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ON THE UNUSUAL FUČÍK SPECTRUM

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Abstract. We construct the spectra for some second order boundary value problem of Fučík type. This spectrum differs essentially from the known Fučík spectra.

1. **Introduction.** In this paper we study spectra for some second order equation with piece-wise linear right-hand sides.

Investigations of Fučík spectra have started in sixties of XX century [1]. A number of authors have studied the specific cases. Let us mention the cases of the Dirichlet [1] and the Sturm-Liouville [5] boundary conditions. There are some papers on higher order equations. Habets and Gaudenzi have studied the third order equation $x''' = -\mu x^+ + \lambda x^-$ with the boundary conditions $x(0) = x'(0) = 0 = x(1)$ in the work [3], where many useful references on the subject can be found. Fučík spectra for the fourth order equations were considered by Krejčí [2] and Pope [4].

The paper is organized as follows. In Section 2 we formulate the second order boundary value problems. In Section 3 we give results on the spectrum for the well-known Fučík problem, but in Section 4 we construct the Fučík spectrum for the second order problem with the boundary conditions $y(a) = 0$, $\int_a^b y(s)ds = 0$. This spectrum differs essentially from the known Fučík spectra. To the best of our knowledge Fučík spectra for problems with nonlocal boundary conditions were not considered previously. These are the main results of the work. A connection between the spectra is discussed in Section 5.

2. **Formulations of the boundary value problems.** Consider the second order differential equation

$$-y'' = \mu y^+ - \lambda y^-, \quad \mu, \lambda > 0, \tag{1}$$

$$y^+ = \max\{y, 0\}, \quad y^- = \max\{-y, 0\},$$

with the boundary conditions

$$y(a) = 0, \quad y(b) = 0 \tag{2}$$

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and with the boundary conditions

$$y(a) = 0, \quad \int_a^b y(s)ds = 0. \tag{3}$$

We call the problems (1), (2) and (1), (3) the problem II_A and II_B, respectively.

We are looking for such points (μ, λ) so that the problem II_A (II_B) has nontrivial solutions (we will call them the Fučík spectrum).

The first result describes decomposition of the spectrum into branches F_i^+ and F_i^- $(i = 0, 1, 2, \ldots)$ for the problems II_A and II_B.

Proposition 1. *The Fučík spectrum consists of the set of branches*
$F_i^+ = \{(\mu, \lambda)|\ y'(a) > 0,$ *the nontrivial solution* $y(t)$ *of the problem has exactly* i *zeroes in* $(a, b)\}$;
$F_i^- = \{(\mu, \lambda)|\ y'(a) < 0,$ *the nontrivial solution* $y(t)$ *of the problem has exactly* i *zeroes in* $(a, b)\}$.

Proof of Proposition 1 by construction of the spectra in Theorems 1 and 2.

3. Spectrum of the problem II_A.

Theorem 1. *The Fučík spectrum for the problem* II_A *consists of the branches given by*

$$F_0^+ = \Big\{\Big(\big(\frac{\pi}{b-a}\big)^2, \lambda\Big)\Big\},$$
$$F_0^- = \Big\{\Big(\mu, \big(\frac{\pi}{b-a}\big)^2\Big)\Big\},$$

$$\begin{aligned} F_{2i-1}^+ &= \Big\{(\mu, \lambda)\Big|\ \frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} = b - a\Big\}, \\ F_{2i-1}^- &= \Big\{(\mu, \lambda)\Big|\ (\lambda, \mu) \in F_{2i-1}^+\Big\}, \\ F_{2i}^+ &= \Big\{(\mu, \lambda)\Big|\ \frac{(i+1)\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}} = b - a\Big\}, \\ F_{2i}^- &= \Big\{(\mu, \lambda)\Big|\ (\lambda, \mu) \in F_{2i}^+\Big\}, \end{aligned}$$

where $i = 1, 2, \ldots$.

Notice that the problem II_A is the classical Fučík problem, which was investigated in the work [1], the proof of this theorem is given in the book.

Figure 1 illustrates the Fučík spectrum structure for the problem II_A in the case of $b - a = 1$.

4. Spectrum of the problem II_B.

Theorem 2. *The Fučík spectrum for the problem* II_B *consists of the branches given by*

$$\begin{aligned} F_{2i-1}^+ = \ & \Big\{(\mu, \lambda)\Big|\ \frac{2i\sqrt{\lambda}}{\sqrt{\mu}} - \frac{(2i-1)\sqrt{\mu}}{\sqrt{\lambda}} - \frac{\sqrt{\mu}\cos(\sqrt{\lambda}(b-a) - \frac{\sqrt{\lambda}\pi i}{\sqrt{\mu}} + \pi i)}{\sqrt{\lambda}} = 0, \\ & \tfrac{i\pi}{\sqrt{\mu}} + \tfrac{(i-1)\pi}{\sqrt{\lambda}} \le b - a,\ \tfrac{i\pi}{\sqrt{\mu}} + \tfrac{i\pi}{\sqrt{\lambda}} > b - a\Big\}, \end{aligned}$$

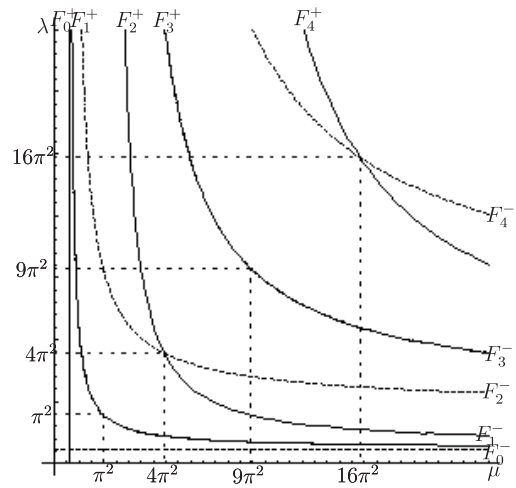

FIGURE 1. The spectrum for the problems II_A.

$$F_{2i-1}^- = \Big\{(\mu,\lambda)\Big|\ (\lambda,\mu)\in F_{2i-1}^+\Big\},$$

$$F_{2i}^+ = \Big\{(\mu,\lambda)\Big|\ \frac{(2i+1)\sqrt{\lambda}}{\sqrt{\mu}} - \frac{2i\sqrt{\mu}}{\sqrt{\lambda}} - \frac{\sqrt{\lambda}\cos(\sqrt{\mu}(b-a) - \frac{\sqrt{\mu}\pi i}{\sqrt{\lambda}} + \pi i)}{\sqrt{\mu}} = 0,$$
$$\tfrac{i\pi}{\sqrt{\mu}} + \tfrac{i\pi}{\sqrt{\lambda}} \le b-a,\ \tfrac{(i+1)\pi}{\sqrt{\mu}} + \tfrac{i\pi}{\sqrt{\lambda}} > b-a\Big\},$$

$$F_{2i}^- = \Big\{(\mu,\lambda)\Big|\ (\lambda,\mu)\in F_{2i}^+\Big\},$$

where $i = 1, 2, \ldots$.

Proof. Consider the problem II_B. We are looking for the classical C^2-solution $y(t)$, so the derivative $y'(t)$ is a continuous function and we must look for continuity of $y'(t)$ at zero points of $y(t)$.

It is clear that $y(t)$ must have zeroes in (a,b). That is why $F_0^\pm = \emptyset$.

We will prove the theorem for the case of F_1^+. Suppose that $(\mu,\lambda)\in F_1^+$ and let $y(t)$ be the corresponding nontrivial solution of the problem II_B. The solution has only one zero in (a,b) and $y'(a) > 0$. Let this zero be denoted by τ.

Consider a solution of the problem II_B in the interval (a,τ) and in the interval (τ,b). We obtain that the problem II_B in these intervals reduces to the linear eigenvalue problems. So in the interval (a,τ) we have the problem $y'' = -\mu y$ with boundary conditions $y(a) = y(\tau) = 0$. In the interval (τ,b) we have the problem $y'' = -\lambda y$ with boundary condition $y(\tau) = 0$, notice that $y'(\tau) < 0$. In view of (3) a solution $y(t)$ must satisfy the condition

$$\int_a^\tau y(s)ds = \Big|\int_\tau^b y(s)ds\Big|. \tag{4}$$

Since $y(t) = A\sin(\sqrt{\mu}(t-a))$ for $A > 0$ (this relation holds only for $t \in (a,\tau)$) and $y(\tau) = 0$ we obtain $\tau = \frac{\pi}{\sqrt{\mu}} + a$. In view of this equality it is easy to get that

$$\int_a^\tau y(s)ds = \frac{A}{\sqrt{\mu}}\big(1 - \cos\sqrt{\mu}(\tau - a)\big) = \frac{2A}{\sqrt{\mu}}.$$

We have also

$$y'(\frac{\pi}{\sqrt{\mu}} + a) = -\sqrt{\mu}A. \tag{5}$$

Now we consider a solution of the problem II_B in (τ, b). Since $y(t) = -B\sin(\sqrt{\lambda}(t - \frac{\pi}{\mu} - a))$ $(B > 0)$ we obtain

$$\Big|\int_\tau^b y(s)ds\Big| = \frac{B}{\sqrt{\lambda}}\big(1 - \cos(\sqrt{\lambda}(b - a - \frac{\pi}{\mu}))\big).$$

We have also that

$$y'(\frac{\pi}{\sqrt{\mu}} + a) = -\sqrt{\lambda}B. \tag{6}$$

It follows from (5) and (6) that $A = \sqrt{\frac{\lambda}{\mu}}B$.

In view of the last equality and (4) we obtain

$$\frac{2\sqrt{\lambda}B}{\mu} = \frac{B}{\sqrt{\lambda}}\big(1 - \cos(\sqrt{\lambda}(b - a - \frac{\pi}{\mu}))\big).$$

Multiplying both sides by $\frac{\sqrt{\mu}}{B}$, we obtain

$$\frac{2\sqrt{\lambda}}{\sqrt{\mu}} - \frac{\sqrt{\mu}}{\sqrt{\lambda}} + \frac{\sqrt{\mu}\cos(\sqrt{\lambda}(b-a) - \frac{\sqrt{\lambda}\pi}{\sqrt{\mu}})}{\sqrt{\lambda}} = 0. \tag{7}$$

Considering the solution of the problem II_B it is easy to prove that $\frac{\pi}{\sqrt{\mu}} \le b - a < \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}}$.

This result and (7) prove the theorem for the case of F_1^+. The proof for other branches is analogous. □

Figure 2 illustrates the Fučík spectrum structure for the problem II_B in the case of $b - a = 1$.

At the end of this section we would like to give some properties of the spectra of the problems II_A and II_B:

- the branches of the spectrum for the problem II_B are bounded, but the branches of the spectrum for the problem II_A are infinite;
- all positive branches (F_i^+) constitute a continuous curve, which is located below the bisectrix, similarly all negative branches (F_i^-) constitute a continuous curve, which is located above the bisectrix for the problem II_B;
- the branches $F_{2n-1}^\pm$ and $F_{2n}^\pm$ have a common point which is the eigenvalue of the corresponding linear problem, the branches $F_{2n}^\pm$ and $F_{2n+1}^\pm$ have a common point which is not the eigenvalue of the corresponding linear problem for the problem II_B;

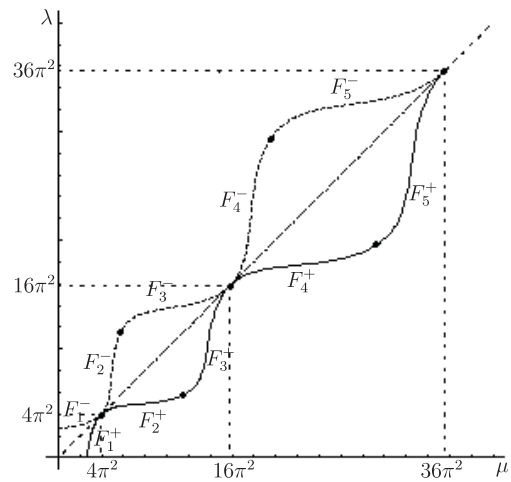

FIGURE 2. The spectrum for the problem II_B.

- easy computations shows that the branches F^-_{2i} and F^-_{2i+1} intersect at the points $\left((\sqrt{i(i+1)}+i)^2\pi^2; (\sqrt{i(i+1)}+i+1)^2\pi^2\right)$. We can obviously see some of these points in Figure 2. The points of intersections of positive branches are symmetric to those of the negative branches with respect of the bisectrix.

5. **Connection between the spectra.** Consider the second order equation (1) together with the boundary conditions

$$\begin{gathered} y(a) = 0; \\ (1-\alpha)y(b) + \alpha \int_a^b y(s)ds = 0, \quad \alpha \in [0,1]. \end{gathered} \tag{8}$$

We call this problem the problem II_C.

Theorem 3. *The spectrum for the problem II_C consists of the branches given by (where $i = 1, 2, \ldots$)*

$$\begin{aligned} F^+_{2i-1} = \quad & \Big\{(\mu,\lambda)\Big| \frac{2i\sqrt{\lambda}}{\sqrt{\mu}}\alpha - \frac{(2i-1)\sqrt{\mu}}{\sqrt{\lambda}}\alpha - \frac{\sqrt{\mu}\alpha\cos\left(\sqrt{\lambda}(b-a) - \frac{\sqrt{\lambda}\pi i}{\sqrt{\mu}} + \pi i\right)}{\sqrt{\lambda}} \\ & + \sqrt{\mu}\sin\left(\sqrt{\lambda}(b-a) - \frac{\sqrt{\lambda}\pi i}{\sqrt{\mu}} + \pi i\right) - \alpha\sqrt{\mu}\sin\left(\sqrt{\lambda}(b-a) - \frac{\sqrt{\lambda}\pi i}{\sqrt{\mu}} + \pi i\right) \\ = \quad & 0, \\ & \tfrac{i\pi}{\sqrt{\mu}} + \tfrac{(i-1)\pi}{\sqrt{\lambda}} \le b-a,\ \tfrac{i\pi}{\sqrt{\mu}} + \tfrac{i\pi}{\sqrt{\lambda}} > b-a\Big\}, \end{aligned}$$

$$F^-_{2i-1} \quad = \Big\{(\mu,\lambda)\Big|\ (\lambda,\mu) \in F^+_{2i-1}\Big\},$$

$$\begin{aligned} F^+_{2i} = \quad & \Big\{(\mu,\lambda)\Big| \frac{(2i+1)\sqrt{\lambda}}{\sqrt{\mu}}\alpha - \frac{2i\sqrt{\mu}}{\sqrt{\lambda}}\alpha - \frac{\sqrt{\lambda}\alpha\cos\left(\sqrt{\mu}(b-a) - \frac{\sqrt{\mu}\pi i}{\sqrt{\lambda}} + \pi i\right)}{\sqrt{\mu}} \\ & + \sqrt{\lambda}\sin\left(\sqrt{\mu}(b-a) - \frac{\sqrt{\mu}\pi i}{\sqrt{\lambda}} + \pi i\right) - \alpha\sqrt{\lambda}\sin\left(\sqrt{\mu}(b-a) - \frac{\sqrt{\mu}\pi i}{\sqrt{\lambda}} + \pi i\right) \\ = \quad & 0, \\ & \tfrac{i\pi}{\sqrt{\mu}} + \tfrac{i\pi}{\sqrt{\lambda}} \le b-a,\ \tfrac{(i+1)\pi}{\sqrt{\mu}} + \tfrac{i\pi}{\sqrt{\lambda}} > b-a\Big\}, \end{aligned}$$

$$F_{2i}^{-} \quad = \left\{ (\mu, \lambda) \middle| \, (\lambda, \mu) \in F_{2i}^{+} \right\}.$$

Proof. The proof of Theorem is similar to that of Theorem 2. □

Remark 1. If $\alpha = 0$ we obtain the problem II_A. In case of $\alpha = 1$ we have the problem II_B.

The branches $F_1^{\pm}$ to $F_5^{\pm}$ of the spectrum for the problem II_C for several values of α are depicted in Figures 3 and 4 in the case of $b - a = 1$.

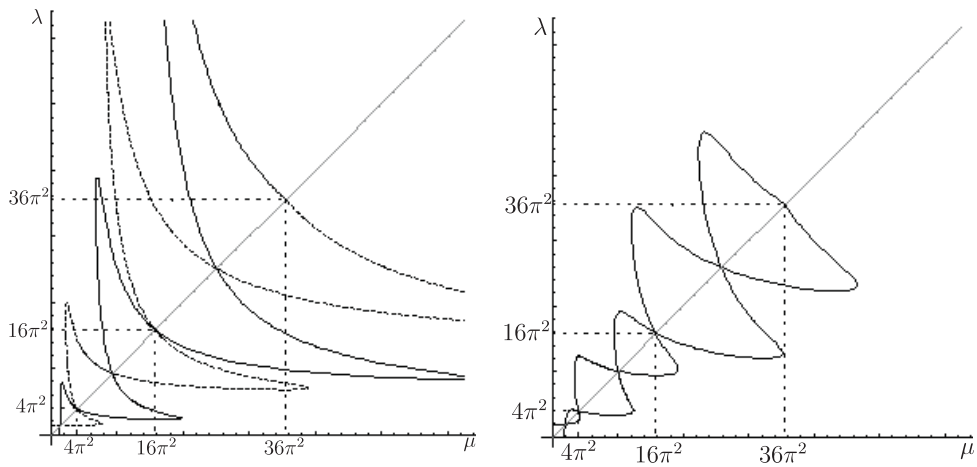

FIGURE 3. The spectrum for the problem II_C for $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.

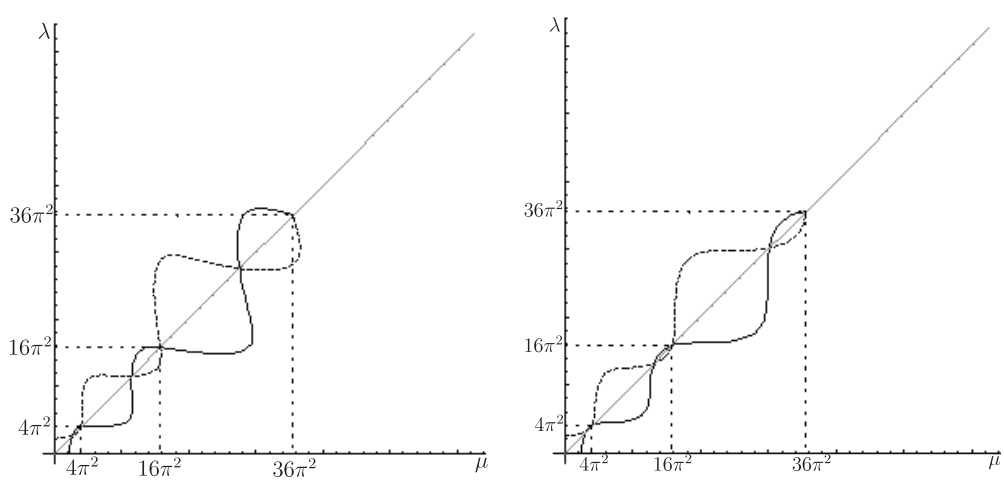

FIGURE 4. The spectrum for the problem II_C for $\alpha = \frac{8}{9}$ and $\alpha = \frac{19}{20}$.

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