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RESULTS RELATED TO GENERALIZATIONS OF HILBERT'S NON-IMMERSIBILITY THEOREM FOR THE HYPERBOLIC PLANE

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Abstract. We discuss generalizations of the well-known theorem of Hilbert that there is no complete isometric immersion of the hyperbolic plane into Euclidean 3-space. We show that this problem is expressed very naturally as the question of the existence of certain homotheties of reflective submanifolds of a symmetric space. As such, we conclude that the only other (non-compact) cases to which this theorem could generalize are the problem of isometric immersions with flat normal bundle of the hyperbolic space H^n into a Euclidean space E^{n+k}, $n \geq 2$, and the problem of Lagrangian isometric immersions of H^n into $\mathbb{C}^n$, $n \geq 2$. Moreover, there are natural compact counterparts to these problems, and for the compact cases we prove that the theorem does, in fact, generalize: local embeddings exist but complete immersions do not.

1. Introduction

Around 1900, D. Hilbert proved that there is no complete isometric immersion of the hyperbolic plane into Euclidean 3-space [9]. Cartan studied the generalization of the problem to higher dimensions in [4, 5], proving that the minimal codimension needed for even a *local* isometric immersion of the hyperbolic n-space H^n into the Euclidean space E^{n+k} is $k = n-1$. In this codimension, a formula exists which gives explicit local solutions. Thus a question, which is still open for $n > 2$, is whether or not *complete* isometric immersions exist $H^n \to E^{2n-1}$. For convenience, we will call this the *hyperbolic non-immersion problem*.

Building on Cartan's work, J.D. Moore showed the existence of global asymptotic coordinates and the flatness of the normal bundle for such an immersion [13]. The existence of these coordinates, which had been used in the proof of Hilbert's theorem, led to the conjecture that the higher dimensional analogue should hold. Moreover, for space forms, these results only depend on the fact that the extrinsic curvature (the sectional curvature of the source minus that of the target) is negative. Thus, a corollary of the existence of asymptotic coordinates, which give a covering of the immersed space by Euclidean space, is that, on topological grounds, there can be no global isometric immersion of a sphere of dimension $n \geq 2$ into a

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sphere of smaller radius and of dimension $2n-1$. We call this the *compact version* of the non-immersion problem.

In fact, there really are only two versions of this negative extrinsic curvature non-immersion problem for space forms, since in the hyperbolic version the target can equivalently be replaced by a hyperbolic space H_K^{2n-1}, with sectional curvature $K > -1$, or a sphere $S^{2n-1}(R)$ of any radius, R. The third possibility, flat immersions into a sphere, does have a global solution in the critical codimension, namely the Clifford torus immersion $E^n \to S^{2n-1}$. One should point out that the *positive* extrinsic curvature problem is not of interest, as global umbilic hypersurfaces exist.

The (hyperbolic) non-immersion problem was studied by various people, such as Terng and Tenenblat [18, 17], Xavier [19], Pedit [15], and in several works of Y. Aminov. In general, the results to date concerning this depend not directly on the codimension, but only on the flatness of the normal bundle. The compact version result mentioned above also holds in arbitrary codimension, with this normal bundle assumption. Concerning the hyperbolic version, Nikolayevsky [14] proved that if M has constant sectional curvature $c < 0$, and the fundamental group of M is non-trivial, then there is no complete isometric immersion with flat normal bundle of M into any Euclidean space. Thus, only the simply connected case remains.

All of the works just mentioned used the special coordinates of Cartan and Moore as the starting point. On the other hand, from a different point of view, Ferus and Pedit [8], gave a representation of constant (non-zero) curvature submanifolds with flat normal bundle of (non-flat) space forms as certain maps into a loop group, ΛG, the group of maps from the unit circle S^1 into a Lie group G. They showed how to produce infinitely many local solutions by solving a collection of commuting ODEs on a certain finite dimensional vector space, a standard feature of so-called "integrable systems" (as is the existence of the Bäcklund transformations studied by Terng and Tenenblat).

This loop group construction was studied further in [1] and [3]. A consequence of the loop group formulation is that it is not hard to see that the whole construction applies much more generally, and is associated to any pair of commuting involutions on a semisimple Lie group. One can see that certain questions, concerning existence of solutions, should only depend on the rank of a symmetric space corresponding to one of these involutions. Thus the general context of the non-immersion problem should be determined by identifying what is the geometric interpretation of the special submanifolds arising in analogue to the constant curvature submanifolds of space forms.

The generalized loop group problem is investigated comprehensively in the article [2]. In this note, we extract the conclusions which are relevant to generalizations of Hilbert's theorem. One concludes that, within this context, the only other possible generalization, besides that of isometric immersions with flat normal bundle of H^n into E^{n+k}, is the problem of isometric Lagrangian immersions of H^n into $\mathbb{C}^n$. For these cases, local solutions can be constructed by integrable systems methods, and the author does not know whether global solutions exist. In all other potential cases (described below) local solutions do not exist.

We also determine all possible generalizations of the compact version of the problem, which are: the problem of isometric immersions with flat normal bundle of a sphere $S^n(R)$ of radius $R > 1$ into a unit sphere S^{n+k}, $k \geq n-1$, and the problem of isometric Lagrangian immersions of $S^n(R)$, $R > 1$, into $\mathbb{C}P^n$. We prove that these have local, but no global, solutions.

Remark 1.1. A stronger version of Hilbert's theorem was proved by Efimov in [6]. He proved that there is no complete isometric immersion in E^3 of a surface whose Gauss curvature is bounded above by some negative constant. Generalizations to higher dimensions of this stronger result have been in the direction of hypersurfaces [16] rather than to codimension $n-1$.

2. Generalizations of the compact version

2.1. **The loop group construction.** We will primarily describe the compact case in this article. Full details of all cases can be found in [2]. The basis of the method is that, given an immersion into a homogeneous space, $f : M \to G/H$, one can lift f to a frame $F : M \to G$. Up to an irrelevant isometry of G/H, the immersion f is completely determined by the pull-back to M of the Maurer-Cartan form of G, denoted by $F^{-1}\mathrm{d}F$, and called the Maurer-Cartan form of F. One can study special submanifolds by choosing an appropriately adapted frame, and the geometry is encoded in the Maurer-Cartan form.

Let G be a complex semisimple Lie group, and U a real form defined as the fixed point subgroup of a complex antilinear involution, ρ, of G. Let σ and τ be a pair of involutions of G, and suppose that all three involutions commute. Let ΛG denote the group of maps from the unit circle S^1 to G, of a suitable class, so that ΛG is a Banach Lie group. Denoting the S^1 parameter by λ, extend the involutions to ΛG by the formulae:

$$(\rho x)(\lambda) := \rho(x(\bar{\lambda})), \qquad (\tau x)(\lambda) := \tau(x(-\lambda^{-1})), \qquad (\sigma x)(\lambda) := \sigma(x(-\lambda)),$$

where $x : S^1 \to G$ is any element of ΛG. Now define a subgroup $\mathcal{H}$, of ΛG as the set of elements which are fixed by all three involutions,

$$\mathcal{H} := \{x \in \Lambda G \mid \rho x = \tau x = \sigma x = x\}.$$

The point of this loop group construction will be that the maps we consider give families of specially adapted frames, F_λ, whose Maurer-Cartan forms have a particular expression (see (1) below). The description in terms of the extended involutions above is important for the problem of *constructing* solutions, as this description fits naturally into very general methods which we shall not describe.

The Lie algebra, $Lie(\mathcal{H})$, of $\mathcal{H}$ consists of Laurent polynomials in λ, $\sum X_i \lambda^i$, with coefficients X_i in certain subspaces of the Lie algebra $\mathfrak{g}$ of G, and with an appropriate convergence condition. Let $\mathcal{H}^0 := \mathcal{H} \cap G$ be the subgroup of constant loops. The type of loop group maps we consider are smooth maps $f : M \to \mathcal{H}/\mathcal{H}^0$, where M is a simply connected manifold, and $\mathcal{H}/\mathcal{H}^0$ is the left coset homogeneous space. Moveover, we impose the restriction that for any lift, $F : M \to \mathcal{H}$, of f, the Maurer-Cartan form $F^{-1}\mathrm{d}F$ of F is a Laurent polynomial (the coefficients of which are $\mathfrak{g}$-valued 1-forms) whose highest and lowest powers of λ are 1 and -1 respectively. Denote the set of such maps by:

$$\mathcal{F}(M) := \{f : M \to \mathcal{H}/\mathcal{H}^0 \mid F^{-1}\mathrm{d}F = \sum_{i=-1}^{1} \alpha_i \lambda^i, \ \forall \text{ lifts } F\}.$$

Clearly, if we fix a value of the loop parameter, λ, an element $f \in \mathcal{F}(M)$, gives a map $f_\lambda : M \to G/\mathcal{H}^0$, with corresponding frames F_λ. The restriction on the Maurer-Cartan form of F ensures that its dependence on λ extends holomorphically to the punctured complex plane $\mathbb{C}^*$, so we can consider f_λ for non-zero real values $\lambda \in \mathbb{R}^*$. The condition $\rho F = F$ implies that, for real values of λ, F_λ takes its values in the real form U. It is clear that $\mathcal{H}^0$ is the fixed point subgroup of G with respect to the three involutions ρ, τ and σ. In fact $\mathcal{H}^0 = K \cap U_+$, where

$$K = U_\tau, \qquad U_+ = U_\sigma,$$

are the fixed point subgroups with respect to τ and σ.

Thus, for $\lambda \in \mathbb{R}^*$, f_λ is a map

$$M \to \frac{U}{K \cap U_+},$$

and one can consider projections to either of the symmetric spaces U/K or U/U_+. We denote these projections by $\bar{f}_\lambda : M \to U/K$ and $\hat{f}_\lambda : M \to U/U_+$.

Set

$$R_\lambda = \left| \frac{\lambda + \lambda^{-1}}{2} \right|.$$

Note that $R_\lambda \geq 1$ for $\lambda \in \mathbb{R}^*$.

Theorem 2.1. [8, 2] *Projection to U/K: Suppose that M has dimension n, and let $f \in \mathcal{F}(M)$. Suppose that the projection of the map obtained at $\lambda = 1$, $\bar{f}_1 : M \to U/K$ is regular (i.e. an immersion). Then so is $\bar{f}_\lambda$ for any other value of $\lambda \in \mathbb{R}^*$, and*

(i) *Suppose $U = SO(n+k+1)$, $\sigma = \mathrm{Ad}_P$, $\tau = \mathrm{Ad}_Q$, where*

$$P = \begin{bmatrix} I_n & 0 \\ 0 & -I_{k+1} \end{bmatrix}, \qquad Q = \begin{bmatrix} I_{n+k} & 0 \\ 0 & -1 \end{bmatrix}$$

and I_l denotes an $l \times l$ identity matrix.

Then $U/K = S^{n+k}$, and $\bar{f}_\lambda : M \to S^{n+k}$, with the induced metric, is an isometric immersion with flat normal bundle of a part of a sphere $S^n(R_\lambda)$, of radius R_λ.

(ii) *Suppose $U = SU(n+1)$ is represented by the matrix subgroup of $SO(2n+2)$ consisting of all matrices of the form $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$, where A and B are $(n+1) \times (n+1)$, and such that $\det(A+iB) = 1$. Let $\sigma = \mathrm{Ad}_P$, for $P = \mathrm{diag}(I_{n+1}, -I_{n+1})$, and $\tau = \mathrm{Ad}_Q$, for $Q = \mathrm{diag}(I_n, -1, I_n, -1)$.*

Then $U/K = \mathbb{C}P^n$, and $\bar{f}_\lambda : M \to \mathbb{C}P^n$, with the induced metric, is a Lagrangian isometric immersion of a part of a sphere $S^n(R_\lambda)$, of radius R_λ.

Conversely, in both cases, any such isometric immersion with $R > 1$ can be represented by such an element, $f \in \mathcal{F}(M)$.

Note that, for the converse, one needs R to be strictly greater than 1. The reasons for this will become clear in the proof outlined below.

Theorem 2.2. [2] *Projection to U/U_+: Let $f \in \mathcal{F}(M)$. For $\lambda \in \mathbb{R}^*$, let $\hat{f}_\lambda : M \to U/U_+$ be the map obtained by projection. In the limit, as $\lambda \to \infty$, $\hat{f}_\lambda$ is asymptotic to a flat: that is, a flat totally geodesic submanifold of the symmetric space U/U_+.*

If the projection $\bar{f}_\lambda : M \to U/K$ is an immersion, then so is $\hat{f}_\lambda$. In this case, M admits a flat metric.

Note that in [2] (Proposition 4.2) it is stated, too strongly, that the projection $\hat{f}_\lambda$ is a curved flat for all real λ. In fact, this is only true in the limit, as λ approaches 0 or ∞.

A flat is obtained by exponentiating an Abelian subalgebra of $\mathfrak{u}_-$. If the symmetric space U/U_+ is Riemannian, that is, if U_+ is compact, then the dimension of such a subalgebra is, by definition, no greater than the rank of U/U_+. Hence, Theorem 2.2 has the following corollary:

Corollary 2.3. *Let $f \in \mathcal{F}(M)$, and suppose that the map $\bar{f}_\lambda : M \to U/K$ is regular. Then, if U_+ is compact, the dimension of M is no greater than the rank of U/U_+.*

Applying this condition to the first case in Theorem 2.1, we conclude that the minimal codimension needed for an isometric immersion $S^n(R) \to S^{n+k}$, where $R > 1$, is $k = n - 1$. Thus Corollary 2.3 gives a natural explanation for this known result.

For the second case in Theorem 2.1, the symmetric space U/U_+ is just $SU(n+1)/SO(n+1)$, which has rank n, thus local solutions are not ruled out. In fact, Corollary 2.3 has a converse:

Theorem 2.4. [2] *If $\mathrm{Dim}(M) \leq \mathrm{Rank}(U/U_+)$, then solutions $f \in \mathcal{F}(M)$ can always be constructed, at least locally, which have regular projections to U/K.*

Hence we conclude that local Lagrangian isometric immersions $S(R)^n \to \mathbb{C}P^n$ do indeed exist for $R > 1$.

Finally, the flat metric of Theorem 2.2 implies the existence of a topological covering by a Euclidean space. We therefore conclude:

Corollary 2.5. *If the dimension of M is greater than 1, then none of the solutions, $\bar{f}_\lambda$, in either case of Theorem 2.1, can be complete.*

2.2. The Proofs of Theorems 2.1 and 2.2.

Let

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{u}_+ \oplus \mathfrak{u}_-$$

be the canonical decompositions of the Lie algebra of U, associated to τ and σ respectively. We have the orthogonal (with respect to the Killing form of $\mathfrak{u}$) decomposition

$$\mathfrak{u} = \mathfrak{u}^{++} \oplus \mathfrak{u}^{+-} \oplus \mathfrak{u}^{-+} \oplus \mathfrak{u}^{--},$$

where

$$\mathfrak{u}^{++} = \mathfrak{k} \cap \mathfrak{u}_+ =: \mathfrak{k}', \qquad \mathfrak{u}^{+-} = \mathfrak{k} \cap \mathfrak{u}_-,$$
$$\mathfrak{u}^{--} = \mathfrak{p} \cap \mathfrak{u}_- =: \mathfrak{p}', \qquad \mathfrak{u}^{-+} = \mathfrak{p} \cap \mathfrak{u}_+ =: \mathfrak{p}'^{\perp}.$$

The starting point for the proof of both theorems is the fact that if $F : M \to \mathcal{H}$ is a lift of an element $f \in \mathcal{F}(M)$, then, for $\lambda \in \mathbb{R}^*$, the Maurer-Cartan form, $\alpha^\lambda := F^{-1}\mathrm{d}F$, of F has the following expansion:

$$\alpha^\lambda = \alpha_0^{++} + \alpha_1^{+-}(\lambda - \lambda^{-1}) + \alpha_1^{--}(\lambda + \lambda^{-1}), \tag{1}$$

where

$$\alpha_0^{++} \in \mathfrak{u}^{++} \otimes \Omega(M), \qquad \alpha_1^{+-} \in \mathfrak{u}^{+-} \otimes \Omega(M), \qquad \alpha_1^{--} \in \mathfrak{u}^{--} \otimes \Omega(M),$$

and $\Omega(M)$ denotes the space of 1-forms on M. The expansion (1) is deduced from the fact that α^λ is fixed by the infinitesimal versions of σ and τ.

2.2.1. *Proof of Theorem 2.1.* Recall that F_λ is U-valued for $\lambda \in \mathbb{R}^*$. We can use the frame, F_λ, to give vector bundle isomorphisms between the tangent and normal bundles for $\bar{f}_\lambda$, $\lambda \neq 1$, and those at $\lambda = 1$, by left translating them to $\mathfrak{p}$ via F_1^{-1} and F_λ^{-1} respectively. This is accomplished as follows: the tangent space to U/K at $f_\lambda(x)$ is identified with $\mathfrak{p}$ via left multiplication by $F_\lambda(x)^{-1}$, in the standard way. Giving the immersions the metric induced from the standard Killing form metric on U/K, the coframe is thus given by the $\mathfrak{p}$ component of this 1-form, namely $\alpha_1^{--}(\lambda+\lambda^{-1})$, from which one deduces that the induced metric for $\bar{f}_\lambda$ is just a scalar multiple of the metric at $\lambda = 1$. In both cases of Theorem 2.1, $\mathrm{Dim}(M) = \mathrm{Dim}(\mathfrak{p}')$, and so it follows that the tangent space to $f_\lambda(M)$ and the normal space are given, respectively, by $\mathfrak{p}'$ and $\mathfrak{p}'^\perp$ under the above identification. The connection 1-form for $\bar{f}_\lambda$ is given by the projection of α^λ to $\mathfrak{k}$, and this splits into the connections on the tangent and normal bundles as well as the second fundamental form. One has the relations

$$[\mathfrak{k} \cap \mathfrak{u}_+, \mathfrak{p}'] \subset \mathfrak{p}', \qquad [\mathfrak{k} \cap \mathfrak{u}_+, \mathfrak{p}'^\perp] \subset \mathfrak{p}'^\perp,$$
$$[\mathfrak{k} \cap \mathfrak{u}_-, \mathfrak{p}'] \subset \mathfrak{p}'^\perp, \qquad [\mathfrak{k} \cap \mathfrak{u}_-, \mathfrak{p}'^\perp] \subset \mathfrak{p}',$$

from which it is not difficult to see that the second fundamental form is given by the $\mathfrak{k} \cap \mathfrak{u}_-$ component, namely $\alpha_1^{+-}(\lambda - \lambda^{-1})$, which implies that, at $\lambda = 1$, we have (part of) a totally geodesic submanifold N of U/K. Since the coframe takes values in $\mathfrak{p}'$, N must be the projection of $\exp(\mathfrak{p}')$ to U/K. In the first case, N is a totally geodesic sphere S^n in $SO(n+k+1)/SO(n+k) = S^{n+k}$. In the second case, N is a totally geodesic Lagrangian submanifold of $\mathbb{C}P^n$.

Finally, the 1-form α_0^{++} is the sum of the tangential and normal connections of $\bar{f}_\lambda$, which does not depend on λ. This means that, under the isomorphisms between the respective tangent and normal bundles for different values of λ given above, the connections on these bundles are also preserved. For the first case, this means the normal bundle is flat, as the totally geodesic sphere N has this property. For the second case, where N is Lagrangian, it follows that, for $\lambda \neq 1$, $\bar{f}_\lambda$ is also Lagrangian, as the complex structure is given on $\mathfrak{p}$, and the tangent and normal bundles for different values of λ are identified in $\mathfrak{p}$.

The converse can be obtained in a fairly straightforward manner, by choosing appropriately adapted frames for the type of immersions required. Note that one cannot have $R = 1$ for the converse, because this corresponds to $\lambda - \lambda^{-1} = 0$ and we would not know what 1-form to insert for α_1^{+-} in (1).

2.2.2. *Proof of Theorem 2.2.* For the projection, $\hat{f}$, to U/U_+, the coframe is given by the $\mathfrak{u}_-$ component of α_λ, namely $\beta = \alpha_1^{+-}(\lambda - \lambda^{-1}) + \alpha_1^{--}(\lambda + \lambda^{-1})$. The limiting coframe, as $\lambda \to \infty$, is proportional to the 1-form

$$\beta_\infty = \alpha_1^{+-} + \alpha_1^{--}.$$

Now for any value of λ, α^λ must satisfy the Maurer-Cartan equation, $\mathrm{d}\alpha + \alpha \wedge \alpha = 0$, this condition being equivalent to the existence of a map F_λ such that $\alpha^\lambda = F_\lambda^{-1}\mathrm{d}F_\lambda$. The fact that this holds for *all* values of λ implies the curved flat equation, $\beta_\infty \wedge \beta_\infty = 0$, or equivalently, that the matrix components of β all commute, from which one can deduce that $\hat{f}_\lambda$ is asymptotic to a flat as $\lambda \to \infty$. The condition that $\bar{f}_\lambda$ be regular is just that its coframe, α_1^{--}, consists of n linearly independent 1-forms. The condition that $\hat{f}_\lambda$ be regular is that β has the same property, and this clearly follows from the regularity of $\bar{f}_\lambda$.

Finally, one can show that the metric given by

$$(X, Y) := \langle \beta_\infty((F_\lambda)_* X), \beta_\infty((F_\lambda)_* Y) \rangle,$$

where $\langle , \rangle$ is the Killing metric on $\mathfrak{u}_-$, is a flat metric on M. One way to see this is that, as shown in [2], there is, locally associated to F_λ, a curved flat $f_+ : M \to U/U_+$ (see the proof of Proposition 5.2), and the coframe of f_+ is given by $\psi = \mathrm{Ad}_C \beta_\infty$, where C takes values in U_+. Since U_+ acts by isometries on $\mathfrak{u}_-$, the metric $(,)$ is the same as that induced by the curved flat. It is shown in [7] that such a metric is flat.

2.3. **Generalizations to other symmetric spaces.** It is known that such a pair of involutions, τ and σ define a *reflective submanifold* N of the symmetric space U/K; that is, a totally geodesic submanifold which has an external symmetry. This is just the projection to U/K of $\exp(\mathfrak{p}')$, mentioned in the previous section. In fact, all connected totally geodesic submanifolds of symmetric spaces are given by $\exp(\mathfrak{p}')$ for some vector subspace $\mathfrak{p}'$ of $\mathfrak{p}$ which is closed under the Lie triple product, $[\mathfrak{p}', [\mathfrak{p}', \mathfrak{p}']] \subset \mathfrak{p}'$. A reflective submanifold has the additional property that the orthogonal complement in $\mathfrak{p}$, denoted $\mathfrak{p}'^{\perp}$, is also closed under the triple product. This is equivalent to the existence of the second involution σ.

The argument outlined above applies in all cases, assuming that $\mathrm{Dim}(M) = \mathrm{Dim}(\mathfrak{p}')$. In general, the projections $\bar{f}_\lambda$ of elements $f \in \mathcal{F}(M)$ correspond to certain homotheties of the reflective submanifold $\bar{f}_1$ obtained at $\lambda = 1$, keeping the normal bundle isomorphic. We have shown that, in the compact case, globally, there is no such homothety for any reflective submanifold. To check whether local solutions exist for other cases is just a matter of comparing the dimension of the reflective submanifold with the rank of the associated second symmetric space, U/U_+. Now reflective submanifolds of symmetric spaces were studied and classified by D.S.P. Leung in [10, 11, 12], and there are many cases. However, it turns out that, in all the other cases, the rank is too small for local solutions to exist. Hence we conclude that Corollary 2.5 contains all possible generalizations (to reflective submanifolds of simply connected, compact, irreducible, Riemannian symmetric spaces) of the compact version of the Hilbert theorem.

3. The hyperbolic case

For the hyperbolic case, the problem in which we are interested, namely, negative extrinsic curvature, corresponds to homotheties of the reflective submanifold by a factor $R < 1$, rather than greater than 1. This problem also has a loop group formulation, which differs from that described above only in that the loops are real-valued for values of the parameter λ in S^1, rather than $\mathbb{R}^*$. However, by evaluating such a loop group map for values of λ in $\mathbb{R}^*$, instead of in S^1, one obtains a similar situation to that of the compact case (although for a different, non-Riemannian, symmetric space $\tilde{U}/\tilde{K}$) and one can obtain analogous results to those described in the compact case, with the exception of the non-existence of global solutions, which remains an open problem. In particular, it is shown in [2] that the only cases (of reflective submanifolds of simply connected, non-compact, irreducible, Riemannian symmetric spaces) where local solutions exist are:

(i) $U/K = H^{n+k}$, and $\bar{f}_\lambda : M \to H^{n+k}$, $k \geq n-1$, with the induced metric, is an isometric immersion with flat normal bundle of a part of a hyperbolic space H^n_c of constant sectional curvature $c < -1$.

(ii) $U/K = \mathbb{C}H^n$, and $\bar{f}_\lambda : M \to \mathbb{C}H^n$, with the induced metric, is a Lagrangian isometric immersion of a part of a hyperbolic space H^n_c of constant sectional curvature $c < -1$.

In both cases local solutions exist and can be constructed using integrable systems methods.

Equivalently, one can replace the target spaces in cases (i) and (ii) with, respectively, the Euclidean space, E^{n+k}, and complex Euclidean space $\mathbb{C}^n$, by an argument which was given in [3]. That article dealt with the first case only, but the proof is easily adapted to the second case. It essentially involves dilating the target space, while keeping the metric on the immersed space constant, until, in the limit, an immersion into flat space is obtained.

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