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THE EXISTENCE OF A GLOBAL ATTRACTOR FOR A KURAMOTO-SIVASHINSKY TYPE EQUATION IN 2D

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Abstract. We consider a variation of the Kuramoto-Sivashinsky Equation in space dimension two. We show that under some assumptions the equation is globally well-posed and posseses a global attractor in the periodic case. The analysis is based on the Lyapunov function approach, point dissipativeness and asymptotic compactness.

The Kuramoto-Sivashinsky Equation $\phi_t = -\Delta^2\phi - \Delta\phi - \frac{1}{2}\|\nabla\phi\|^2$ has been introduced three decades ago as a model of nonlinear evolution of linearly unstable interfaces in various contexts, e.g. hydrodynamics and combustion theory. In the case of one space dimension, the equation has been studied extensively. For the case of bounded domain $[-L, L]$, using Lyapunov function approach, in [7] the first long-time bound was given as $\limsup_{t\to\infty} \|u\|_{L^2} \leq CL^{5/2}$ for odd initial data. Then, in [3] the exponent was improved to $\frac{8}{5}$ for any mean-zero initial data. Recently, in [1] the exponent was improved to $\frac{3}{2}$.

In the case of two space dimensions, the equation is very challenging and even the global well-posedness is still an open problem. We mention several interesting related works here. In [11] the existence of a bounded local absorbing set and an attractor is shown in thin 2D domain with restricted initial data. In [6] this result is improved by showing that $\limsup_{t\to\infty} \|\vec{u}\|_{L^2} \leq CL_x^{8/5}L_y^{1/2}$ on the bounded domain $(0, L_x) \times (0, L_y)$ with the assumption $L_y \leq CL_x^{-67/35}$. Combining the results in [6] with their results for 1D case, the authors of [1] showed that $\limsup_{t\to\infty} \|\vec{u}\| \leq CL_x^{3/2}L_y^{1/2}$ with the assumption of $L_y \leq CL_x^{13/7}$. In [16] the existence of global solution with Newmann boundary conditions is proved for a radial function defined on an annulus $\Omega = \{x \in R^2 : 0 < r_0 < \|x\| < R_1\}$. In [8] the following variation of KS equation is studied

$$u_t = -\Delta^2 u - u_{xx} - uu_x$$

and the existence of an attractor with periodic boundary conditions is proved.

We consider the following variation of the Kuramoto-Sivashinsky equation in 2D:

$$u_t = -\Delta^2 u - \Delta u - uu_x - uu_y + g(x) \tag{1}$$

$$u(0; x, y) = u_0(x, y) \tag{2}$$

$$u(t; x, y) = u(t; x + 2L, y) = u(t; x, y + 2L) \qquad \forall (x, y) \in R^2, \;\; t \geq 0 \tag{3}$$

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We prove the existence of a global attractor for this equation assuming that u is a mean-zero solution with

$$\frac{d^k u}{dx^i dy^j}(x, \pm L) = \frac{d^k u}{dx^i dy^j}(\pm L, y) = \frac{d^k u}{dx^i dy^j}(L, L) \quad k = i + j = 0, 1, 2, 3 \tag{4}$$

$\int_{[-L,L]^2} u_0(x, y)dxdy = 0, \quad (x, y) \in (-L, L) \times (-L, L)$. We also assume that the external force $g(\vec{x})$ is in L^2 and $\int_{[-L,L]^2} g(x, y)dxdy = 0$. In section 1 we introduce some notations and preliminary results. In section 2 we prove the local well-posedness in the periodic case. In order to prove the global well-posedness in the periodic case, we use the potential function ϕ_x introduced in [1], which gives the following result

$$\limsup_{t\to\infty} \|u\|_{L^2([-L,L]^2)} \leqslant CL^2 \tag{5}$$

In section 3 we show that the solution u is point dissipative and asymptotically compact in the periodic case with the assumption of the initial solution u_0 being in the class of L^2. Thus we conclude the existence of a global attractor in $L^2([-L, L]^2)$.

1. **Preliminaries.**

1.1. **Littlewood-Paley projections and function spaces.**

We will define the *Littlewood-Paley operators* acting on $L^2([-L, L]^d)$ via Fourier transform. The Fourier transform $L^2([-L, L]^d) \to l^2(\mathbb{Z}^d)$ is given by $f \to \{a_k\}_{k\in\mathbb{Z}^d}$, where

$$a_k = \frac{1}{(2L)^{d/2}} \int_{[-L,L]^d} f(x)e^{-2\pi ik\cdot x/L}dx$$

The inverse Fourier transform is the Fourier expansion

$$f(x) = \frac{1}{(2L)^{d/2}} \sum_{k\in\mathbb{Z}^d} a_k e^{2\pi ik\cdot x/L}$$

and the Plancherel identity is $\|f\|_{L^2}^2 \sim \sum_{k\in\mathbb{Z}^d} |a_k|^2$.

The **Littlewood-Paley operators** on $L^2([-L, L]^d)$ for a function f are

$$P_{\leq n} f(x) = \frac{1}{(2L)^{d/2}} \sum_{k:|k|\leq 2^n L} a_k e^{2\pi ik\cdot x/L}$$

The projection operator $P_{\leq n}$ truncates the terms in the Fourier series expansion with frequencies $k : |k| > 2^n L$. More generally, we may define for all $0 \leq n < m \leq \infty$

$$P_{n\leq\cdot\leq m} f(x) = \frac{1}{(2L)^{d/2}} \sum_{k:2^n L\leq|k|\leq 2^m L} a_k e^{2\pi ik\cdot x/L}.$$

Next, we introduce the Sobolev spaces

$$\dot{H}^s([-L, L]^d) = \{f : [-L, L]^d \to \mathcal{C} : \left(\sum_{k\in Z^d} |a_k|^2 \left(\frac{|k|}{L}\right)^{2s} \right)^{1/2} < \infty\},$$

$$H^s_{per}([-L, L]^d) = \{\phi : \phi \in \dot{H}^s([-L, L]^d \cap L^2([-L, L]^d)) \text{ and } \phi \text{ satisfies (4)}\}$$

$$\bar{H}^s_{per}([-L, L]^d) = \{\phi : \phi \in H^s_{per}([-L, L]^d) \text{ and } \int_{[-L,L]^d} \phi = 0\}$$

One may also find convenient to work with the equivalent norm

$$\|f\|_{\dot{H}^s([-L,L]^2)} \sim \left(\sum_{j\in\mathcal{Z}} 2^{2sj}(\sum_{|k|\sim 2^j L} |a_k|^2)\right)^{1/2} \sim \left(\sum_{j\in\mathcal{Z}} 2^{2sj}\|P_j f\|^2_{L^2([-L,L]^2)}\right)^{1/2} \tag{6}$$

We will need some properties of this operator while we are working on the following sections. The following lemma will be helpful in the proof of the asymptotic compactness of the solution u. Note that throughout this paper, we will use $\|\cdot\|_2$ to denote $|\cdot|_{([-L,L]^2)}$.

Lemma 1.1. *For the Littlewood-Paley operator P_k defined by*

$$P_k f(x) = \frac{1}{(2L)^{d/2}} \sum_{|n|\sim 2^k L} a_n e^{2\pi i n\cdot x/L}$$

we have

$$\|P_k f\|_2 \lesssim 2^k \|f\|_{L^1([-L,L]^2)} \tag{7}$$

Proof. The proof is in [14]. ☐

1.2. **Attractors.** In this section, we will give some definitions and elementary properties of attractors. For an initial value problem for well-posed evolution equation,

$$\frac{d}{dt}u(t) = F(u(t)), \qquad u(0) = u_0$$

defined on a Hilbert Space H, the solution semigroup $\{S(t)\}_{t\geq 0}$ defined by

$$u(t) = S(t)u(0)$$

$$u(t+s) = S(t)u(s) = S(s)u(t), \qquad \forall s,t \geq 0$$

maps H into itself and enjoys the usual semigroup properties.

$$S(t+s) = S(t)S(s), \qquad \forall s,t \geq 0$$

We will also assume that $S(t)$ is a continuous operator in the initial data for each $t \geq 0$. The above properties are equivalent with well-posedness of the equation.

Definition 1.2. *$\mathcal{A} \in H$ is called a* ***global attractor*** *for the evolution equation if it is compact, invariant ($S(t)\mathcal{A} = \mathcal{A}, t \geq 0$) and attracts every bounded set X. ($S(t)X \to \mathcal{A}, t \to \infty$).*

Definition 1.3. *Let $S(t)$ be a solution semigroup, acting on a normed space H.*
• *$S(t)$ is called* ***point dissipative*** *if there is a bounded set $B \subset H$ such that for any $u_0 \in H$, $S(t)u_0 \in B$ for all sufficiently large $t \geq 0$. That is*

$$\sup_{u_0\in H} \limsup_{t\to\infty} \|S(t)u_0\|_H < \infty \tag{8}$$

• *$S(t)$ is called* ***asymptotically compact*** *in H if $S(t_n)u_n$ has a convergent subsequence for any bounded sequence u_n and any $t_n \to \infty$*

A classical result in dynamical systems is that point dissipativeness and asymptotic compactness of $S(t)$ implies the existence of an attractor, see [10].

Next, we recall the Riesz-Rellich Criteria for precompactness, see Theorem XIII. 66, P248, in [9]. As shown in [12] and [13], we may replace condition (3) in the Riesz-Rellich Criteria above by an equivalent condition, which basically says that the mass of the high-frequency component has to go uniformly to zero. The following proposition is the exact formulation.

Proposition 1. *Assume that*

$$\bullet \sup_n \|u_n(t_n,\cdot)\|_{L^2} \leqslant C \tag{9}$$

$$\bullet \lim_{N\to\infty} \limsup_{n\to\infty} \|u_n(t_n,\cdot)\|_{L^2(|x|>N)} = 0 \tag{10}$$

$$\bullet \limsup_n \|P_{>N}u_n(t_n,\cdot)\|_{L^2} \to 0 \text{ as } N \to \infty \tag{11}$$

Then the sequence $\{u_n(t_n,\cdot)\}$ *is precompact in* $L^2(\mathbb{R}^n)$.

Remark 1. If we are in a bounded domain, then (10) is authomatically satisfied.

2. **Global Well-Posedness for the Kuramoto Sivashinsky Type Equation.** In this section, we first show the local well-posedness for (1) and then iterate the local well-posedness result to a global one by using the apriori bound for the solution.

2.1. **Local well-posedness for (1) in** $L^2([-L,L]^2)$**.** We will show that $\Lambda : L^2([-L,L]^2) \to L^2([-L,L]^2)$ defined by

$$\Lambda u = e^{-t\triangle^2}u(0) + \int_0^t e^{-(t-s)\triangle^2}(-\triangle u - uu_x - uu_y + g)ds \tag{12}$$

has a fixed point in $X_{R,T} = \{u \in L^\infty((0,T), L^2([-L,L]^2)) : \sup \|u(t,\cdot)\|_2 \leqslant R\}$. We need some estimates which we collect in the following lemma.

Lemma 2.1. *Let* $f \in L^2([-L,L]^2)$ *then we have*

$$\|e^{-t\triangle^2}f\|_{\dot{H}^1([-L,L]^2)} \leqslant \frac{C}{t^{1/2}}\|f\|_{L^1([-L,L]^2)} \tag{13}$$

$$\|e^{-t\triangle^2}f\|_2 \leqslant C\|f\|_2 \tag{14}$$

$$\|e^{-t\triangle^2}f\|_{\dot{H}^2([-L,L]^2)} \leqslant \frac{C}{t^{1/2}}\|f\|_2 \tag{15}$$

Proof of (13):

From duality, it is enough to show that $\|\nabla e^{-t\triangle^2}f\|_{L^\infty([-L,L]^2)} \leqslant \frac{C}{t^{1/2}}\|f\|_2$

$$\|\nabla e^{-t\triangle^2}f\|_{L^\infty([-L,L]^2)} \leqslant \|\nabla \widehat{e^{(-D^4)t}f}\|_{L^1([-L,L]^2)} = \frac{1}{2L}\sum_{n\in\mathbb{Z}^2}\frac{2\pi}{L}ne^{-t(\frac{2\pi}{L})^4n^4}|a_n|$$

$$\leqslant \sum_{|n|:n\lesssim t^{-1/4}L}\frac{\pi}{L^2}ne^{-t(\frac{2\pi}{L})^4n^4}|a_n| + \sum_{m\in\mathbb{Z}}\sum_{|n|\sim 2^m t^{-1/4}L}\frac{\pi}{L^2}ne^{-2^{4m}(2\pi)^4}|a_n|$$

$$\leqslant (\sum_n |a_n|^2)^{1/2}\frac{\pi}{t^{1/2}} + \sum_m\left((\pi 2^m t^{-1/4})(e^{-2^{4m}(2\pi)^4})(\sum_n |n|^2|a_n|^2)^{1/2}\right)$$

$$\lesssim (\sum_n |a_n|^2)^{1/2}\frac{1}{t^{1/2}} + \sum_m(\sum_n |a_n|^2)^{1/2}(2^m t^{-1/4})(2^m t^{-1/4})(e^{-2^{4m}16\pi^4})$$

$$= \|f\|_2\frac{1}{t^{1/2}} + \|f\|_2\frac{1}{t^{1/2}}\sum_m 2^{2m}e^{-2^{4m}16\pi^4}$$

$$\leqslant \frac{C}{t^{1/2}}\|f\|_2 \qquad \text{(since } \sum_m 2^{2m}e^{-2^{4m}16\pi^4} \text{ converges.)}$$

Proof of (14): $\|e^{-t\triangle^2}f\|_2 = \|\widehat{e^{-t\triangle^2}f}\|_2 = \left(\sum_{n\in\mathbb{Z}^2} e^{-2t(\frac{2\pi}{L})^4 n^4}|a_n|^2\right)^{1/2} \leqslant C\|f\|_2$

Proof of (15):

$$\|e^{-t\triangle^2}f\|_{\dot{H}^2([-L,L]^2)} = \|\triangle e^{-t\triangle^2}f\|_2 = \left(\sum_{n\in\mathbb{Z}^2}(\frac{2\pi}{L})^4 n^4 e^{-2t(\frac{2\pi}{L})^4 n^4}|a_n|^2\right)^{1/2}$$
$$= \frac{1}{t^{1/2}}(\sum_{n\in\mathbb{Z}^2} t(\frac{2\pi}{L})^4 n^4 e^{-2t(\frac{2\pi}{L})^4 n^4}|a_n|^2)^{1/2} \leqslant \frac{1}{t^{1/2}}\sup_m me^{-2m^2}(\sum|a_n|^2)^{1/2}$$
$$\leqslant \frac{C}{t^{1/2}}\|f\|_2 \qquad (m = t(\frac{2\pi}{L})^4 n^4 \text{ and } \sup_m me^{-2m^2} \text{ exists.})$$

Now we are ready to prove the local well-posedness in $L^2[-L,L]^2$. We have

$$\|\Lambda u\|_2 \lesssim \|e^{-t\triangle^2}u(0)\|_2 + \int_0^t \|e^{-(t-s)\triangle^2}u\|_{\dot{H}^2([-L,L]^2)}$$
$$+ \|e^{-(t-s)\triangle^2}(u^2)\|_{\dot{H}^1([-L,L]^2)} + \|e^{-(t-s)\triangle^2}g\|_2 ds,$$

Applying Lemma 2.1 we get

$$\|\Lambda u\|_2 \leqslant C\|u(0)\|_2 + \int_0^t \frac{C_1}{(t-s)^{1/2}}\|u\|_2 + \frac{C_2}{(t-s)^{1/2}}\|u^2\|_{L^1([-L,L]^2)} + C_3\|g\|_2 ds$$

Since t is in $[0,T]$ we have

$$\|\Lambda u\|_2 \leqslant C\|u(0)\|_2 + 2C_1T^{1/2}\|u\|_2 + 2C_2T^{1/2}\|u\|_2^2 + C_3T\|g\|_2$$

If we choose R such that $C\|u(0)\|_2 \leqslant R/2$ and T such that

$$2C_1T^{1/2}\|u\|_2 + 2C_2T^{1/2}\|u\|_2^2 + C_3T\|g\|_2 \leqslant R/2,$$

we have $\|\Lambda u\|_2 \leqslant R$. Similarly one can show that Λ is a contraction.

2.2. **Global well-posedness for (1) in $L^2([-L,L]^2)$.** In order to prove global well-posedness in $L^2([-L,L]^2)$, it is enough to show that there is a time-independent bound for the solution

$$\sup_{0\leq t\leq T}\|u(t,\cdot)\|_2 \leqslant C.$$

In the following section, we will see that C will depend on $\|g\|_2$ and L only.

We will show that for the local L^2 solution $u(t,.)$ there exists a Lyapunov function $\phi = \phi(x) \in H^2([-L,L]^2)$ such that one has the estimate

$$\|u(t,\cdot)\|_2 \leqslant \|\phi\|_2 + \sqrt{e^{-\frac{\lambda_0}{2}t}\|u(0)\|_2^2 + \frac{2P^2}{\lambda_0}} \tag{16}$$

for some constants $\lambda_0 > 0$ and P, and for every $0 < t < T$, where T is its life span.

3. **Global Attractor for the equation in the periodic case.** As we discussed in Proposition 1, we need to verify that for any $t_n \to \infty$, $B > 0$ and any sequence of initial data $\{u_n\} \in L^2([-L,L]^2)$ with $\sup_n \|u_n\|_2 \leq B$, we have

$$\sup_{u_0\in L^2([-L,L]^2)} \limsup_{t\to\infty} \|S(t)u_0\|_2 \leq C(g,L) \tag{17}$$

$$\sup_n \|S(t_n)u_n)\|_2 \leqslant C(g,B,L) \tag{18}$$

$$\lim_N \limsup_n \|P_{>N}S(t_n)u_n\|_2 = 0 \text{ as } N \to \infty \tag{19}$$

3.1. **Point dissipativeness.** In this section, our aim will be to prove (17). The lemmas and the theorem in this section will be based on Lyapunov approach.

Lemma 3.1. *Given $u = u(t;x,y) \in L^2([-L,L]^2)$ for all $t \geq 0$, and $\phi(t;x,y) = \phi(x) \in L^2([-L,L])$ satisfying the following inequality:*

$$\frac{d}{dt}\|u-\phi\|_2^2 \leqslant -\lambda_0\|u\|_2^2 + P^2 \tag{20}$$

for some constants $\lambda_0 > 0$ and P, then $B(O,R^{**})$, the ball of radius R^{**} centered about the origin, is an attracting region, where the radius R^{**} is given by

$$R^{**} = \sqrt{2\|\phi\|_2^2 + \frac{2P^2}{\lambda_0}} + \|\phi\|_2 \tag{21}$$

Proof. The proof for 1D is in [1], and it also works for 2D. By the parallelogram law $-\lambda_0\|u-\phi\|_2^2 \geqslant -2\lambda_0\|u\|_2^2 - 2\|\phi\|_2^2$, which gives

$$\frac{d}{dt}\|u-\phi\|_2^2 + \frac{\lambda_0}{2}\|u-\phi\|_2^2 \leqslant \lambda_0\|\phi\|_2^2 + P^2$$

If we multiply each side by $e^{\frac{\lambda_0}{2}t}$, we get $\frac{d}{dt}(e^{\frac{\lambda_0}{2}t}\|u-\phi\|_2^2) \leqslant e^{\frac{\lambda_0}{2}t}(\lambda_0\|\phi\|_2^2 + P^2)$. By integrating we get $\|u-\phi\|_2^2 \leqslant e^{-\frac{\lambda_0}{2}t}\|u(0)\|_2^2 + \frac{2}{\lambda_0}(\lambda_0\|\phi\|_2^2 + P^2)$. Thus we have the following result

$$\|(u(t,\cdot)\|_2 \leqslant \|\phi\|_2 + \sqrt{e^{-\frac{\lambda_0}{2}t}\|u(0)\|_2^2 + \frac{2P^2}{\lambda_0}} \tag{22}$$

It is clear that $B(\phi,R^*)$, the ball of radius R^* centered about ϕ, is exponentially attracting, with ${R^*}^2 = 2\|\phi\|_2^2 + (\frac{2P^2}{\lambda_0})$. The triangle inequality implies $B(\phi,R^*) \subset B(0,R^{**})$. This will guarantee the existence of an absorbing set. □

Lemma 3.2. *For any $\phi(t;x,y) = \phi(x) \in \bar{H}^2_{per}[-L,L]$ and $u(t;x,y)$ solving (1) we have the inequality*

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int\limits_{[-2L,2L]^2}(u-8\tilde{\phi})^2 d\tilde{x}d\tilde{y} \leqslant 4\int\limits_{[-2L,2L]^2}(\tilde{\nabla}u)^2 - (\tilde{\triangle}u)^2 + (\frac{1}{4} - \tilde{\phi}_{\tilde{x}})u^2 d\tilde{x}d\tilde{y} \\ + \int\limits_{[-2L,2L]^2} 32(\tilde{\phi}_{\tilde{x}})^2 + 256(\tilde{\phi}_{\tilde{x}\tilde{x}})^2 + 16\tilde{\phi}^2 + \frac{g^2}{2}d\tilde{x}d\tilde{y}\end{aligned} \tag{23}$$

Proof. Our proof will be similar to the one for the space dimension one given in [1]. A straightforward calculation gives

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = \int\limits_{[-L,L]^2} u_t(u-\phi) = \int\limits_{[-L,L]^2}(-\Delta^2 u - \Delta u - uu_x - uu_y + g)(u-\phi).$$

After integration by parts and applying periodic boundary conditions this becomes

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = \int\limits_{[-L,L]^2}(\nabla u)^2 - (\triangle u)^2 - \phi_x u_x + \phi_{xx}\triangle u - \frac{1}{2}\phi_x u^2 + gu - g\phi.$$

Applying the Cauchy-Schwartz inequality in the form $\langle f,g\rangle \leq p/2\langle f,f\rangle + 1/2p\langle g,g\rangle$ and making substitution $\phi = 8\tilde{\phi}, \tilde{x} = 2x, \tilde{y} = 2y$, we get (23). □

Note that (20) and (23) show that if we can construct $\phi \in \bar{H}^2_{per}[-L,L]^2$ such that the coercivity estimate

$$\langle u, Ku\rangle = \int\limits_{[-L,L]^2} \left((\triangle u)^2 - (\nabla u)^2 + (\phi_x - \frac{1}{4})u^2\right) dxdy \geq \lambda_0 \|u\|_2^2 > 0 \tag{24}$$

holds for some λ_0 independent of L, then we get an estimate of the form

$$\limsup_{t\to\infty} \|u\|_2 \leqslant R^{**} = \sqrt{c_1\|\phi\|_2^2 + c_2\|\phi_x\|_2^2 + c_3\|\phi_{xx}\|_2^2 + c_4\|g\|_2^2} + \|\phi\|_2 \tag{25}$$

$$\leqslant C(\|\phi\|_{\bar{H}^2_{per}}, \|g\|_2) < \infty \tag{26}$$

In order to prove (24), we will use the same potential function $\phi(x)$ as constructed in [1]. We will also use some results from [1] such as

$$\int_{-L}^{L} u_{xx}^2 - u_x^2 + (\phi_x - \frac{1}{2})u^2 dx \geqslant \frac{1}{4}\int_{-L}^{L} u_{xx}^2 + u^2 dx \tag{27}$$

for all $u \in C^3[-L,L]$ with $u(0) = 0$. In fact (27) is not the exact inequality that is proved in [1]. However one can reconstruct the potential $\phi(x)$ so that (27) holds.

Lemma 3.3. *For $u(t;x,y)$ solving (1) we have the inequality*

$$\int\limits_{[-L,L]^2} u_{yy}^2 + 2u_{xx}u_{yy} - u_y^2 + \frac{1}{4}u^2 \geqslant 0 \tag{28}$$

Proof.

$$\int\limits_{[-L,L]^2} u_{yy}^2 + 2u_{xx}u_{yy} - u_y^2 + \frac{1}{4}u^2 = \int\limits_{[-L,L]^2} u_{yy}^2 + \frac{1}{4}u^2 - u_y^2 + \int\limits_{[-L,L]^2} 2u_{xx}u_{yy} \tag{29}$$

Applying Plancherel's Theorem, integration by parts with the periodic boundary conditions we have

$$\int\limits_{[-L,L]^2} u_{yy}^2 + 2u_{xx}u_{yy} - u_y^2 + \frac{1}{4}u^2 = \left(\frac{4\pi^2n^2}{L^2} - \frac{1}{2}\right)^2 \|u\|_2^2 + 2\|u_{xy}\|_2^2 \geqslant 0 \tag{30}$$

□

Proof of (24):

$$\int\limits_{[-L,L]^2} \left((\triangle u)^2 - (\nabla u)^2 + (\phi_x - \frac{1}{4})u^2\right) =$$

$$= \int\limits_{[-L,L]^2} \left(u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2 - u_x^2 - u_y^2 + (\phi_x - \frac{1}{4})u^2\right) =$$

$$= \int\limits_{[-L,L]^2} u_{xx}^2 - u_x^2 + (\phi_x - \frac{1}{2})u^2 + \int\limits_{[-L,L]^2} u_{yy}^2 + 2u_{xx}u_{yy} - u_y^2 + \frac{1}{4}u^2 \geqslant$$

$$\geqslant \frac{1}{4}\int_{-L}^{L}\int_{-L}^{L}(u_{xx}^2 + u^2)dxdy \geqslant \frac{1}{4}\|u\|_2^2$$

Lemma 3.4. *The potential ϕ satisfies $\|\phi\|_{\bar{H}^2_{per}([-L,L]^2)} \leqslant CL^2$.*

Proof. From [1], we know that $\|\phi\|_{\bar{H}^2_{per}([-L,L])} \leqslant CL^{3/2}$, thus $\int_{-L}^{L} \|\phi\|^2_{H^2([-L,L])}dy \leqslant CL^4$ □

Remark 2. Extension to arbitrary initial data The results claimed above are for odd initial data with the assumption of the external force to be odd. Since the theorem proved in [1] requires the assumption of u(0)=0, it is clear that it holds for any odd initial data. These results can be extended to arbitrary mean-zero initial data in the manner done by ([3]) or ([4]). So we can conclude that one can construct a potential function ϕ satisfying $\|\phi\|_{\bar{H}^2_{per}} \leqslant CL^2$.

Proof of (17):
Fix the initial data u_0 with $\|u_0\|_2 \leq B$, and define $u(t,\cdot) = S(t)u_0$ we can conclude (17) because from (16) we have the following result.

$$\|S(t)u_0\|_2 \leqslant \|\phi\|_2 + \sqrt{e^{-\frac{\lambda_0}{2}t}\|u(0)\|_2^2 + \frac{2P^2}{\lambda_0}} \tag{31}$$

It follows that

$$\limsup_{t\to\infty} \|u(t,\cdot)\|_2 \leqslant R^{**} \leqslant C_1\|\phi\|_{H^2([-L,L]^2)} + C_2\|g\|_2 \leq C(g,L) \tag{32}$$

which is the point dissipativeness of $S(t)$.

3.2. **Asymptotic compactness.** Since we are in a bounded domain, our aim will be just to prove the first two statements (18) and (19) of Proposition 1.

Proof of (18): The uniform boundedness follows from (16) as well. If $u(t_n.\cdot) = S(t_n)u_n$, where $u_n \in L^2([-L,L]^2)$ and $\|u_n\|_2 \leq B$ then from (16)

$$\|S(t_n)u_n\|_2 \leqslant \|\phi\|_2 + \sqrt{e^{-\frac{\lambda_0 t_n}{2}}\|u_n(0)\|_2^2 + \frac{2P^2}{\lambda_0}} \leqslant C(g,B,L)$$

Proof of (19): In Section 1.1, we introduced Littlewood-Paley Projection operators P_k and now we will define $u_k := P_k u$ and $g_k = P_k(g(x))$. Next apply P_k to (1) to get the $P_k u_t = P_k(-\triangle^2 u) - P_k(\triangle u) - P_k(uu_x + uu_y) + P_k(g(x))$. Rewrite this as

$$(u_k)_t = -\triangle^2 u_k - \triangle u_k - P_k(uu_x + uu_y) + g_k.$$

Multiplying each side by u_k, integrating over the domain $[-L,L]^2$ and applying integration by parts gives:

$$\partial_t \frac{1}{2}\|u_k(t,\cdot)\|_2^2 + \int (\triangle u_k)^2 - \int (\nabla u_k)^2 + \int P_k(uu_x + uu_y)u_k = \int g_k u_k.$$

Now from (6), we have

$$\int (\triangle u_k)^2 \geqslant C_1 2^{4k}\|u_k\|_2^2 \text{ and } \int (\nabla u_k)^2 \leqslant C_2 2^{2k}\|u_k\|_2^2 \tag{33}$$

We can also find a bound for $\|g_k\|_2\|u_k\|_2$.

$$\|g_k\|_2\|u_k\|_2 \leqslant \frac{C_3}{2}2^{4k}\|u_k\|_2^2 + \frac{1}{2C_3}2^{-4k}\|g_k\|_2^2 \tag{34}$$

$$\textbf{Claim}: |\int P_k(uu_x+uu_y)u_k dx| \leqslant C\|P_k(u^2)\|_2\|\nabla u_k\|_2 \tag{35}$$

Proof. Define $v = \frac{1}{2}[\partial_x(u^2) + \partial_y(u^2)]$. Thus we have

$$\int P_k(v)u_k dx = -\frac{1}{2}\int (u^2)_k[\partial_x u_k + \partial_y u_k]dx \leqslant C\left\|P_k(u^2)\right\|_2 \|\nabla u_k\|_2 .$$

□

From (7),(33) and (35), we can say that

$$|\int P_k(uu_x+uu_y)u_k| \leqslant C\|P_k(u^2)\|_2\|\nabla u_k\|_2 \lesssim 2^k\|u^2\|_{L_1}2^k\|u_k\|_2 \tag{36}$$

Finally using Cauchy- Schwartz inequality, we get

$$|\int P_k(uu_x+uu_y)u_k| \leqslant \varepsilon 2^{5k}\|u_k\|_2^2 + \frac{\|u\|_2^4}{\varepsilon 2^k} \tag{37}$$

Thus from (34),(35) and (36), we have $\partial_t \frac{1}{2}\|u_k(t,\cdot)\|_2^2 + C_1 2^{4k}\|u_k\|_2^2 \leqslant$

$$\leqslant C_2 2^{2k}\|u_k\|_2^2 + \varepsilon 2^{5k}\|u_k\|_2^2 + \frac{\|u\|_2^4}{\varepsilon 2^k} + \frac{C_3}{2}2^{4k}\|u_k\|_2^2 + \frac{1}{2C_3}2^{-4k}\|g_k\|_2^2 \tag{38}$$

Now defining $I_k(t)$ by $I_k(t) = \|u_k(t,\cdot)\|_2^2$, choosing $\varepsilon = \frac{C_1}{2}2^{-k}$ and rewriting (38), we have the following inequality:

$$\partial_t I_k(t) + C_4 2^{4k} I_k(t) \leqslant \frac{2}{C_1}\|u\|_2^4 + \frac{1}{2C_3}2^{-4k}\|g_k\|_2^2 \tag{39}$$

Since $\limsup_{t\to\infty}\|u\|_2 = C(g,L)$ from (25) and using Gronwall inequality, we get

$$I_k(t) \leqslant I_k(0)e^{C_4 2^{-4k}t} + \frac{2^{-4k}}{C_4}\left(\frac{2}{C_1}C(g,L)^4 + \frac{1}{2C_3}2^{-4k}\|g_k\|_2^2\right)$$

Since $g \in L^2([-L,L]^2)$, we have that $(\frac{2}{C_1}C(g,L)^4 + \frac{1}{2C_3}2^{-4k}\|g_k\|_2^2)$ is bounded. Thus $\|P_{>N}u_n\|_2^2 \approxeq \sum_{k:2^kL\geq N}\|u_k\|_2^2 = \sum_{k:2^kL\geq N} I_k$

$\leqslant \sum_{k:2^kL\geq N} I_k(0)e^{C_4 2^{-4k}t} + \frac{2^{-4k}}{C_4}\tilde{C}(g,L)$, which tends to 0 as $N \to \infty$.

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