcal{T}f, \mathcal{T}g) \in \mathcal{S}$ , then*

$$\mathcal{T}(\hat{\max}(f, g)) = \hat{\min}(\mathcal{T}f, \mathcal{T}g).$$

The proof is straightforward, see e.g. Lemma 4 in [3].

It follows, of course, that if  $\mathcal{T}$  satisfies  $\mathcal{T}\mathcal{T}\phi = \phi$  for all  $\phi$  (which implies 1-1 and onto) and that  $\phi \leq \psi$  implies  $\mathcal{T}\phi \geq \mathcal{T}\psi$ , then the conclusion of the above proposition holds.

Similarly, we can show

**Proposition 2.2.** *Let  $\mathcal{T} : \mathcal{S} \rightarrow \mathcal{S}$  be a 1-1 and onto transform satisfying*

- (1)  $\phi \leq \psi$  implies  $\mathcal{T}\phi \geq \mathcal{T}\psi$ ,
- (2)  $\mathcal{T}\phi \leq \mathcal{T}\psi$  implies  $\phi \geq \psi$ .

*Then for any family  $f_\alpha \in \mathcal{S}$  for which  $\hat{\inf}_\alpha(f_\alpha) := \sup\{h(x) : h \in \mathcal{S}, h \leq f_\alpha \forall \alpha\}$  belongs to  $\mathcal{S}$  and  $\hat{\sup}_\alpha(\mathcal{T}f_\alpha) := \inf\{h(x) : h \in \mathcal{S}, h \geq \mathcal{T}f_\alpha \forall \alpha\}$  belong to  $\mathcal{S}$  as well, we have*

$$\mathcal{T}(\hat{\inf}(f_\alpha)) = \hat{\sup}(\mathcal{T}f_\alpha),$$

*and vice versa.*

### 3. THE CLASS $Cvx(\mathbb{R}^n)$ AND THEOREM 1

To prove Theorem 1 above, one actually proves a similar theorem with a weaker assumption, namely that both  $\mathcal{T}$  and  $\mathcal{T}^{-1}$  are the order reversing. The conclusion will be a slightly more general formula for the transform, but still á-la-Legendre, and to arrive at the conclusion of Theorem 1 one invokes the involution assumption on the formula to particularize it to the form given in Theorem 1. More precisely, we may show

**Theorem 5.** *Let  $\mathcal{T} : Cvx(\mathbb{R}^n) \rightarrow Cvx(\mathbb{R}^n)$  be a 1-1 and onto transform satisfying*

- (1)  $\phi \leq \psi$  implies  $\mathcal{T}\phi \geq \mathcal{T}\psi$ ,
- (2)  $\mathcal{T}\phi \leq \mathcal{T}\psi$  implies  $\phi \geq \psi$ .

*Then there exist constants  $C_0 \in \mathbb{R}, C_1 > 0$ , two vectors  $v_0, v_1 \in \mathbb{R}^n$ , and an invertible linear transformation  $B \in GL_n$  such that*

$$(\mathcal{T}\phi)(x) = C_0 + \langle v_1, x \rangle + C_1(\mathcal{L}\phi)(B(x + v_0)).$$

The full proof of Theorem 5 appears in [2], and here we explain the three main parts in the proof. The first part follows from Proposition 2.2, namely the above conditions imply that

$$\mathcal{T}(\sup_\alpha f_\alpha) = \hat{\inf}_\alpha \mathcal{T}f_\alpha \quad \text{and} \quad \mathcal{T}(\hat{\inf}_\alpha f_\alpha) = \sup_\alpha \mathcal{T}f_\alpha.$$

The second part of the proof consists of the simple observation that, to know the general form of the transform, it is enough to determine it on elementary functions such as affine linear functions or delta-type functions. For simplicity of notation we will denote the function  $-\log \delta_\theta(x)$  by  $D_\theta(x)$ , that is,

$$D_\theta(\theta) = 0 \text{ and } D_\theta(y) = +\infty \text{ for } y \neq \theta.$$

These functions clearly belong to  $Cvx(\mathbb{R}^n)$ , and so do their parallels (or shifts)  $D_\theta + c$ . Moreover, any function can be expressed as the infimum of such shifted functions, namely  $\phi(x) = \inf_y (D_y(x) + \phi(y))$ . Taking this into account, the following lemma is almost straightforward.

**Lemma 6.** *Let  $\mathcal{T} : Cvx(\mathbb{R}^n) \rightarrow Cvx(\mathbb{R}^n)$  be a 1-1 and onto transform (defined on the whole domain  $Cvx(\mathbb{R}^n)$ ) satisfying*

- (1)  $\phi \leq \psi$  implies  $\mathcal{T}\phi \geq \mathcal{T}\psi$ ,
- (2)  $\mathcal{T}\phi \leq \mathcal{T}\psi$  implies  $\phi \geq \psi$ ,
- (3) *There exist  $C_0, \in \mathbb{R}, C_1 > 0, B \in GL_n$  and  $v_0, v_1 \in \mathbb{R}^n$  such that for any  $\theta$  and  $c$  we have*

$$\mathcal{T}(D_\theta + c) = \langle B\theta + v_1, \cdot \rangle + \langle v_0, \theta \rangle - C_1c + C_0.$$

Then  $\mathcal{T}$  is a variant of the Legendre transform defined by

$$(\mathcal{T}\phi)(x) = C_0 + \langle v_1, x \rangle + C_1(\mathcal{L}\phi)(B'x + v'_0).$$

for  $B' = B^*/C_1 \in GL_n$  and  $v'_0 = v_0/C_1$ .

A similar Lemma can be shown with the condition (3), namely, knowledge about  $\mathcal{T}(D_\theta + c)$ , replaced by knowledge about  $\mathcal{T}(\langle \cdot, \theta \rangle + c)$ , since any convex l.s.c. function can be written as the supremum of such affine linear functions.

Finally, the third and most involved part is to determine the formula for  $\mathcal{T}(D_\theta + c)$ , and to show that it must be of the form given in (3) above. To do this, one first shows that these delta-type functions must be mapped to affine linear functions and vice versa, this is done by using the fact that any two functions in  $Cvx(\mathbb{R}^n)$  that are greater than  $D_\theta + c$  are comparable, which means that any two functions below  $\mathcal{T}(D_\theta + c)$  are comparable, which can hold only if its gradient is constant, namely if it is (affine) linear. This implies that

$$\mathcal{T}(D_\theta + c) = \langle u, \cdot \rangle + c',$$

where  $u = u(\theta)$  and  $c' = c'(\theta, c)$  (the fact that  $u$  does not depend on  $c$  is easy). Then, to determine the formulae for these functions, we show that the function  $F(\theta, c) = (u(\theta), c'(\theta, c))$  preserves intervals, and then use the fundamental theorem of projective geometry (or rather, its version for affine geometry) that implies that  $F$  must be linear. This already provides enough information to determine the function  $F$ , and thus  $\mathcal{T}(D_\theta + c)$ , completely.

#### 4. THE CLASS OF INDICATORS OF CONVEX BODIES

The following result is another manifestation of the abstract concept of duality. As we already noted, it turns out that even in the simple and well studied case of convex bodies, the result was not known, and recently this question, which was asked by the second named author, was answered in a paper of Böröczky and Schneider [6]. Denote by  $\Xi_{(0)}(\mathbb{R}^n)$  the class of indicator functions of compact convex bodies in  $\mathbb{R}^n$  including 0 in their interior. The same result also holds when the class is restricted to include only symmetric sets, in which case it follows from the results of Gruber [10].

**Theorem 7** (Böröczky-Schneider [6]). *Let  $n \geq 2$ . Assume we are given a transform  $\mathcal{T} : \Xi_{(0)}(\mathbb{R}^n) \rightarrow \Xi_{(0)}(\mathbb{R}^n)$  (defined on the whole domain  $\Xi_{(0)}(\mathbb{R}^n)$ ) satisfying*

- (1)  $\mathcal{T}\mathcal{T}\chi_K = \chi_K$
- (2)  $\mathcal{T}(\chi_{K_1}\chi_{K_2}) = \max(\mathcal{T}\chi_{K_1}, \mathcal{T}\chi_{K_2})$

*Then, up to a symmetric linear transformation  $B \in GL_n$ ,  $\mathcal{T}$  is the usual duality transform  $\mathcal{D}$  taking the indicator function  $\chi_K$  of a body  $K$  to the indicator function of its polar body  $\mathcal{D}K = K^\circ$  defined by  $K^\circ = \{x : \sup_{y \in K} \langle x, y \rangle \leq 1\}$ . That is,*

$$\mathcal{T}\chi_K = \chi_{BK^\circ}.$$

**Remark 8.** *Clearly by Proposition 2.1 one may replace the second condition by the formally weaker*

- (2')  $\chi_{K_1} \leq \chi_{K_2}$  implies  $\mathcal{T}\chi_{K_1} \geq \mathcal{T}\chi_{K_2}$ .

In [3] we have shown that the same result holds also if we consider the larger class of convex sets that include 0 (possibly on their boundary), and are not necessarily compact. Namely, denote by  $\Xi(\mathbb{R}^n)$  the class of indicator functions of closed convex sets in  $\mathbb{R}^n$  including 0 (possibly on their boundary). We showed

**Theorem 9.** *Let  $n \geq 2$  and  $\mathcal{T} : \Xi(\mathbb{R}^n) \rightarrow \Xi(\mathbb{R}^n)$  satisfy*

- (1)  $\mathcal{T}\mathcal{T}\chi_K = \chi_K$
- (2)  $\chi_{K_1} \leq \chi_{K_2}$  implies  $\mathcal{T}\chi_{K_1} \geq \mathcal{T}\chi_{K_2}$

*Then there exists a symmetric  $B \in GL_n$  such that*

$$\mathcal{T}\chi_K = \chi_{BK^\circ} = \chi_{\{x: \langle x, By \rangle \leq 1 \ \forall y \in K\}}.$$

The proof is quite similar to that of Theorem 3 described in Section 5 below, and, in fact, slightly simpler. However, it turns out that the condition of involution can be weakened, and in fact, the following is true

**Theorem 10.** *Let  $n \geq 2$  and a 1-1 and onto  $\mathcal{T} : \Xi(\mathbb{R}^n) \rightarrow \Xi(\mathbb{R}^n)$  satisfy*

$$\chi_{K_1} \leq \chi_{K_2} \Leftrightarrow \mathcal{T}\chi_{K_1} \geq \mathcal{T}\chi_{K_2}.$$

*Then there exists  $B \in GL_n$  such that*

$$\mathcal{T}\chi_K = \chi_{BK^\circ} = \chi_{\{x: \langle x, By \rangle \leq 1 \ \forall y \in K\}}.$$

The main ideas of the proof are presented in [3], where a detailed proof of a similar result is given. A simple corollary regarding order *preserving* transformations on the class of closed convex sets in  $\mathbb{R}^n$  including 0, (or, equivalently,  $\Xi(\mathbb{R}^n)$ ) is given in Section 7.

## 5. THE CLASS OF $s$ -CONCAVE FUNCTIONS AND THEOREM 3

The proof of Theorem 3 is presented in [3], and we only remark on its main ingredients, which are slightly different from those of Theorem 1 because for  $s$ -concave functions (as for convex bodies) 0 plays a special role.

Firstly, it is enough to limit the discussion to the case  $s = 1$ , since for any  $s > 0$  and any concave  $g$  we have that  $(\mathcal{L}_s(g^s))^{1/s} = \mathcal{L}_1 g$ , and so any transform  $\mathcal{T}$  on  $s$ -concave functions can be translated to a transform on 1-concave functions by  $\mathcal{T}'g = (\mathcal{T}g^s)^{1/s}$ , which, as we will show below, will imply that  $\mathcal{T}'$  is, up to a linear transform and a constant,  $\mathcal{L}_1$ , which in turn means  $\mathcal{T}$  is, up to a linear transform and a constant,  $\mathcal{L}_s$ .

Second, it turns out that the proof for dimension  $n \geq 2$  is different from the one in dimension 1. Here we do not remark at all on the case  $n = 1$ , see [3] for the details.

Thirdly, we first prove the theorem for a subclass of functions, those attaining their maximum at 0 (which we denote by  $\text{Conc}^{(m)}(\mathbb{R}^n)$ ). To then pass to general 1-concave functions, we start by showing that the class of concave functions with maximum at 0 is an invariant subclass under the transform  $\mathcal{T}$ . After this, we explain why the transform on the whole class is given by the same formula. To this end, we compose the transform  $\mathcal{T}$  with the standard  $\mathcal{L}_1$  transform (modified by  $B$ ), get a transform that is identity on the subclass of functions attaining their maximum at 0, which is order *preserving* and invertible, and its inverse is order *preserving* as well. We then show that this is enough to imply it is the identity transform.

As for the proof of Theorem 3 for the subclass of functions attaining their maximum at 0, it is quite long and technical. The elementary functions we use are triangle functions  $\triangleleft_x$  defined on  $[0, x]$  by  $\triangleleft_x(tx) = 1 - t$  for  $0 \leq t \leq 1$ , and zero elsewhere. This is the smallest concave function with support  $[0, x]$  and which is

greater than  $\delta_0$ , the function attaining 1 at 0 and 0 elsewhere. Each concave function attaining its maximum at 0 is the supremum of such functions. We also use Proposition 2.2, of course, and the fundamental theorem of affine geometry, as in the case of Legendre transform.

## 6. INF-CONVOLUTION

Recall the definition of the inf-convolution of two functions  $f, g \in Cvx(\mathbb{R}^n)$ , given by

$$(f \square g)(z) = \inf_{x+y=z} (f(x) + g(y)).$$

We need some convention to decide what to do with  $-\infty + \infty$ . We omit the justification for the following reasonable agreement:  $(-\infty) \square f \equiv -\infty$  for all  $f$ , and  $(-\infty) + f = -\infty$  for all  $f \not\equiv +\infty$ , and, as functions,  $-\infty + \infty \equiv +\infty$ .

**Theorem 11.** *Let  $\mathcal{T} : Cvx(\mathbb{R}^n) \rightarrow Cvx(\mathbb{R}^n)$  be a transform (defined on the whole domain) satisfying*

- (1)  $\mathcal{T}\mathcal{T}f = f$
- (2)  $\mathcal{T}f + \mathcal{T}g = \mathcal{T}(f \square g)$

*Then there exists a symmetric linear transformation  $B \in GL_n$  such that for all  $f$*

$$(\mathcal{T}f)(x) = \sup_{y \in \mathbb{R}^n} (\langle Bx, y \rangle - f(y)).$$

*Proof. Step Zero:* The transform of the constant function  $+\infty$  must be the constant function  $-\infty$  and vice versa.

Indeed, since for any function  $\varphi \not\equiv -\infty$  we have that  $+\infty \square \varphi = +\infty$  then  $\mathcal{T}(+\infty) + \mathcal{T}\varphi = \mathcal{T}(+\infty)$ . However,  $\mathcal{T}\varphi$  can be any function, and thus  $(\mathcal{T}(+\infty))(z) \in \{+\infty, -\infty\}$  and thus from convexity it is one of the two constant functions. (Similarly for the constant function  $-\infty$ ). However, it cannot be that  $\mathcal{T}(+\infty) = +\infty$  since, for example,  $D_0 + D_1 = +\infty$  so that  $\mathcal{T}D_0 \square \mathcal{T}D_1 = \mathcal{T}(+\infty)$ , but since there exist  $x_0$  and  $x_1$  such that  $(\mathcal{T}D_0)(x_0) \neq \infty$  and  $(\mathcal{T}D_1)(x_1) \neq \infty$  (because we know that either  $\mathcal{T}(+\infty) = +\infty$  or  $\mathcal{T}(-\infty) = +\infty$ ). But then  $((\mathcal{T}D_0) \square (\mathcal{T}D_1))(x_0 + x_1) < +\infty$ . Thus, the only option is that  $\mathcal{T}(+\infty) = -\infty$  (and so,  $\mathcal{T}(-\infty) = +\infty$ ).

**Step One:**  $\mathcal{T}D_0 = 0$ . Indeed, fix some  $z$  and pick a function  $f$  with  $f(z) = 1$ . Then, as  $D_0 \square f = f$ , we have that  $\mathcal{T}D_0 + \mathcal{T}f = \mathcal{T}f$ , and, in particular,  $\mathcal{T}D_0(z) = 0$ . Since  $z$  was arbitrary, we are done.

**Step Two:** There exists an additive function  $A : \mathbb{R} \rightarrow \mathbb{R}$  (we will later show it is, in fact, multiplication by -1) such that for any constant  $c$  we have (denoting by  $c$  also the function that is identically equal to  $c$ )  $\mathcal{T}c = A(c) + D_0$ . Indeed,

$$(0 \square c) = c,$$

and thus  $\mathcal{T}0 + \mathcal{T}c = \mathcal{T}c$ , but from Step One (and the involution property)  $\mathcal{T}0 = D_0$ , so that

$$\mathcal{T}c = \mathcal{T}0 + \mathcal{T}c = A + D_0$$

with  $A = (\mathcal{T}c)(0)$  (note that  $(\mathcal{T}c)(0) \neq +\infty$  since then we would get from the above equation that  $\mathcal{T}c \equiv +\infty$ , which would contradict 1-1). Moreover, if  $\mathcal{T}c_1 = A_1 + D_0$  and  $\mathcal{T}c_2 = A_2 + D_0$ , then  $\mathcal{T}(c_1 + c_2) = \mathcal{T}(c_1 \square c_2) = \mathcal{T}c_1 + \mathcal{T}c_2 = (A_1 + A_2 + D_0)$ , so the additivity of the function  $A$  follows.

**Step Three:** Consider the linear functionals  $(\phi_u)(x) = \langle u, x \rangle$ . We claim first that there is some additive map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\mathcal{T}\phi_u = D_{F(u)}$  (we will later show that  $F$  is, in fact, linear). First note that  $\phi_u \square \phi_u = \phi_u$ , thus  $\mathcal{T}\phi_u = 2\mathcal{T}\phi_u$ ,

which implies  $(\mathcal{T}\phi_u)(x) \in \{0, +\infty\}$ , i.e., there is some closed convex set  $A_u$  such that  $\mathcal{T}\phi_u = D_{A_u}$ , where we define for a set  $A$  the function

$$D_A(x) = 0 \text{ if } x \in A, \text{ and } +\infty \text{ otherwise}$$

(which is in  $Cvx(\mathbb{R}^n)$  if  $A$  is convex and closed). Moreover, for  $u \neq v$  we have  $\phi_u \square \phi_v = -\infty$ , so that  $\mathcal{T}\phi_u + \mathcal{T}\phi_v = \infty$ , that is,  $A_u \cap A_v = \emptyset$ . More importantly,  $\phi_u + \phi_v = \phi_{u+v}$ , so that  $\mathcal{T}\phi_u \square \mathcal{T}\phi_v = \mathcal{T}\phi_{u+v}$ , which implies directly that  $A_{u+v} = A_u + A_v$ . We also know that  $A_0 = \{0\}$ , so, in particular,  $A_u + A_{-u} = \{0\}$ , which, of course, means that  $A_u$  is a point  $F(u)$  for every  $u$ . The function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is additive,  $F(u+v) = F(u) + F(v)$ . In particular, we have that  $F|_{\mathbb{Q}^n}$  is linear, say, is given by  $F|_{\mathbb{Q}^n} = B \in GL_n(\mathbb{Q})$ .

**Step Four:** We see that also  $\mathcal{T}D_{x_0} = \phi_{F^{-1}x_0}$  from the involution property. However, for any  $f$  we have that  $c + f = (c + D_0) \square f$  and thus

$$\mathcal{T}(c + f) = \mathcal{T}(c + D_0) + \mathcal{T}f = A(c) + \mathcal{T}f,$$

so that shifts are mapped to shifts. Moreover, we saw in Step Two that the relation between  $c$  and  $A$  is additive. It is also involutive since we may reiterate

$$\mathcal{T}(\mathcal{T}(c + f)) = \mathcal{T}(A(c) + \mathcal{T}f) = A(A(c)) + f,$$

and so  $A(A(c)) = c$ .

**Step Five:** Consider again  $\phi_u = \langle \cdot, u \rangle$ , and let  $g = \phi_u \square D_{x_0}$ . Then

$$g(x) = \inf_y (\phi_u(y) + D_{x_0}(x - y)) = \phi_u(x - x_0) = \phi_u(x) - \langle x_0, u \rangle.$$

Apply the transform: on the one hand, we get

$$\mathcal{T}\phi_u + \mathcal{T}D_{x_0} = D_{F(u)} + \langle F^{-1}(x_0), \cdot \rangle = D_{F(u)} + \langle F^{-1}(x_0), F(u) \rangle.$$

On the other hand, we get

$$\mathcal{T}(\phi_u - \langle x_0, u \rangle) = \mathcal{T}\phi_u - A(\langle x_0, u \rangle) = D_{F(u)} - A(\langle x_0, u \rangle).$$

(The fact that  $A(-c) = -A(c)$  follows from the additivity of  $A$ .)

Thus, we see that  $-A(\langle x_0, u \rangle) = \langle F^{-1}(x_0), F(u) \rangle$  for every  $u$  and every  $x_0$ .

**Step Six:** Consider  $g = f + D_{x_0}$  for a general  $f$ . Then we know that  $g = f(x_0) + D_{x_0}$ , so

$$A(f(x_0)) + \langle F^{-1}(x_0), \cdot \rangle = \mathcal{T}f \square \mathcal{T}D_{x_0} = \inf_y ((\mathcal{T}f)(y) + \langle F^{-1}(x_0), \cdot - y \rangle).$$

Rewrite  $\psi = \mathcal{T}f$  to get that for every  $x$

$$A((\mathcal{T}\psi)(x_0)) = -\langle F^{-1}(x_0), x \rangle + \inf_y (\psi(y) + \langle F^{-1}(x_0), x - y \rangle),$$

which, in turn, means, letting  $x = 0$ , that

$$(\mathcal{T}\psi)(x_0) = -A \left( \sup_y (-\psi(y) + \langle F^{-1}(x_0), y \rangle) \right)$$

(we have used again that  $A(-c) = -A(c)$ ).

Because  $F$  is only additive, and not known in advance to be linear, to be able to use an associated linear map, we begin by restricting the considerations to rational  $x_0$  only (i.e., with all rational coordinates). On the subset  $\mathbb{Q}^n$  we have  $F = B \in GL_n(\mathbb{Q})$  so that

$$(\mathcal{T}\psi)(x_0) = -A \left( \sup_y (-\psi(y) + \langle B^{-1}x_0, y \rangle) \right).$$

The part inside the brackets is a Legendre-type transform of  $\psi$  (restricted to  $\mathbb{Q}^n$ ). Assume by contradiction that  $A$  is not linear. Since an additive non-linear function is, in particular, not monotone, we may find two values  $c_1 < c_2$  such that  $A(c_1) < A(c_2)$ . Then we choose  $\psi \in Cvx(\mathbb{R}^n)$  such that for some pre-chosen three vectors  $x, y$  and  $z \in [x, y]$  we have that

$$\mathcal{L}'\psi = \sup_y(-\psi(y) + \langle B^{-1}, y \rangle)$$

satisfies  $(\mathcal{L}'\psi)(x) = (\mathcal{L}'\psi)(y) = c_2$  and  $(\mathcal{L}'\psi)(z) = c_1$ .

By the above formula we have that

$$(\mathcal{T}\psi)(x) = (\mathcal{T}\psi)(y) = -A(c_1) \text{ and } (\mathcal{T}\psi)(z) = -A(c_2) > -A(c_1)$$

which is a contradiction to  $(\mathcal{T}\psi) \in Cvx(\mathbb{R}^n)$ . We conclude that  $A$  is linear, and so (since  $A(A(c)) = c$ ) we have  $A(c) = A(1)c$  for all  $c$ , where  $A(1) = -1$  or  $A(1) = 1$ . Going back to the formula above we see that for all  $\psi$

$$(\mathcal{T}\psi)(x_0) = \pm \sup_y(-\psi(y) + \langle F^{-1}(x_0), y \rangle).$$

Clearly, to preserve convexity the sign next to the expression on the right hand side should be positive, hence  $A(c) = -c$ .

Moreover, a function in  $Cvx(\mathbb{R}^n)$  is determined by its values on  $\mathbb{Q}^n$ , and since this function coincides with the function

$$\sup_y(-\psi(y) + \langle B^{-1}(x_0), y \rangle)$$

on  $\mathbb{Q}^n$ , we conclude that they are the same on all  $\mathbb{R}^n$ .

Finally, the identity  $-A(\langle x_0, u \rangle) = \langle F^{-1}(x_0), F(u) \rangle$  is now translated into  $\langle x_0, u \rangle = \langle B^{-1}x_0, Bu \rangle$ , which means that  $B = B^*$ .  $\square$

## 7. COROLLARIES AND ADDITIONAL RESULTS

First, we describe a corollary regarding order preserving transformations.

**Corollary 12.** *Assume a 1-1 and onto transform  $\mathcal{F} : Cvx(\mathbb{R}^n) \rightarrow Cvx(\mathbb{R}^n)$  satisfies*

- (1)  $\phi \leq \psi$  implies  $\mathcal{F}\phi \leq \mathcal{F}\psi$ ,
- (2)  $\mathcal{F}\phi \leq \mathcal{F}\psi$  implies  $\phi \leq \psi$ .

*Then there exist constants  $C_0 \in \mathbb{R}, C_1 > 0$ , two vectors  $v_0, v_1 \in \mathbb{R}^n$ , and an invertible linear transformation  $B \in GL_n$  such that*

$$(\mathcal{F}\phi)(x) = C_0 + \langle v_1, x \rangle + C_1\phi(Bx + v_0).$$

For the proof, consider  $\mathcal{T} = \mathcal{F} \circ \mathcal{L}$ , it satisfies the conditions of Theorem 5, and we are done.

**Theorem 13.** *Let  $n \geq 2$  and a 1-1 and onto transform  $\mathcal{F} : \mathcal{K}^n \rightarrow \mathcal{K}^n$  satisfies*

$$K_1 \subset K_2 \Leftrightarrow \mathcal{F}K_1 \subset \mathcal{F}K_2.$$

*Then there exists  $B \in GL_n$  such that*

$$\mathcal{F}K = BK.$$

Again, for the proof define  $\mathcal{T}$  by  $\mathcal{T}(K) = \mathcal{F}(K^\circ)$ , then it satisfies the conditions of Theorem 10, and we are done.

This result should be compared with the results of P. Gruber [9], [10], who studied the endomorphisms of the lattice of compact convex bodies in  $\mathbb{R}^n$  with respect to the operations of intersection and of convex hull of union.

Secondly, each of the theorems regarding the class  $Cvx(\mathbb{R}^n)$  has as an immediate corollary a corresponding theorem for the class  $LC(\mathbb{R}^n)$  of log-concave functions. For example, if the Asplund product of two functions is defined by  $(f \star g)(z) = \sup_{x+y=z} f(x)g(y)$ , Theorem 11 implies the following

**Theorem 14.** *Let  $\mathcal{T} : LC(\mathbb{R}^n) \rightarrow LC(\mathbb{R}^n)$  be a transform (defined on the whole domain) satisfying*

- (1)  $\mathcal{T}\mathcal{T}f = f$  for all  $f$ ,
- (2)  $\mathcal{T}f \cdot \mathcal{T}g = \mathcal{T}(f \star g)$  for all  $f, g$ .

*Then there exists a symmetric linear transformation  $B \in GL_n$  such that for all  $f$*

$$(\mathcal{T}f)(x) = \inf_{y \in \mathbb{R}^n} e^{-\langle Bx, y \rangle} / f(y).$$

## 8. TRANSFORMS PRESERVING COMPARABILITY

In this section we show the strengthening of Theorem 5 mentioned in the introduction, namely, that if we know that comparable pairs of functions are mapped to comparable pairs, then all we need to know is whether the order is preserved or reversed for *one* pair of (non-identical) comparable functions, and we know what the transform is (up to linear terms). We prove:

**Theorem 15.** *Let  $\mathcal{T} : Cvx(\mathbb{R}^n) \rightarrow Cvx(\mathbb{R}^n)$  be a 1-1 and onto transform that satisfies that for every  $\phi$  and  $\psi$  such that  $\phi \leq \psi$ , either  $\mathcal{T}\phi \geq \mathcal{T}\psi$  or  $\mathcal{T}\phi \leq \mathcal{T}\psi$ , and also for every  $\phi$  and  $\psi$  such that  $\mathcal{T}\phi \leq \mathcal{T}\psi$ , either  $\phi \leq \psi$  or  $\phi \geq \psi$ . Then there exist constants  $C_0 \in \mathbb{R}, C_1 > 0$ , two vectors  $v_0, v_1 \in \mathbb{R}^n$ , and  $B \in GL_n$  such that either for every  $\phi$  that is not identically  $+\infty$  or identically  $-\infty$*

$$(\mathcal{T}\phi)(x) = C_0 + \langle v_1, x \rangle + C_1(\mathcal{L}\phi)(Bx + v_0),$$

*or for every  $\phi$  that is not identically  $+\infty$  or identically  $-\infty$*

$$(\mathcal{T}\phi)(x) = C_0 + \langle v_1, x \rangle + C_1\phi(Bx + v_0),$$

*and we have either  $\mathcal{T}(+\infty) = +\infty$  and  $\mathcal{T}(-\infty) = -\infty$  or  $\mathcal{T}(+\infty) = -\infty$  and  $\mathcal{T}(-\infty) = +\infty$ .*

We start with a simpler statement.

**Lemma 16.** *Let  $\mathcal{T} : Cvx(\mathbb{R}^n) \rightarrow Cvx(\mathbb{R}^n)$  be a 1-1 and onto transform such that for every  $\phi$  one of the two following statements is true:*

- (1) *For every  $\psi$  except, possibly, the functions identically  $+\infty$  or identically  $-\infty$ , we have  $\phi \leq \psi \Leftrightarrow \mathcal{T}\phi \geq \mathcal{T}\psi$  and  $\psi \leq \phi \Leftrightarrow \mathcal{T}\psi \geq \mathcal{T}\phi$ ,*

*or*

- (2) *For every  $\psi$  except, possibly, the functions identically  $+\infty$  or identically  $-\infty$ , we have  $\phi \leq \psi \Leftrightarrow \mathcal{T}\phi \leq \mathcal{T}\psi$  and  $\psi \geq \phi \Leftrightarrow \mathcal{T}\psi \geq \mathcal{T}\phi$ .*

Then there exist constants  $C_0 \in \mathbb{R}, C_1 > 0$ , two vectors  $v_0, v_1 \in \mathbb{R}^n$ , and  $B \in GL_n$  such that either for every  $\phi$  that is not identically  $+\infty$  or identically  $-\infty$

$$(\mathcal{T}\phi)(x) = C_0 + \langle v_1, x \rangle + C_1(\mathcal{L}\phi)(Bx + v_0),$$

or for every  $\phi$  that is not identically  $+\infty$  or identically  $-\infty$

$$(\mathcal{T}\phi)(x) = C_0 + \langle v_1, x \rangle + C_1\phi(Bx + v_0),$$

and we have either  $\mathcal{T}(+\infty) = +\infty$  and  $\mathcal{T}(-\infty) = -\infty$  or  $\mathcal{T}(+\infty) = -\infty$  and  $\mathcal{T}(-\infty) = +\infty$ .

The reader may have noticed a strange ‘‘omission’’ in the statement of the Proposition, namely, we require each condition to hold for all functions excluding the identically infinite functions. It is easy to see that the same theorem holds without this omission, however it is in the version written above that we use the theorem. In the proof we give below, we will first prove the theorem with the extra assumption, that (1) and (2) also holds for the identically infinite functions, and then show how this can be avoided.

*Proof of Lemma 16.* The proof is a simple combination of Theorem 5 and Corollary 12, with one additional trick. Indeed, looking back at Theorem 5 and Corollary 12, all we need to do is to show that if for any function  $\phi$  either (1) or (2) holds, then actually either (1) holds for all  $\phi$  or (2) does. If it is (1) that holds for all  $\phi$ , we use Theorem 5, and if it is (2), we use Corollary 12 to arrive at the conclusion.

To do this, notice first that if a function  $f$  satisfies (1), then so does any function  $g$  comparable to it (since it cannot satisfy (2)). Similarly, if  $f$  satisfies (2) then so does any  $g$  comparable to it. However, notice that if we allow the constant infinity functions, then any two functions have a third function, which is comparable to both of them (say the constant  $+\infty$  function). Thus, the question of whether (1) or (2) holds turns out to be equivalent to the question whether  $\mathcal{T}(+\infty) = +\infty$  or  $\mathcal{T}(+\infty) = -\infty$ .

If this last argument left you with a taste of ‘‘cheating’’ in the mouth, do not worry. Notice that we did not actually need to use the identically infinite (or  $-\infty$ ) functions. Indeed, for any two functions  $f_1$  and  $f_2$  we may easily find a set of three (not identically  $\pm\infty$ ) functions  $g_1, g_2, g_3$  such that  $f_1 \leq g_1, f_2 \leq g_2$  and  $g_1, g_2 \geq g_3$ . Then, if  $f_1$  satisfies (1) (resp. (2)) then so must  $g_1$ , and thus also  $g_3$ , and thus also  $g_2$ , and consequently also  $f_2$ . Indeed, to construct the  $g_i$  we may simply look at a point  $x$  where  $f_1(x) \neq +\infty$  and a point  $y$  where  $f_2(y) \neq +\infty$ , take  $g_1(z) = D_x(z) + f_1(x)$ ,  $g_2(z) = D_y + f_2(y)$  and  $g_3 = \hat{\inf}(g_1, g_2)$ .

We have thus proven Lemma 16.  $\square$

We are now ready to begin the proof of Theorem 15. Before we begin, since we can see in the statement of the Theorem that the identically  $\pm\infty$  functions have a special role, let us agree that whenever not specifically mentioned, when below we write ‘‘function’’ we mean a function *not* identically  $\pm\infty$ .

*Proof of Theorem 15.* We start with simple functions, namely linear functions  $h(x) = \langle u, x \rangle + c$  and delta-type function  $D(x) = D_u(x) + c$  (where  $D_u$  is the function, which equals to 0 at  $u$  and  $+\infty$  elsewhere). We first show that the inverse image of these simple functions must satisfy the condition in Lemma 16.

**Part 1:** Fix some affine linear  $h$ , and let  $f$  be such that  $\mathcal{T}f = h$ . Assume first that  $f$  has two different points where it is  $\neq +\infty$ . Then we claim that for any function  $+\infty \neq g \geq f$  we have  $\mathcal{T}g \geq \mathcal{T}f$ . To do this, for every  $g \geq f$  (with  $g \neq f$ ) we construct another function, greater than  $f$ , which is incomparable to  $g$ . Indeed, any such function  $g$  has a point  $y$  where  $f(y) < g(y)$ . If  $g$  is not equal to  $D_y + c$ , then  $g$  is not comparable to  $D_y + f(y) (\geq f)$ . If  $g$  is of the form  $D_y + c$ , then we simply use the fact that  $f$  has at least one more point  $y'$  in its support, to get an incomparable function  $D_{y'} + f(y') (\geq f)$ . We denote this other function that is greater than  $f$  and incomparable to  $g$  by  $g'$ . By the condition in the theorem,  $\mathcal{T}g$  and  $\mathcal{T}g'$  are also incomparable. If we had  $\mathcal{T}g \leq \mathcal{T}f = h$ , we would have  $\mathcal{T}g = h - c'$  for some  $c' > 0$ , which, in turn, would mean that  $\mathcal{T}g'$  is comparable to it, since either  $\mathcal{T}g' \geq \mathcal{T}f \geq \mathcal{T}g$  or  $\mathcal{T}g' \leq \mathcal{T}f$ , so  $\mathcal{T}g' = h - c''$  is again comparable to  $\mathcal{T}g$ . Thus, we conclude that  $\mathcal{T}g \geq \mathcal{T}f$  for all  $g \geq f$ , provided  $f$  has “support” containing more than one point.

Similarly, if  $f$  has two different supporting linear functions, then for all  $-\infty \neq g \leq f$  we have  $\mathcal{T}g \geq \mathcal{T}f = h$ . Indeed, this goes along similar lines: Under this condition on  $f$ , we can construct for each function  $g \leq f$  (with  $g \neq f$ ), an incomparable function  $g' \leq f$ : we look at a point  $y$  where  $g(y) < f(y)$ . Assume first that  $f(y) \neq +\infty$ ; then we may take the supporting linear function at  $y$ . Clearly it is not smaller than  $g$ ; for  $g$  to be smaller than it,  $g$  must be linear. In this case, we can take another, different, supporting linear function (which we assumed to exist), and it is incomparable to  $g$ . Second, what if  $f(y) = +\infty$  for every point  $y$  where  $f(y) > g(y)$ ? In this case, we may simply consider  $f - 1$ ; it is clearly smaller than  $f$ , but since it has the same “support” as  $f$ , there is some point  $x$ , which is in the support of  $g$  and not in its support, hence it is not smaller than  $g$ . Of course, it cannot be larger than  $g$ , since for all the points  $y$  in the support of  $f$ ,  $f(y) = g(y) > f(y) - 1$ .

Once we have two incomparable functions below  $f$ , their images are comparable to  $f$  but not to each other, and so must both lie above  $\mathcal{T}f = h$ , as claimed.

However, from 1-1 and onto, both things cannot hold simultaneously, that is, it cannot be that all functions below  $f$ , and all functions above  $f$ , are mapped to functions above  $\mathcal{T}f = h$ . Indeed, it leaves no functions to be mapped to  $h - c$ , because these must be comparable to  $f$ , and we have ruled them all out.

To conclude what we have shown so far: For the function  $f$  such that  $\mathcal{T}f = h$ , either  $f$  is of the form  $D_y + c$ , and all functions below it are mapped to functions above  $h$ , or  $f$  is of the form  $\langle \cdot, y \rangle + c$  and all functions above it are mapped to functions above  $h$ .

Moreover, assume that the first case holds. Then all functions above  $f = D_y + c$  must be mapped to functions below  $h$ . Indeed, these functions are comparable to any function comparable to  $f$ , which, in turn, means that their image is comparable to any function comparable with  $h$ , and this does not hold for functions larger than  $h$ . Similarly, if the second case holds, then all functions below  $f = \langle \cdot, y \rangle + c$  must be mapped to functions below  $h$ . Indeed, again these functions are comparable to any function comparable to  $f$ , which, in turn, means that their image is comparable to any function comparable with  $h$ , and this does not hold for functions larger than  $h$ .

In other words, we have shown that for any affine linear  $h$ , for the special function  $f$  such that  $\mathcal{T}f = h$ , one direction in the condition of Lemma 16 holds.

Before showing that the other direction in the condition holds as well, we will show the same direction but for the inverse images of delta-type functions.

**Part 2:** We may repeat the argument with  $f$  being the inverse image of  $D = D_y + c$ , using the fact that functions larger than  $D$  are comparable to any function comparable with  $D$ . For the function  $f$  such that  $\mathcal{T}f = D$ , either  $f$  is of the form  $D_y + c$ , all functions below it are mapped to functions below  $D$ , and all functions above it to functions above  $D$ , or  $f$  is of the form  $\langle \cdot, y \rangle + c$ , all functions above it are mapped to functions below  $D$ , and all functions below it are mapped to functions above  $D$ . That is, for  $f$ , one direction in the condition of Lemma 16 holds.

**Part 3:** To show that the other direction in the condition holds as well, both for  $f = \mathcal{T}^{-1}h$  and  $f = \mathcal{T}^{-1}D$ , simply notice that there is no difference, in the data assumed, between  $\mathcal{T}$  and  $\mathcal{T}^{-1}$ ; so, everything that we have shown so far remains true if we replace  $\mathcal{T}$  by  $\mathcal{T}^{-1}$ . Thus, if  $f = \mathcal{T}h$ , also either the order is reversed between  $f$  and all functions comparable to it, or the order is preserved. Moreover, either  $f$  is of the form  $D_y + c$  or of the form  $\langle \cdot, y \rangle + c$ . A similar argument works for  $f = \mathcal{T}D$ . Thus, we may use the first part again to get the following: If  $\mathcal{T}f = h$ , then, one possible case is that  $f = D_y + c$ , all functions below  $f$  are mapped to functions above  $h$ , all functions above  $f$  are mapped to functions below  $h$ , and also, since  $\mathcal{T}^{-1}h = f$ , by Part 2 applied to  $\mathcal{T}^{-1}$  (and using that we know by choice of  $h$  that it is affine linear), all functions above  $h$  are mapped by  $\mathcal{T}^{-1}$  to functions below  $f$  and all functions below  $h$  are mapped by  $\mathcal{T}^{-1}$  to functions above  $f$ . Thus, in this case both directions of condition (1) are satisfied. Or, the second possible case is that  $f = \langle \cdot, y \rangle + c$ , all functions below  $f$  are mapped to functions below  $h$ , all functions above  $f$  are mapped to functions above  $h$ , and also  $\mathcal{T}^{-1}h = f$  so by Part 1 applied to  $\mathcal{T}^{-1}$  (and using that  $h$  is affine linear), all functions above  $h$  are mapped by  $\mathcal{T}^{-1}$  to functions above  $f$  and all functions below  $h$  are mapped by  $\mathcal{T}^{-1}$  to functions below  $f$ . Thus, in this case both directions of condition (2) are satisfied.

Similarly, for  $f$  with  $\mathcal{T}f = D$  we repeat the same argument to get also the other direction in the theorem.

To conclude, in the first three parts of the proof we have established that for a linear function and its image (which is either linear or delta-type) either condition (1) in Lemma 16 is satisfied and the order is reversed with *any* comparable function, or condition (2) in Lemma 16 is satisfied and the order is preserved with *any* comparable function. A similar argument works for a delta-type function and its image.

**Part 4:** Next we claim that either condition (1) holds for *all* the elementary functions described above (by elementary functions we mean either delta-type or affine linear), or condition (2) does. Indeed, this is very similar to the proof of Lemma 16, since for comparable functions clearly the same condition must hold for both, and any two elementary functions can be “connected” by a “chain” of comparable functions: two linear functions have some delta function above both, and two delta functions have a linear function below both, also, any linear function has some delta function above it and any delta function has some linear function below it.

**Part 5:** Next we claim that either the order is reversed for all comparable functions  $f \leq g$  or the order is preserved for all of them, depending on which of the two happens to all elementary functions. Indeed, assume the order is reversed for all

elementary functions. Consider  $f \leq g$  (with  $f \neq g$ ), and assume by contradiction that  $\mathcal{T}f \leq \mathcal{T}g$ . Of course, this rules out the option that  $g$  is of delta-type. We may thus let  $D$  be some delta function  $f \leq D \not\leq g$ . Then, because for delta functions the order is reversed, we have that  $\mathcal{T}D \leq \mathcal{T}f$ , which implies  $\mathcal{T}D \leq \mathcal{T}g$  but  $D$  and  $g$  are not comparable and this is a contradiction.

Similarly, assume the order is preserved for all elementary functions. Consider  $f \leq g$  (with  $f \neq g$ ), and assume by contradiction  $\mathcal{T}g \leq \mathcal{T}f$ . Of course this again rules out the option that  $g$  is of delta-type. We may thus let  $D$  be some delta function  $f \leq D \not\leq g$ . Then, because for delta functions the order is preserved, we have  $\mathcal{T}f \leq \mathcal{T}D$ , which implies  $\mathcal{T}g \leq \mathcal{T}D$ , but as before  $D$  and  $g$  are not comparable and this is a contradiction.

We have thus shown that either the order is preserved for all non-infinite comparable functions, or the order is reversed for all comparable non-infinite functions, and in the same way as in the proof of Lemma 16, we are done.  $\square$

A simple Corollary of Theorem 15 corresponding to the class  $LC(\mathbb{R}^n)$  holds, and we leave the formulation to the reader. A theorem about “comparable pairs” also holds for convex bodies, the proof is similar to the proof above and uses, of course, Theorem 10 and Theorem 13. We omit the proof. Denote by  $\mathcal{K}^n$  the class of closed convex sets in  $\mathbb{R}^n$  that include 0 (possibly on the boundary). We remark that  $\mathcal{K}^n$  is not standard notation, and is sometimes used to denote the class of convex bodies in  $\mathbb{R}^n$ , however we wish to be consistent with [2] and [3], and in this paper the point 0 is always assumed to be contained in the convex body.

**Theorem 17.** *Assume we are given a 1-1 and onto transformation  $\mathcal{T} : \mathcal{K}^n \rightarrow \mathcal{K}^n$  (defined on the whole domain) such that for every  $K_1 \subset K_2$ , either  $\mathcal{T}K_1 \subset \mathcal{T}K_2$  or  $\mathcal{T}K_1 \supset \mathcal{T}K_2$ , and also for every  $K_1$  and  $K_2$  such that  $\mathcal{T}K_1 \subset \mathcal{T}K_2$ , either  $K_1 \subset K_2$  or  $K_1 \supset K_2$ . (Notice that this condition can be stated, perhaps more elegantly, using the intersections of the boundaries of the bodies).*

*Then there exists  $B \in GL_n$  such that either for all  $K$  which is not  $\mathbb{R}^n$  or  $\{0\}$ ,*

$$\mathcal{T}K = BK^\circ,$$

*or for all  $K$  which is not  $\mathbb{R}^n$  or  $\{0\}$ ,*

$$\mathcal{T}K = BK,$$

*and we have either  $\mathcal{T}(\mathbb{R}^n) = \{0\}$  and  $\mathcal{T}(\{0\}) = \mathbb{R}^n$  or  $\mathcal{T}(\mathbb{R}^n) = \mathbb{R}^n$  and  $\mathcal{T}(\{0\}) = \{0\}$ .*

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