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REGULAR WEAK SOLUTIONS TO 3–D CAHN-HILLIARD SYSTEM IN ELASTIC SOLIDS

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Abstract. In this paper we prove the existence and regularity of weak solutions to a three-dimensional (3–D) Cahn-Hilliard system coupled with nonstationary elasticity. Such nonlinear parabolic-hyperbolic system arises as a model of phase separation in deformable alloys.

1. **Introduction.** The present paper is concerned with the existence and regularity of weak solutions to a three-dimensional (3–D) Cahn-Hilliard system coupled with nonstationary elasticity. Such system arises as a model of phase separation in a binary deformable alloy quenched below a critical temperature. The problem under consideration has the following form:

$$\begin{aligned} &\boldsymbol{u}_{tt} - \nabla \cdot W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = \boldsymbol{b} && \text{in } \Omega^T = \Omega \times (0,T), \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1 && \text{in } \Omega, \\ &\boldsymbol{u} = \boldsymbol{0} && \text{on } S^T = S \times (0,T), \end{aligned} \tag{1}$$

$$\begin{aligned} &\chi_t - \nabla \cdot \boldsymbol{M} \nabla \mu = 0 && \text{in } \Omega^T, \\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \boldsymbol{M} \nabla \mu = 0 && \text{on } S^T, \end{aligned} \tag{2}$$

$$\begin{aligned} &\mu = -\nabla \cdot \boldsymbol{\Gamma} \nabla \chi + \psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) && \text{in } \Omega^T, \\ &\boldsymbol{n} \cdot \boldsymbol{\Gamma} \nabla \chi = 0 && \text{on } S^T. \end{aligned} \tag{3}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary S, occupied by a body in a reference configuration with constant mass density $\varrho = 1$; $\boldsymbol{n}$ is the unit outward normal to S, and $T > 0$ is an arbitrary fixed time. The body is a binary $a - b$ alloy.

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The unknowns are the fields $\boldsymbol{u}, \chi$ and μ, where $\boldsymbol{u} : \Omega^T \to \mathbb{R}^3$ is the *displacement vector*, $\chi : \Omega^T \to \mathbb{R}$ is the *order parameter* (phase ratio) and $\mu : \Omega^T \to \mathbb{R}$ is the *chemical potential*. The second order tensor $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$ denotes the linearized strain tensor.

In case of a binary a–b alloy the order parameter is related to the volumetric fraction of one of the two phases, characterized by different crystalline structures of the components, e.g. $\chi = -1$ is identified with the phase a and $\chi = 1$ with the phase b. The function $W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)$ denotes the elastic energy defined by

$$W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = \frac{1}{2}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)) \cdot \boldsymbol{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)). \tag{4}$$

Then $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = \boldsymbol{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi))$, $W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = -\bar{\boldsymbol{\varepsilon}}'(\chi) \cdot \boldsymbol{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi))$ represent respectively the stress tensor and the elastic part of the chemical potential. The fourth order tensor $\boldsymbol{A} = (A_{ijkl})$ denotes a constant elasticity tensor given by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) \mapsto \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \bar{\lambda} tr \boldsymbol{\varepsilon}(\boldsymbol{u}) \boldsymbol{I} + 2\bar{\mu}\boldsymbol{\varepsilon}(\boldsymbol{u}), \tag{5}$$

where $\boldsymbol{I} = (\delta_{ij})$ is the identity tensor, and $\bar{\lambda}, \bar{\mu}$ are the Lamé constants. The energy (4) with $\boldsymbol{A}$ given by (5) refers to the isotropic, homogeneous medium with the same elastic properties of the phases. The second order tensor $\bar{\boldsymbol{\varepsilon}}(\chi)$ denotes the eigenstrain defined by

$$\bar{\boldsymbol{\varepsilon}}(\chi) = (1 - z(\chi))\bar{\boldsymbol{\varepsilon}}_a + z(\chi)\bar{\boldsymbol{\varepsilon}}_b, \tag{6}$$

with $\bar{\boldsymbol{\varepsilon}}_a, \bar{\boldsymbol{\varepsilon}}_b$ constant eigenstrains of the phases a, b, and $z : \mathbb{R} \to [0, 1]$ is some smooth function satisfying

$$z(\chi) = 0 \quad \text{for} \quad \chi \leq -1 \quad \text{and} \quad z(\chi) = 1 \quad \text{for} \quad \chi \geq 1. \tag{7}$$

Furthermore, the function $\psi : \mathbb{R} \to \mathbb{R}$ denotes the chemical energy of the system at zero stress:

$$\psi(\chi) = \frac{1}{4}(1 - \chi^2)^2. \tag{8}$$

System (1)–(3) represents respectively the linear momentum balance, the mass balance and a generalized equation for the chemical potential. In a thermodynamical theory due to Gurtin [8] equation (3) is identified with a microforce balance. The free energy density underlying system (1)–(3) has the Landau-Ginzburg-Cahn-Hilliard form

$$f(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi, \nabla\chi) = W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) + \psi(\chi) + \frac{1}{2}\nabla\chi \cdot \boldsymbol{\Gamma}\nabla\chi, \tag{9}$$

with the three terms on the right-hand side representing respectively the elastic, chemical and interfacial energy.

The remaining quantities in (1)–(3) have the following meaning: $\boldsymbol{b} : \Omega^T \to \mathbb{R}^3$ represents the external body force, and $\boldsymbol{u}_0, \boldsymbol{u}_1 : \Omega \to \mathbb{R}^3$, $\chi_0 : \Omega \to \mathbb{R}$ are the initial conditions respectively for the displacement, the velocity and the order parameter.

In recent years Cahn-Hilliard systems in elastic solids have been the subject of many different modelling, mathematical and numerical studies (see [14], [1], [15]). The most general setting of the Cahn-Hilliard system coupled with elasticity, accounting for additional anisotropic, heterogeneous and kinetic effects, was derived by Gurtin [8] within the frame of his thermodynamical theory of phase transitions based on a microforce balance. System (1)–(3) represents a simplified version of Gurtin's model with neglected anisotropic, heterogeneous and kinetic effects; for more details see [1] where the full Gurtin's model was studied.

In view of the fact that the mechanical equilibrium is usually attained on a much faster time scale than diffusion, in most of the studies a quasi-stationary approximation of $(1)_1$, obtained by neglecting the inertial term $\boldsymbol{u}_{tt}$, was assumed. Various variants of the Cahn-Hilliard system with quasi-stationary elasticity were analyzed by Garcke [5], [6], [7], Bonetti et al. [2], Carrive et al. [3], [4], Miranville [10], [11], [12], [13].

The Cahn-Hilliard system with nonstationary elasticity was studied in [11], [1] where the existence and properties of weak solutions were examined, and in [15] where the classical solvability was proved in 1-D case. We point out that the study of the Cahn-Hilliard system with nonstationary elasticity – apart from the mathematical interest on its own – is of special importance for the initial stages of phase separation at which the formation of the microstructure is on a very fast time scale.

The goal of the present paper is to outline the proof of the existence and some regularity of weak solutions to system (1)–(3). The proof is based on the analysis of the time-differentiated system. The obtained result constitutes a first step towards the existence and uniqueness of a strong solution to (1)–(3), i.e. such a solution that its all derivatives appearing in the equations are at least in L_2. The strong solvability theory will be presented separately in [16]. It is based on the regularity results proved in the present paper together with some additional time regularity. More precisely, having sufficiently high time regularity we shall apply the standard elliptic regularity theory to conclude further space regularity and consequently the strong solvability. The key idea of the regularity results presented here and in [16] consists in the analysis of time-differentiated versions of problem (1)–(3) which yield solutions with sufficiently high time derivatives. The analysis is performed with the help of the Faedo-Galerkin approximation.

For further analysis it is convenient to introduce a simplified formulation of problem (1)–(3) which results on account of particular constitutive equations (4)–(6) and (9). Let $\boldsymbol{Q}$ be the linear elasticity operator defined by

$$\boldsymbol{u} \mapsto \boldsymbol{Q}\boldsymbol{u} = \nabla \cdot \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \bar{\mu}\Delta\boldsymbol{u} + (\bar{\lambda} + \bar{\mu})\nabla(\nabla \cdot \boldsymbol{u}). \tag{10}$$

Moreover, let us denote

$$\boldsymbol{B} = -\boldsymbol{A}(\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a), \quad D = -\boldsymbol{B} \cdot (\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a), \quad E = -\boldsymbol{B} \cdot \bar{\boldsymbol{\varepsilon}}_a, \tag{11}$$

where $\boldsymbol{B} = (B_{ij})$ is a symmetric second order tensor, and D, E are two scalars. For simplicity we assume that $\boldsymbol{M}$ and $\boldsymbol{\Gamma}$ are unit matrices. With such notation (1)–(3) simplifies to

$$\begin{aligned} &\boldsymbol{u}_{tt} - \boldsymbol{Q}\boldsymbol{u} = z'(\chi)\boldsymbol{B}\nabla\chi + \boldsymbol{b} && \text{in } \Omega^T, \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1 && \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } S^T, \end{aligned} \tag{12}$$

$$\chi_t - \Delta\mu = 0 \quad \text{in } \Omega^T, \quad \chi|_{t=0} = \chi_0 \quad \text{in } \Omega, \quad \boldsymbol{n} \cdot \nabla\mu = 0 \quad \text{on } S^T, \tag{13}$$

$$\mu = -\Delta\chi + \psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) \quad \text{in } \Omega^T, \quad \boldsymbol{n} \cdot \nabla\chi = 0 \quad \text{on } S^T, \tag{14}$$

with $W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = z'(\chi)(\boldsymbol{B} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E)$.

The summation convention over repeated indices is used and vector- and tensor-valued mappings are denoted by bold letters.

We use the standard Sobolev spaces notation $H^m(\Omega) = W_2^m(\Omega)$ for $m \in \mathbb{N}$,

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : \ v = 0 \text{ on } S\}, \quad H_N^2(\Omega) = \{v \in H^2(\Omega) : \ \boldsymbol{n} \cdot \nabla v = 0 \text{ on } S\}.$$

The dual of the space V is denoted by V', and $\langle\cdot,\cdot\rangle_{V',V}$ stands for the duality pairing between V' and V. By c and $c(T)$ we denote generic positive constants different in various instances, depending on the data of the problem and domain Ω. The argument T indicates the time horizon dependence.

2. **Assumptions and main results.** System (1)–(3) (in simplified form (12)–(14)) is studied under the following assumptions:

(A1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with the boundary S of class at least C^2; $T > 0$ is an arbitrary final time.

(A2) The coefficients of the elasticity operator $\boldsymbol{Q}$ satisfy

$$\bar{\mu} > 0, \quad 3\bar{\lambda} + 2\bar{\mu} > 0 \ \ \text{(elasticity range)}.$$

These two conditions assure the following:

(i) Coercivity and boundedness of the operator $\boldsymbol{A}$

$$\underline{c}|\boldsymbol{\varepsilon}|^2 \le \boldsymbol{\varepsilon}\cdot\boldsymbol{A}\boldsymbol{\varepsilon} \le \bar{c}|\boldsymbol{\varepsilon}|^2 \quad \text{for all} \ \ \boldsymbol{\varepsilon} \in \boldsymbol{S}^2,$$

where $\boldsymbol{S}^2$ denotes the set of symmetric second order tensors in $\mathbb{R}^3$.

(ii) Strong ellipticity of the operator $\boldsymbol{Q}$:

$$c\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \le \|\boldsymbol{Q}\boldsymbol{u}\|_{\boldsymbol{L}_2(\Omega)} \quad \text{for} \quad \boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$$

with constant c depending on Ω.

The next two assumptions concern the ingredients of the free energy

$$f(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi, \nabla\chi) = W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) + \psi(\chi) + \frac{1}{2}|\nabla\chi|^2. \tag{15}$$

(A3) The elastic energy $W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)$ is given by (4)–(6). The interpolation function $z : \mathbb{R} \to [0,1]$ is at least of class C^1 with the property (7). Hence,

$$0 \le z(\chi) \le 1 \quad \text{and} \quad |z'(\chi)| \le c \quad \text{for all} \quad \chi \in \mathbb{R}. \tag{16}$$

(A4) The chemical energy $\psi(\chi)$ is given by (8), so

$$\psi'(\chi) = \chi^3 - \chi, \quad \psi''(\chi) = 3\chi^2 - 1, \quad \psi'''(\chi) = 6\chi. \tag{17}$$

Moreover, for simplicity we assume that

(A5) The mobility tensor $\boldsymbol{M}$ and the interfacial tensor $\boldsymbol{\Gamma}$ are the identities matrices $\boldsymbol{M} = \boldsymbol{I}$, $\boldsymbol{\Gamma} = \boldsymbol{I}$. The second order symmetric tensor $\boldsymbol{B}$ and the scalars D, E are defined in (11).

We note that assumptions (A3) and (A4) imply the following bounds for all $\boldsymbol{\varepsilon} \in \boldsymbol{S}^2$ and $\chi \in \mathbb{R}$:

$$\begin{gathered}|\bar{\boldsymbol{\varepsilon}}(\chi)| + |\bar{\boldsymbol{\varepsilon}}'(\chi)| \le c, \quad |\psi(\chi)| \le c(\chi^4 + 1), \quad |\psi'(\chi)| \le c(|\chi|^3 + 1),\\ |W(\boldsymbol{\varepsilon}, \chi)| \le c(|\boldsymbol{\varepsilon}|^2 + 1), \quad |W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \chi)| + |W_{,\chi}(\boldsymbol{\varepsilon}, \chi)| \le c(|\boldsymbol{\varepsilon}| + 1)\end{gathered}$$

with some positive constant c. Moreover,

$$f(\boldsymbol{\varepsilon}, \chi, \nabla\chi) \ge c_f(|\boldsymbol{\varepsilon}|^2 + \chi^4 + |\nabla\chi|^2) - c'_f$$

with constants $c_f > 0$ and c'_f. This bound plays the key role in the derivation of energy estimates for problem (1)–(3).

We state now the main results of the paper.

Theorem 2.1. Weak solutions
Let assumptions (A1)–(A5) hold true. Moreover, let the data satisfy

$$\boldsymbol{b} \in \boldsymbol{L}_2(\Omega^T), \quad \boldsymbol{u}_0 \in \boldsymbol{H}_0^1(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{L}_2(\Omega), \quad \chi_0 \in H^1(\Omega).$$

Then there exist functions $(\boldsymbol{u}, \chi, \mu)$ *such that*

$$\begin{aligned}
&\boldsymbol{u} \in L_\infty(0,T;\boldsymbol{H}_0^1(\Omega)), \quad \boldsymbol{u}_t \in L_\infty(0,T;\boldsymbol{L}_2(\Omega)), \quad \boldsymbol{u}_{tt} \in L_2(0,T;(\boldsymbol{H}_0^1(\Omega))'),\\
&\chi \in L_\infty(0,T;H^1(\Omega)) \cap L_2(0,T;H_N^2(\Omega)), \quad \chi_t \in L_2(0,T;(H^1(\Omega))'),\\
&\mu \in L_2(0,T;H^1(\Omega)), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \chi(0) = \chi_0,
\end{aligned}$$

which satisfy problem (12)–(14) in the following weak sense

$$\begin{aligned}
&\int_0^T \langle \boldsymbol{u}_{tt}, \boldsymbol{\eta}\rangle_{(\boldsymbol{H}_0^1(\Omega))',\boldsymbol{H}_0^1(\Omega)} dt + \int_0^T (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{\eta}))dt\\
&= \int_0^T (z'(\chi)\boldsymbol{B}\nabla\chi + \boldsymbol{b}, \boldsymbol{\eta})dt \qquad \forall \boldsymbol{\eta} \in L_2(0,T;\boldsymbol{H}_0^1(\Omega)),\\
&\int_0^T \langle \chi_t, \xi\rangle_{(H^1(\Omega))',H^1(\Omega)} dt + \int_0^T (\nabla\mu, \nabla\xi)dt = 0 \qquad \forall \xi \in L_2(0,T;H^1(\Omega)),\\
&\int_0^T (\mu, \zeta)dt = -\int_0^T (\Delta\chi, \zeta)dt + \int_0^T (\psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi), \zeta)dt \quad \forall \zeta \in L_2(0,T;L_2(\Omega)).
\end{aligned} \tag{18}$$

Moreover, $(\boldsymbol{u}, \chi, \mu)$ *satisfy a priori estimates*

$$\begin{aligned}
&\|\boldsymbol{u}_t\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\chi\|_{L_\infty(0,T;L_4(\Omega))}\\
&\quad + \|\nabla\chi\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\nabla\mu\|_{\boldsymbol{L}_2(\Omega^T)} + \|\chi_t\|_{L_2(0,T;(H^1(\Omega))')} \le c_0,\\
&\|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{H}_0^1(\Omega))} + \|\chi\|_{L_\infty(0,T;H^1(\Omega))} \le c(c_0),\\
&\|\chi\|_{L_2(0,T;H_N^2(\Omega))} + \|\mu\|_{L_2(0,T;H^1(\Omega))} + \|\boldsymbol{u}_{tt}\|_{L_2(0,T;(\boldsymbol{H}_0^1(\Omega))')} \le c(c_0, T),
\end{aligned} \tag{19}$$

where $c_0 = c(\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}, \|\boldsymbol{u}_1\|_{\boldsymbol{L}_2(\Omega)}, \|\chi_0\|_{H^1(\Omega)}, \|\boldsymbol{b}\|_{L_1(0,T;\boldsymbol{L}_2(\Omega))}, c_f, c_f')$,

The second theorem states a time-regularity result which is concluded from a time-differentiated version of problem (12)–(14).

In compatibility with equations $(12)_1$, $(13)_1$ and $(14)_1$, we define the following initial conditions corresponding respectively to $\boldsymbol{u}_{tt}(0)$, $\chi_t(0)$ and $\mu(0)$:

$$\begin{aligned}
&\boldsymbol{u}_2 := \boldsymbol{Q}\boldsymbol{u}_0 + z'(\chi_0)B\nabla\chi_0 + \boldsymbol{b}(0), \quad \chi_1 := \Delta\mu_0,\\
&\mu_0 := -\Delta\chi_0 + \psi'(\chi_0) + z'(\chi_0)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + Dz(\chi_0) + E).
\end{aligned} \tag{20}$$

Theorem 2.2. Time regularity
Let (A1)–(A5) hold, the boundary S *of domain* Ω *be of class* C^4*, and*

$$z : \mathbb{R} \to [0,1] \text{ be of class } C^2 \text{ with } \quad |z'(\chi)| + |z''(\chi)| \le c \text{ for all } \chi \in \mathbb{R}. \tag{21}$$

Moreover, let the data satisfy

$$\begin{aligned}
&\boldsymbol{b} \in H^1(0,T;\boldsymbol{L}_2(\Omega)), \quad \boldsymbol{u}_0 \in \boldsymbol{H}^3(\Omega) \cap \boldsymbol{H}_0^1(\Omega), \quad \boldsymbol{u}_1 \in \boldsymbol{H}_0^1(\Omega),\\
&\chi_0 \in H^4(\Omega) \cap H_N^2(\Omega), \quad \boldsymbol{u}_2 \in \boldsymbol{L}_2(\Omega), \quad \chi_1 \in L_2(\Omega), \quad \mu_0 \in H_N^2(\Omega).
\end{aligned} \tag{22}$$

Then there exist functions $(\boldsymbol{u},\chi,\mu)$ *such that*

$$\begin{aligned}
&\boldsymbol{u}\in L_\infty(0,T;\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega)),\quad \boldsymbol{u}_t\in L_\infty(0,T;\boldsymbol{H}^1_0(\Omega)),\\
&\boldsymbol{u}_{tt}\in L_\infty(0,T;\boldsymbol{L}_2(\Omega)),\quad \boldsymbol{u}_{ttt}\in L_2(0,T;(\boldsymbol{H}^1_0(\Omega))'),\\
&\chi\in C^{1/2}([0,T];H^2_N(\Omega)),\quad \chi_t\in L_\infty(0,T;L_2(\Omega))\cap L_2(0,T;H^2_N(\Omega)),\\
&\chi_{tt}\in L_2(0,T;(H^2_N(\Omega))'),\quad \mu\in L_\infty(0,T;H^2_N(\Omega)),\quad \mu_t\in L_2(\Omega^T),\\
&\boldsymbol{u}(0)=\boldsymbol{u}_0,\ \boldsymbol{u}_t(0)=\boldsymbol{u}_1,\ \boldsymbol{u}_{tt}(0)=\boldsymbol{u}_2,\ \chi(0)=\chi_0,\ \chi_t(0)=\chi_1,\ \mu(0)=\mu_0,
\end{aligned}\tag{23}$$

which satisfy problem (12)–(14) in the sense of the identities:

$$\begin{aligned}
&\int_0^T\langle\boldsymbol{u}_{ttt},\boldsymbol{\eta}\rangle_{(\boldsymbol{H}^1_0(\Omega))',\boldsymbol{H}^1_0(\Omega)}dt+\int_0^T(\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t),\boldsymbol{\varepsilon}(\boldsymbol{\eta}))dt\\
&=\int_0^T([z'(\chi)\boldsymbol{B}\nabla\chi]_{,t}+\boldsymbol{b}_t,\boldsymbol{\eta})dt\quad\forall\boldsymbol{\eta}\in L_2(0,T;\boldsymbol{H}^1_0(\Omega)),\\
&\int_0^T\langle\chi_{tt},\xi\rangle_{(H^2_N(\Omega))',H^2_N(\Omega)}dt=\int_0^T(\mu_t,\Delta\xi)dt\quad\forall\xi\in L_2(0,T;H^2_N(\Omega)),\\
&\int_0^T(\mu_t,\zeta)dt=-\int_0^T(\Delta\chi_t,\zeta)dt+\int_0^T([\psi'(\chi)+W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)]_{,t},\zeta)dt,\\
&\forall\zeta\in L_2(0,T;L_2(\Omega)),
\end{aligned}\tag{24}$$

where

$$\begin{aligned}
&[z'(\chi)\boldsymbol{B}\nabla\chi]_{,t}=z''(\chi)\chi_t\boldsymbol{B}\nabla\chi+z'(\chi)\boldsymbol{B}\nabla\chi_t,\\
&[\psi'(\chi)+W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)]_{,t}=\psi''(\chi)\chi_t+z''(\chi)\chi_t(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})+Dz(\chi)+E)\\
&\quad+z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t)+Dz'(\chi)\chi_t).
\end{aligned}\tag{25}$$

In addition, $(\boldsymbol{u},\chi,\mu)$ *satisfy estimates (19) and*

$$\begin{aligned}
&\|\boldsymbol{u}\|_{L_\infty(0,T;\boldsymbol{H}^2(\Omega))}+\|\boldsymbol{u}_t\|_{L_\infty(0,T;\boldsymbol{H}^1(\Omega))}+\|\boldsymbol{u}_{tt}\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))}\\
&+\|\chi\|_{C^{1/2}([0,T];H^2_N(\Omega))}+\|\chi_t\|_{L_\infty(0,T;L_2(\Omega))}+\|\chi_t\|_{L_2(0,T;H^2_N(\Omega))}\\
&+\|\mu\|_{L_\infty(0,T;H^2_N(\Omega))}+\|\boldsymbol{u}_{ttt}\|_{L_2(0,T;(\boldsymbol{H}^1_0(\Omega))')}+\|\chi_{tt}\|_{L_2(0,T;(H^2_N(\Omega))')}\\
&+\|\mu_t\|_{L_2(\Omega^T)}\le c(E_1,T),
\end{aligned}\tag{26}$$

where $E_1(T)=T^{1/2}\|\boldsymbol{b}_t\|_{\boldsymbol{L}_2(\Omega^T)}+\|\boldsymbol{u}_2\|_{\boldsymbol{L}_2(\Omega)}+\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1)\|_{\boldsymbol{L}_2(\Omega)}$.

3. **Estimates for approximation.** In this section we introduce a Faedo-Galerkin approximation of problem (1)–(3) (in simplified form (12)–(14)) and derive basic energy estimates. These estimates are used to prove the existence of weak solutions in Theorem 2.1. Let us consider the following two eigenvalue problems

$$-\boldsymbol{Q}\boldsymbol{v}_j=\lambda_j\boldsymbol{v}_j\quad\text{in }\ \Omega,\qquad \boldsymbol{v}_j=\boldsymbol{0}\quad\text{on }\ S,\quad j\in\mathbb{N},\tag{27}$$

where $\boldsymbol{Q}$ is the elliptic operator defined by (10), and

$$-\Delta w_j=\lambda_j w_j\quad\text{in }\ \Omega,\qquad \boldsymbol{n}\cdot\nabla w_j=0\quad\text{on }\ S,\quad j\in\mathbb{N}.\tag{28}$$

We recall that, by virtue of the elliptic regularity theory, if the domain Ω has the boundary of class C^l, $l \in \mathbb{N}$, then the solutions of (27) and (28) satisfy

$$\boldsymbol{v}_j \in \boldsymbol{H}^l(\Omega), \quad w_j \in H^l(\Omega).$$

We take the family $\{\boldsymbol{v}_j\}_{j\in\mathbb{N}}$ as a basis of the space $\boldsymbol{H}^1_0(\Omega)$ and the family $\{w_j\}_{j\in\mathbb{N}}$ as a basis of the space $H^2_N(\Omega)$.

For $m \in \mathbb{N}$ we denote by $\boldsymbol{V}_{0m} = \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$, $V_m = \operatorname{span}\{w_1, \dots, w_m\}$ the finite dimensional subspaces, respectively of $\boldsymbol{H}^1_0(\Omega)$ and $H^2_N(\Omega)$, spanned by $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ and $\{w_1, \dots, w_m\}$.

Now, let us introduce the following approximation of problem (12)–(14): For any $m \in \mathbb{N}$ find a triple of functions $(\boldsymbol{u}^m, \chi^m, \mu^m)$ of the form

$$\boldsymbol{u}^m(\boldsymbol{x},t) = \sum_{i=1}^m e_i^m(t)\boldsymbol{v}_i(\boldsymbol{x}), \;\; \chi^m(\boldsymbol{x},t) = \sum_{i=1}^m c_i^m(t)w_i(\boldsymbol{x}), \;\; \mu^m(\boldsymbol{x},t) = \sum_{i=1}^m d_i^m(t)w_i(\boldsymbol{x}),$$

with $e_i^m(t)$, $c_i^m(t)$, $d_i^m(t)$ being determined so that

$$\begin{aligned}
&(\boldsymbol{u}^m_{tt}, \boldsymbol{v}_j) + (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \boldsymbol{\varepsilon}(\boldsymbol{v}_j)) = (z'(\chi^m)\boldsymbol{B}\nabla\chi^m + \boldsymbol{b}, \boldsymbol{v}_j),\\
&(\chi^m_t, w_j) + (\nabla\mu^m, \nabla w_j) = 0,\\
&(\mu^m, w_j) = -(\Delta\chi^m, w_j) + (\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m), w_j), \quad j = 1, \dots, m,\\
&\boldsymbol{u}^m(0) = \boldsymbol{u}^m_0, \quad \boldsymbol{u}^m_t(0) = \boldsymbol{u}^m_1, \quad \chi^m(0) = \chi^m_0,
\end{aligned} \tag{29}$$

where $\boldsymbol{u}^m_0, \boldsymbol{u}^m_1 \in \boldsymbol{V}_{0m}$ and $\chi^m_0 \in V_m$ are the projections respectively of $\boldsymbol{u}_0, \boldsymbol{u}_1$ and χ_0 satisfying for $m \to \infty$ the strong convergences

$$\boldsymbol{u}^m_0 \to \boldsymbol{u}_0 \;\text{ in } \boldsymbol{H}^1_0(\Omega), \quad \boldsymbol{u}^m_1 \to \boldsymbol{u}_1 \;\text{ in } \boldsymbol{L}_2(\Omega), \quad \chi^m_0 \to \chi_0 \;\text{ in } H^1(\Omega).$$

Clearly, (29) can be expressed as a system of first order ordinary differential equations for the coefficients $(e^m_1, \dots, e^m_m)$, $(e^m_{1,t}, \dots, e^m_{m,t})$, $(c^m_1, \dots, c^m_m)$, with the right-hand sides being by assumptions continuous functions of their arguments. Thus (29) has a solution local in time on an interval $[0, T_m]$, $T_m > 0$. The uniform in m a priori estimates proved in lemmas below show that $T_m = T$, i.e. (29) has a solution on the interval $[0, T]$.

Lemma 3.1. *Let (A1)-(A5) hold and let* $\boldsymbol{u}_0 \in \boldsymbol{H}^1_0(\Omega)$, $\boldsymbol{u}_1 \in \boldsymbol{L}_2(\Omega)$, $\chi_0 \in H^1(\Omega)$, $\boldsymbol{b} \in L_1(0,T;\boldsymbol{L}_2(\Omega))$. *Then solutions to (29) satisfy*

$$\begin{aligned}
&\|\boldsymbol{u}^m_t\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\boldsymbol{\varepsilon}(\boldsymbol{u}^m)\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\chi^m\|_{L_\infty(0,T;L_4(\Omega))}\\
&\quad + \|\nabla\chi^m\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\nabla\mu^m\|_{\boldsymbol{L}_2(\Omega^T)} \le c_0,\\
&\|\chi^m\|_{L_\infty(0,T;H^1(\Omega))} \le c(c_0),\\
&\|\chi^m\|_{L_2(0,T;H^2_N(\Omega))} \le c(c_0, T),\\
&\|\boldsymbol{u}^m_{tt}\|_{L_2(0,T;(\boldsymbol{H}^1_0(\Omega))')} + \|\chi^m_t\|_{L_2(0,T;(H^1(\Omega))')} \le c(c_0, T).
\end{aligned} \tag{30}$$

Proof. Estimate $(30)_1$ follows from (29) by testing with $\boldsymbol{u}^m_t$, μ^m and $-\chi^m_t$, respectively, and by applying the classical energy method. $(30)_3$ follows from $(29)_2$ by setting $w_j = 1$. Assuming $w_j = 1$ in $(29)_3$ we obtain $(30)_3$. Finally, $(30)_4$ follows from $(29)_1$ and $(29)_2$ by standard duality arguments. □

Now we derive uniform in m time-regularity estimates for solutions of approximate system (29). These estimates result from time-differentiated version of (29) and lead to the existence result of Theorem 2.2. We assume that the domain Ω has the boundary S of class C^4.

Let us differentiate system $(29)_1$–$(29)_3$ with respect to t and rewrite it in the following form:

$$\begin{aligned}
&(\boldsymbol{u}^m_{ttt}, \boldsymbol{v}_j) + (\boldsymbol{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m_t), \boldsymbol{\varepsilon}(\boldsymbol{v}_j)) = ([z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,t} + \boldsymbol{b}_t, \boldsymbol{v}_j),\\
&(\chi^m_{tt}, w_j) - (\mu^m_t, \Delta w_j) = 0,\\
&(\mu^m_t, w_j) = -(\Delta\chi^m_t, w_j) + ([\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)]_{,t}, w_j), \quad j = 1, \dots, m,
\end{aligned} \tag{31}$$

where the explicit expressions for $[z'(\chi)\boldsymbol{B}\nabla\chi]_{,t}$ and $[\psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)]_{,t}$ are given in (25). The above system is considered with the initial conditions

$$\boldsymbol{u}^m(0) = \boldsymbol{u}^m_0,\ \boldsymbol{u}^m_t(0) = \boldsymbol{u}^m_1,\ \chi^m(0) = \chi^m_0,\ \boldsymbol{u}^m_{tt}(0) = \boldsymbol{u}^m_2,\ \chi^m_t(0) = \chi^m_1,\ \mu^m(0) = \mu^m_0,$$

where $\boldsymbol{u}^m_2 \in \boldsymbol{V}_{0m}$ and $\chi^m_1, \mu^m_0 \in V_m$ are the projections respectively of the data $\boldsymbol{u}_2, \chi_1$ and μ_0, defined in (20).
We assume that the following convergences in the strong sense are satisfied:

$$\begin{array}{llll}
\boldsymbol{u}^m_0 \to \boldsymbol{u}_0 \ \text{ in } \ \boldsymbol{H}^3(\Omega) \cap \boldsymbol{H}^1_0(\Omega), & & \boldsymbol{u}^m_1 \to \boldsymbol{u}_1 \ \text{ in } \ \boldsymbol{H}^1_0(\Omega), & \\
\chi^m_0 \to \chi_0 \ \text{ in } \ H^4(\Omega) \cap H^2_N(\Omega), & & \boldsymbol{u}^m_2 \to \boldsymbol{u}_2 \ \text{ in } \ \boldsymbol{L}_2(\Omega), & \\
\chi^m_1 \to \chi_1 \ \text{ in } \ L_2(\Omega), & & \mu^m_0 \to \mu_0 \ \text{ in } \ H^2_N(\Omega). &
\end{array} \tag{32}$$

Lemma 3.2. *Let the assumptions of Lemma 3.1 be satisfied. Let $S \in C^4$, $z \in C^2(\mathbb{R}; [0,1])$ with properties (16) and (21) being satisfied. Then*

$$\begin{aligned}
&\|u^m_{tt}\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|u^m_t\|_{L_\infty(0,T;\boldsymbol{H}^1(\Omega))}\\
&\quad + \|\chi^m_t\|_{L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;H^2_N(\Omega))} \le c(c_0, T).
\end{aligned} \tag{33}$$

Proof. Testing $(31)_1$ by $\boldsymbol{u}^m_{tt}$ and testing the equation

$$(\chi^m_{tt}, w_j) = -(\Delta\chi^m_t, \Delta w_j) + ([\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)]_{,t}, \Delta w_j),$$

which follows from $(31)_{1,2}$, by χ^m_t we obtain (33) with the help of the energy method and interpolation inequalities. □

Lemma 3.3. *Let the assumptions of Lemma 3.2 hold. Then*

$$\begin{aligned}
&\|\chi^m\|_{C^{1/2}([0,T];H^2_N(\Omega))} \le c(c_0, T),\\
&\|\boldsymbol{u}^m\|_{L_\infty(0,T;\boldsymbol{H}^2(\Omega))} \le c(c_0, T),\\
&\|\mu^m\|_{L_\infty(0,T;H^2_N(\Omega))} + \|\mu^m_t\|_{L_2(\Omega^T)} \le c(c_0, T),\\
&\|u^m_{ttt}\|_{L_2(0,T;(H^1_0(\Omega))')} + \|\chi^m_{tt}\|_{L_2(0,T;(H^2_N(\Omega))')} \le c(c_0, T).
\end{aligned} \tag{34}$$

Proof. Estimate (33) implies $(34)_1$. Equation $(29)_1$ with estimate (33) implies $(34)_2$. Expressing $(29)_2$ in the form $(\chi^m_t, \Delta w_j) = (\Delta\mu^m, \Delta w_j)$, testing by $\mu^m(t)$ and using (33) we obtain the first bound in $(34)_3$. The second bound in $(34)_3$ follows from $(31)_3$ testing by μ^m_t and using (33).
From $(31)_{1,2}$ with test functions from $L_2(0,T; H^1_0(\Omega))$ and $L_2(0,T; H^2_N(\Omega))$, respectively, we conclude $(34)_4$. □

4. **Proof of Theorem 2.2.** From Lemmas 3.1–3.3 it follows that there exists a triple $(\boldsymbol{u}, \chi, \mu)$ with

$$\begin{aligned}
&\boldsymbol{u} \in L_\infty(0,T; \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)), \quad \boldsymbol{u}_t \in L_\infty(0,T; \boldsymbol{H}^1(\Omega)),\\
&\boldsymbol{u}_{tt} \in L_\infty(0,T; \boldsymbol{L}_2(\Omega)), \quad \boldsymbol{u}_{ttt} \in L_2(0,T; (\boldsymbol{H}^1_0(\Omega))'),\\
&\chi \in C^{1/2}([0,T]; H^2_N(\Omega)), \quad \chi_t \in L_\infty(0,T; L_2(\Omega)) \cap L_2(0,T; H^2_N(\Omega)),\\
&\chi_{tt} \in L_2(0,T; (H^2_N(\Omega))'), \quad \mu \in L_\infty(0,T; H^2_N(\Omega)), \quad \mu_t \in L_2(\Omega^T),
\end{aligned} \tag{35}$$

and a subsequence of solutions $(\boldsymbol{u}^m, \chi^m, \mu^m)$ to problem (29) (which we still denote by the same indices) such that as $m \to \infty$:

$$\begin{array}{lll}
\boldsymbol{u}^m \to \boldsymbol{u} & \text{weakly} -^* \text{ in } & L_\infty(0,T;\boldsymbol{H}^2(\Omega)),\\
\boldsymbol{u}^m_t \to \boldsymbol{u}_t & \text{weakly} -^* \text{ in } & L_\infty(0,T;\boldsymbol{H}^1(\Omega)),\\
\boldsymbol{u}^m_{tt} \to \boldsymbol{u}_{tt} & \text{weakly} -^* \text{ in } & L_\infty(0,T;\boldsymbol{L}_2(\Omega)),\\
\boldsymbol{u}^m_{ttt} \to \boldsymbol{u}_{ttt} & \text{weakly in } & L_2(0,T;(\boldsymbol{H}^1_0(\Omega))'),\\
\chi^m \to \chi & \text{weakly} -^* \text{ in } & L_\infty(0,T;H^2_N(\Omega)),\\
\chi^m_t \to \chi_t & \text{weakly} -^* \text{ in } & L_\infty(0,T;L_2(\Omega)) \text{ and}\\
 & \text{weakly in } & L_2(0,T;H^2_N(\Omega)),\\
\chi^m_{tt} \to \chi_{tt} & \text{weakly in } & L_2(0,T;(H^2_N(\Omega))'),\\
\mu^m \to \mu & \text{weakly} -^* \text{ in } & L_\infty(0,T;H^2_N(\Omega)),\\
\mu^m_t \to \mu_t & \text{weakly in } & L_2(\Omega).
\end{array} \tag{36}$$

Using the compactness results [17] it follows that for a subsequence

$$\begin{array}{lll}
\boldsymbol{u}^m \to \boldsymbol{u} & \text{strongly in } & C([0,T];\boldsymbol{H}^1_0(\Omega)), \text{ a.e. in } \Omega^T,\\
\boldsymbol{u}^m_t \to \boldsymbol{u}_t & \text{strongly in } & C([0,T];\boldsymbol{L}_2(\Omega)), \text{ a.e. in } \Omega^T,\\
\boldsymbol{u}^m_{tt} \to \boldsymbol{u}_{tt} & \text{strongly in } & C([0,T];(\boldsymbol{H}^1_0(\Omega))'),\\
\chi^m \to \chi & \text{strongly in } & C([0,T];H^1(\Omega)), \text{ a.e. in } \Omega^T,
\end{array} \tag{37}$$

$$\begin{array}{lll}
\chi^m_t \to \chi_t & \text{strongly in } & L_2(0,T;H^1(\Omega)) \cap C([0,T];(H^2_N(\Omega))'), \text{ a.e. in } \Omega^T,\\
\mu^m \to \mu & \text{strongly in } & C([0,T];H^1(\Omega)), \text{ a.e. in } \Omega^T.
\end{array}$$

Hence, we have the strong convergences

$$\begin{array}{llllll}
\boldsymbol{u}^m(0) = \boldsymbol{u}^m_0 \to \boldsymbol{u}(0) & \text{in } & \boldsymbol{H}^1_0(\Omega), & \boldsymbol{u}^m_t(0) = \boldsymbol{u}^m_1 \to \boldsymbol{u}_t(0) & \text{in } & \boldsymbol{L}_2(\Omega),\\
\boldsymbol{u}^m_{tt}(0) = \boldsymbol{u}^m_2 \to \boldsymbol{u}_{tt}(0) & \text{in } & (\boldsymbol{H}^1_0(\Omega))', & \chi^m(0) = \chi^m_0 \to \chi(0) & \text{in } & H^1(\Omega),\\
\chi^m_t(0) = \chi^m_1 \to \chi_t(0) & \text{in } & (H^2_N(\Omega))', & \mu^m(0) = \mu^m_0 \to \mu(0) & \text{in } & H^1(\Omega),
\end{array}$$

which together with convergences (32), imply that

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1, \quad \boldsymbol{u}_{tt}(0) = \boldsymbol{u}_2, \chi(0) = \chi_0, \quad \chi_t(0) = \chi_1, \quad \mu(0) = \mu_0. \tag{38}$$

The relations (35) and (38) imply assertion (23) of the theorem.

We introduce now the following weak formulation of system (31):

$$\begin{aligned}
&\int_0^T \langle \boldsymbol{u}^m_{ttt}, \boldsymbol{\eta}\rangle_{(\boldsymbol{H}^1_0(\Omega))',\boldsymbol{H}^1_0(\Omega)} dt + \int_0^T (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}^m_t), \boldsymbol{\varepsilon}(\boldsymbol{\eta}))dt\\
&= \int_0^T ([z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,t} + \boldsymbol{b}_t, \boldsymbol{\eta})dt \quad \forall \boldsymbol{\eta} \in L_2(0,T;\boldsymbol{V}_{0m}),\\
&\int_0^T \langle \chi^m_{tt}, \xi\rangle_{(H^2_N(\Omega))',H^2_N(\Omega)} dt = \int_0^T (\mu_t, \Delta\xi)dt \quad \forall \xi \in L_2(0,T;V_m),\\
&\int_0^T (\mu^m_t, \zeta)dt = -\int_0^T (\Delta\chi^m_t, \zeta)dt + \int_0^T ([\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)]_{,t}, \zeta)dt\\
&\forall \zeta \in L_2(0,T;V_m).
\end{aligned} \tag{39}$$

We pass to the limit $m \to \infty$. Clearly, due to the weak convergences (36), all linear terms in identities (39) converge to the corresponding limits. It remains to

examine the convergence of the nonlinear terms $[z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,t}$ and $[\psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)]_{,t}$ whose explicit expressions are given in (25).

Recalling assumptions on z, ψ and using Lemmas 3.1–3.3 we obtain the bounds:

$$\|[z'(\chi^m)\boldsymbol{B}\nabla\chi^m]_{,t}\|_{\boldsymbol{L}_2(\Omega^T)} + \|[\psi'(\chi^m)]_{,t}\|_{L_2(\Omega^T)} + \|[W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}^m), \chi^m)]_{,t}\|_{L_2(\Omega^T)} \leq c(c_0, T).$$

Thanks to these uniform in m bounds and the pointwise convergences (37) we can apply the nonlinear convergence lemma (see [9], Chap. 1, Lemma 1.3) to pass to the limit $m \to \infty$ in identities (39). Thereby we conclude (24) and (26). This completes the proof of the theorem.

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