

HETEROCLINIC SOLUTIONS FOR NON-AUTONOMOUS BOUNDARY VALUE PROBLEMS WITH SINGULAR Φ -LAPLACIAN OPERATORS

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ABSTRACT. We prove the solvability of the following boundary value problem on the real line

$$\begin{cases} \Phi(u'(t))' = f(t, u(t), u'(t)) & \text{on } \mathbb{R}, \\ u(-\infty) = -1, \quad u(+\infty) = 1, \end{cases}$$

with a singular Φ -Laplacian operator.

We assume f to be a continuous function that satisfies suitable symmetry conditions. Moreover some growth conditions in a neighborhood of zero are imposed.

1. Introduction. The study of the existence of travelling wave solutions for reaction diffusion equations has motivated in the recent years many papers concerning the existence of heteroclinic solutions for second order equations (see for instance [1, 10, 11, 13]).

In the recent paper [4] Bianconi and Papalini study the non-autonomous problem

$$\begin{cases} \Phi(u'(t))' = f(t, u(t), u'(t)), & \text{a.e. on } \mathbb{R}, \\ u(-\infty) = 0, \quad u(+\infty) = 1, \end{cases}$$

where the usual linear second order operator u'' is replaced by the nonlinear one $\Phi(u'(t))'$. Here $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing homeomorphism with $\Phi(0) = 0$. The paradigm for this operator is the classical one-dimensional p -Laplacian

$$\Phi_p(s) = |s|^{p-2}s, \quad p > 1.$$

The p -Laplacian operator arises in non-Newtonian fluid theory (as well as in the diffusion of flows in porous media or in nonlinear elasticity) and has become a very popular subject in the last decades (see [8, 12, 14, 15] and references therein). Some existence results for the p -Laplacian in the presence of lower and upper solutions were extended for arbitrary increasing homeomorphisms Φ with different kinds of boundary conditions in [5, 6].

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Recently some papers have appeared where the authors consider Φ -Laplacian type equations with homeomorphisms $\Phi : (-a, a) \rightarrow (-b, b)$ for $0 < a, b \leq +\infty$ (see [2, 3, 7, 9]). When $b < +\infty$ the Φ -Laplacian is said to be *bounded* or *non-surjective* and the classical model is the mean curvature operator $\Phi(s) = \frac{s}{\sqrt{1+s^2}}$ for $s \in \mathbb{R}$. On the other hand if $a < +\infty$ then the Φ -Laplacian is said to be *singular*, in the terminology of Bereanu and Mawhin [3], and in this case the model is the relativistic operator $\Phi(s) = \frac{s}{\sqrt{1-s^2}}$ for $s \in (-1, 1)$.

In this paper we contribute to the literature studying the following boundary value problem on the real line

$$\begin{cases} \Phi(u'(t))' = f(t, u(t), u'(t)), & \text{on } \mathbb{R}, \\ u(-\infty) = -1, \quad u(+\infty) = 1, \end{cases}$$

where Φ is singular.

In [3] Bereanu and Mawhin have proven the striking result that for a singular Φ -Laplacian the Dirichlet problem

$$\begin{cases} \Phi(u'(t))' = f(t, u(t), u'(t)), & \text{for all } t \in [0, T], \\ u(0) = 0 = u(T), \end{cases}$$

is always solvable for every continuous function f and every $T > 0$ without additional assumptions (see also [7]). This “universal” solvability is related with the fact that all solutions of this problem have their derivatives a priori bounded. In this paper we exploit this fact in order to perform an approximation procedure to deal with our infinite interval problem.

2. Preliminaries. We shall deal with the problem

$$\Phi(u'(t))' = f(t, u(t), u'(t)) \quad \text{on } \mathbb{R}, \tag{1}$$

$$u(-\infty) = -1, \quad u(+\infty) = 1, \tag{2}$$

under the following assumptions:

(h0) $\Phi : (-a, a) \rightarrow \mathbb{R}$ is an increasing homeomorphism, with $\Phi(0) = 0$ and $0 < a < +\infty$ (i.e., Φ is singular).

(f0) $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous and satisfies the symmetry condition

$$f(t, x, y) = -f(-t, -x, y) \quad \text{for all } (t, x, y) \in \mathbb{R}^3.$$

(f1) $f(t, 1, y) = 0 = f(t, -1, y)$ for all $t, y \in \mathbb{R}$.

(f2) $f(t, x, y) < 0$ for all $t > 0, -1 < x < 1$ and $y \in \mathbb{R}$. Moreover for every compact set of the form $K = [-r, r] \times [-\varepsilon, \varepsilon]$, where $0 < r < 1$ and $0 < \varepsilon < 1$, there exist $t_K \geq 0$ and a continuous function $h_K : [t_K, \infty) \rightarrow \mathbb{R}$ such that

$$f(t, x, y) \leq h_K(t) \quad \text{for all } t \geq t_K \text{ and } (x, y) \in K,$$

and

$$\int_{t_K}^{+\infty} h_K(s) ds = -\infty.$$

A *solution* of (1)-(2) is a function $u \in C^1(\mathbb{R})$ such that $u' \in (-a, a)$, $\phi \circ u' \in C^1(\mathbb{R})$ and u satisfies the differential equation (1) and the boundary conditions (2).

We shall approximate problem (1)-(2) by problems defined on compact intervals. The following result shall be very useful for us.

Theorem 2.1. [3, Corollary 1] *Suppose that $\Phi : (-a, a) \rightarrow \mathbb{R}$ is an increasing homeomorphism with $0 < a < +\infty$ and $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous. Then the Dirichlet problem*

$$\Phi(u'(t))' = f(t, u(t), u'(t)), \quad u(0) = 0 = u(T),$$

has at least one solution. (Notice that in particular $\|u'\|_\infty < a$).

3. Main results. Next we prove the solvability of our problem.

Theorem 3.1. *If conditions (h0), (f0), (f1) and (f2) hold then problem (1)-(2) has an odd increasing solution $u : \mathbb{R} \rightarrow \mathbb{R}$.*

Proof. By the symmetry condition of (f0) it suffices to prove the existence of a solution $u : [0, +\infty) \rightarrow \mathbb{R}$ of (1) satisfying $u(0) = 0$ and $\lim_{t \rightarrow +\infty} u(t) = 1$, since its odd extension solves (1)-(2).

Claim 1.- For each $n \in \mathbb{N}$ the Dirichlet boundary value problem

$$\Phi(u'(t))' = f(t, u, u'), \quad u(0) = 0, \quad u(n) = 0, \quad (3)$$

has a solution $u_n : [0, n] \rightarrow \mathbb{R}$ satisfying $0 \leq u_n(t) \leq 1$ and $\|u_n'\|_\infty < a$.

Consider the continuous function

$$\tilde{f}(t, x, y) = \begin{cases} f(t, x, y), & \text{if } -1 \leq x \leq 1 \\ 0, & \text{in other case.} \end{cases}$$

For each $n \in \mathbb{N}$ the modified problem

$$\Phi(u'(t))' = \tilde{f}(t, u(t), u'(t)), \quad u(0) = 0 = u(n),$$

has by Theorem 2.1 a solution $u_n : [0, n] \rightarrow \mathbb{R}$ with $\|u_n'\|_\infty < a$. Moreover it is easy to show that $-1 \leq u_n(t) \leq 1$ and therefore u_n is also a solution of (3). On the other hand (f2) implies that u_n is concave and then $0 \leq u_n(t) \leq 1$.

Claim 2.- There exists a bounded nondecreasing solution $u : [0, +\infty) \rightarrow \mathbb{R}$ of (1) such that $u(0) = 0$ and $0 \leq u(t) \leq 1$.

Since u_n and u_n' are uniformly bounded then it is easy to prove that a subsequence of u_n converges uniformly on compact sets to a solution $u : [0, +\infty) \rightarrow \mathbb{R}$ of (1). Clearly $u(0) = 0$ and $0 \leq u(t) \leq 1$.

On the other hand, from the uniform continuity of function Φ^{-1} on compact sets it follows that the sequence $\{u_n'\}$ is an equicontinuous family, and as consequence it is verified that $\Phi(u'(t))' = f(t, u(t), u'(t)) \leq 0$. So we deduce that u' is nonincreasing. If $u'(t_0) < 0$ at some point $t_0 \geq 0$ then $u'(t) \leq u'(t_0) < 0$ for all $t \geq t_0$ and consequently $\lim_{t \rightarrow +\infty} u(t) = -\infty$, a contradiction. Thus $u'(t) \geq 0$ for all $t \geq 0$ and then u is nondecreasing.

Claim 3.- $\lim_{t \rightarrow +\infty} u'(t) = 0$.

Since u' is decreasing there exists $\lim_{t \rightarrow +\infty} u'(t) \in \mathbb{R} \cup \{-\infty\}$. But as u is bounded we deduce that $\lim_{t \rightarrow +\infty} u'(t) = 0$.

Claim 4.- $\lim_{t \rightarrow +\infty} u(t) = 1$.

Since u is concave and bounded there exists $\lim_{t \rightarrow +\infty} u(t) = l \in (0, 1]$. Suppose that $l < 1$. From (f2) and the facts that $0 \leq u(t) \leq l < 1$ and $\lim_{t \rightarrow +\infty} u'(t) = 0$ it follows that there exist a suitable compact set $K \subset (-1, 1) \times \mathbb{R}$, $t_K > 0$ for which $0 < u'(t_K) < 1$ and a continuous function h_K such that

$$\Phi(u'(t))' = f(t, u(t), u'(t)) \leq h_K(t) \quad \text{for all } t \geq t_K,$$

and $\int_{t_K}^{\infty} h_K(t) = -\infty$. But in this case $\Phi(u'(t)) \rightarrow -\infty$ and then $u'(t) \rightarrow -a < 0$, which is a contradiction. Thus $l = 1$ and the proof is over. \square

Remark 1. With some technical minor modifications the result of Theorem (3.1) also holds for L^1 -Carathéodory nonlinearities instead of continuous ones.

Example 1. Let $n \in \mathbb{N}$ be given. Consider the problem

$$\begin{cases} \left(\frac{u'(t)}{\sqrt{1-u'(t)^2}} \right)' = t^3(u(t)^2 - 1)(u'(t)^{2n} + 1), & \text{on } \mathbb{R}, \\ u(-\infty) = -1, \quad u(+\infty) = 1, \end{cases}$$

where $\Phi(s) = \frac{s}{\sqrt{1-s^2}}$ for all $s \in (-1, 1)$ models mechanical oscillations subject to relativistic effects and $f(t, x, y) = t^3(x^2 - 1)(y^{2n} + 1)$. Clearly conditions of Theorem 3.1 are fulfilled and so its solvability is guaranteed.

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