DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **250–258**

A FAST ITERATIVE SCHEME FOR VARIATIONAL INCLUSIONS

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Abstract. We present an iterative scheme for solving inclusions of the form $f(x) + F(x) \ni 0$ where f is a Lipschitz continuous function admitting a first order divided difference while F stands for a set-valued mapping, both of them acting between Banach spaces. We prove the convergence of our method under several regularity properties for F and without any differentiability assumption on f. We investigate, subsequently, the case when the mapping F is metrically regular, strongly metrically regular and strongly metrically subregular.

1. Introduction.

The very inspiration of this study goes back to the works of Dontchev [6, 7] where a Newton-type method was developed to solve inclusions of the form

$$f(x) + F(x) \ni 0, \tag{1}$$

where $f : X \to Y$ is a continuous mapping acting between two Banach spaces X and Y while $F : X \rightrightarrows Y$ denotes a set-valued mapping with closed graph. Inclusions such as (1), known more specifically as *generalized equations*, were introduced by Robinson in the 1970's and serve as a general tool for describing and solving different problems in a unified manner. For instance, constraint systems (feasibility problems), optimality conditions and variational inequalities can be reformulated as generalized equations. More often, algorithms for solving (1), generate a sequence x_n of iterates by subsequently solving subproblems of the form $A(x_{n+1}, x_n) + F(x_{n+1}) \ni 0$ where A denotes some approximation of the mapping f. When f is a Fréchet differentiable mapping admitting a Lipschitz continuous derivative, Dontchev [6] associates to (1) the following iterative procedure

$$f(x_n) + \nabla f(x_n)(x_{n+1} - x_n) + F(x_{n+1}) \ni 0, \text{ for } n = 0, 1, 2, \dots \tag{2}$$

Under the Aubin continuity of the inverse of some set-valued approximation of $f+F$ he obtains the quadratic convergence of the method (2). Moreover, when $F = \{0\}$, the latter reduces to the classical Newton method for solving the nonlinear equation $f(x) = 0$.

In this paper we propose an iterative scheme for solving (1) which does not require the computation of the derivative of f. More precisely, we consider a second-order approximation of the mapping f, constructed on the basis of a Taylor expansion,

2000 *Mathematics Subject Classification.* Primary: 49J53, 49J40; Secondary: 90C48.
Key words and phrases. Set-valued mappings, Metric regularity, Divided differences.
Authors are supported by Contract EA3591.

where the first and second derivatives of f are replaced with a linear operator called the *divided difference* of f. Before going further, we recall the definition of the first order divided difference we need in the sequel (for more details on the divided difference notion in Banach spaces see, e.g., the book [1]).

Definition 1.1. Let X and Y be two Banach spaces. A linear operator acting from X into Y is called a *first order divided difference* of the operator $g : X \to Y$ on the points x_0, y_0, denoted by $[x_0, y_0; g]$, if both of the following properties hold:

(a) $[x_0, y_0; g]\,(y_0 - x_0) = g(y_0) - g(x_0)$, for $x_0 \neq y_0$;

(b) if g is Fréchet differentiable at $x_0 \in X$ then $[x_0, x_0; g] = g'(x_0)$.

Then, we associate to (1) the following subproblem

$$f(x_n) + [x_n, x_{n+1}, f](x_{n+1} - x_n) \\ +(\,[x_{n-1}, x_{n+1}, f] - [x_{n-1}, x_n, f]\,)(x_{n+1} - x_n) + F(x_{n+1}) \ni 0,$$

where $[x_n, x_{n+1}, f](x_{n+1} - x_n)$ plays the role of $\nabla f(x_n)(x_{n+1} - x_n)$ in the regular Taylor expansion of the mapping f at x_n while the second order derivative $\nabla^2 f(x_n)(x_{n+1}-x_n)$ is replaced with the term $(\,[x_{n-1}, x_{n+1}, f]-[x_{n-1}, x_n, f]\,)(x_{n+1}-x_n)$.

Moreover, thanks to Definition 1.1, we can rewrite the above subproblem in the following way

$$f(x_{n+1}) + (\,[x_{n-1}, x_{n+1}, f] - [x_{n-1}, x_n, f]\,)(x_{n+1} - x_n) + F(x_{n+1}) \ni 0. \tag{3}$$

Our purpose is to study the local behavior of the iterative procedure (3) when the mapping F enjoys some regularity properties. More precisely, we subsequently investigate the local convergence of the method (3) when F is metrically regular, strongly metrically regular and strongly metrically subregular around some solution to the inclusion (1). In the first two cases, starting with two initial guesses sufficiently close to a solution to the problem, we obtain the quadratic convergence of the method while in the last case we prove that any sequence satisfying (3) converges linearly to a solution to (1) and, in addition, any sequence generated by some slight modification of (3) converges superlinearly.

The rest of this paper consists of three sections. In Section 2 we collect some definitions regarding the concept of metric regularity while the main results are established in Sections 3 and 4.

2. **Metric regularity.** Throughout, X and Y stand for real Banach spaces. The closed unit ball is denoted by $I\!B$ while $I\!B_r(a)$ stands for the closed ball of radius r centered at a. We denote by $d(x, C)$ the distance from a point x to a set C, that is, $d(x, C) = \inf_{y \in C} \|x - y\|$.

Let F be a set-valued mapping from X into the subsets of Y, indicated by $F : X \rightrightarrows Y$. Here $\operatorname{gph} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ is the graph of F and the range of F is the set $\operatorname{rge} F = \{y \in Y \mid \exists\, x,\ F(x) \ni y\}$. The inverse of F, denoted by F^{-1}, is defined as $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$.

The concept of metric regularity of set-valued mappings goes back to the end of the 1970s but it sources come from some older classical theorems of differential calculus and linear analysis. The Banach open mapping theorem [4], the tangent space theorem of Lyusternik [18] and the surjection theorem of Graves [14] are among them. The definition of the metric regularity of a set-valued mapping reads as follows.

Definition 2.1. A mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ if $F(\bar{x}) \ni \bar{y}$ and there exist some positive constants κ, a and b such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \text{ for all } x \in \mathbb{B}_a(\bar{x}),\ y \in \mathbb{B}_b(\bar{y}). \tag{4}$$

The infimum of κ for which (4) holds is the *regularity modulus* denoted reg $F(\bar{x}\,|\,\bar{y})$; the case when F is not metrically regular at $\bar{x}$ for $\bar{y}$ corresponds to reg $F(\bar{x}\,|\,\bar{y}) = \infty$. Smaller values of κ correspond to more favorable behavior. The metric regularity of a mapping F at $\bar{x}$ for $\bar{y}$ is known to be equivalent to the Aubin continuity of the inverse F^{-1} at $\bar{y}$ for $\bar{x}$ (see, *e.g.,* [21]). Recall that a set-valued map $\Gamma : Y \rightrightarrows X$ is Aubin continuous at $(\bar{y}, \bar{x}) \in \operatorname{gph} \Gamma$ (see [2]) if there exist positive constants κ, a and b such that

$$e(\Gamma(y') \cap \mathbb{B}_a(\bar{x}), \Gamma(y)) \leq \kappa \|y' - y\| \text{ for all } y, y' \in \mathbb{B}_b(\bar{y}), \tag{5}$$

where $e(A, B)$ denotes the excess from a set A to a set B and is defined as $e(A, B) = \sup_{x \in A} d(x, B)$.

A central result in the theory of metric regularity is the Lyusternik-Graves Theorem of which the Banach open mapping theorem is an immediate consequence.

In the general form of this theorem we present next and which is from [10], we use the following convention: we say that a set is locally closed at one of its points if some neighborhood of that point has closed intersection with the set.

Theorem 2.2. (Extended Lyusternik-Graves) *Consider a mapping* $F : X \rightrightarrows Y$ *and any* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *at which* $\operatorname{gph} F$ *is locally closed. Consider also a function* $g : X \to Y$ *which is Lipschitz continuous near* $\bar{x}$ *with a Lipschitz constant* δ*. If* reg $F(\bar{x} \mid \bar{y}) < \kappa < \infty$ *and* $\delta < \kappa^{-1}$*, then*

$$\text{reg } (g + F)(\bar{x} \mid g(\bar{x}) + \bar{y}) \leq (\kappa^{-1} - \delta)^{-1}.$$

The second notion of regularity we will consider is the strong metric subregularity, it is defined as follows.

Definition 2.3. A mapping $F : X \rightrightarrows Y$ is strongly metrically subregular at $\bar{x}$ for $\bar{y}$ if $F(\bar{x}) \ni \bar{y}$ and there exists $\kappa \in [0, \infty)$ along with a neighborhood U of $\bar{x}$ such that

$$\|x - \bar{x}\| \;\leq\; \kappa d(\bar{y}, F(x)) \text{ for all } x \in U.$$

This property is equivalent to the "local Lipschitz property at a point" of the inverse mapping, a property first formally introduced in [5].

In order to introduce the next regularity property, we need the notion of graphical localization. A *graphical localization* of a mapping $F : X \rightrightarrows Y$ at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is a mapping $\tilde{F} : X \rightrightarrows Y$ such that $\operatorname{gph} \tilde{F} \;=\; (U \times V) \cap \operatorname{gph} F$ for some neighborhood $U \times V$ of $(\bar{x}, \bar{y})$.

Definition 2.4. A mapping $F : X \rightrightarrows Y$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$ if and only if F has the "single-valued Lipschitzian inverse property" there. This means F^{-1} has a graphical localization at $(\bar{y}, \bar{x})$ that is single-valued and Lipschitz continuous on a neighborhood of $\bar{y}$.

Strong regularity implies metric regularity by definition. It turns out, see [11], that these properties are equivalent for mappings of the form of the sum of a smooth function and the normal cone mapping over a polyhedral convex set. Moreover, for any set-valued mapping that is *locally monotone* near the reference point, metric regularity at that point implies, and hence is equivalent to, strong regularity. This is a consequence of a deeper result by Kenderov [16] regarding single-valuedness of lower semicontinuous monotone mappings. A version of this result for mapping having the Aubin property was proved in [9] and later generalized for "premonotone" mappings in [17]. Thus, for locally premonotone mappings, the metric regularity is equivalent to the strong regularity. One should also note that the strong monotonicity of a mapping implies strong regularity.

For more details on metric regularity and applications to variational problems one can refer to [3, 12, 13, 15, 20] and the monographs [19, 21].

3. **Local behavior of the method.** From now on we assume that the solution set to (1) is nonempty, *i.e.,* there is an element $\bar{x} \in X$ such that $f(\bar{x}) + F(\bar{x}) \ni 0$. Moreover, we make the following assumptions :
(i) The mapping f is Lipschitz continuous with a constant l_0.
(ii) The mapping f admits a divided difference satisfying the following property: there exists a constant $l > l_0/2$ such that

$$\|[x_1, y_1, f] - [x_2, y_2, f]\| \le l(\|x_1 - x_2\| + \|y_1 - y_2\|),$$

for any pairs $(x_i, y_i) \in X \times X$ such that $x_i \neq y_i, i = 1, 2$.

Next comes an extension to the set-valued setting of a local version of the Banach fixed-point theorem.

Theorem 3.1. [Dontchev-Hager [8]] *Let (X, d) be a complete metric space, and consider a set-valued mapping $\Phi : X \rightrightarrows X$, a point $\bar{x} \in X$, and nonnegative scalars α and θ such that $0 \le \theta < 1$, the sets $\Phi(x) \cap \mathbb{B}_\alpha(\bar{x})$ are closed for all $x \in \mathbb{B}_\alpha(\bar{x})$ and the following conditions hold:*

(i) *$d(\bar{x}, \Phi(\bar{x})) < \alpha(1 - \theta)$;*

(ii) *$e(\Phi(u) \bigcap \mathbb{B}_\alpha(\bar{x}), \Phi(v)) \le \theta d(u, v)$ for all $u, v \in \mathbb{B}_\alpha(\bar{x})$.*

Then Φ has a fixed point in $\mathbb{B}_\alpha(\bar{x})$. That is, there exists $x \in \mathbb{B}_\alpha(\bar{x})$ such that $x \in \Phi(x)$. If Φ is single-valued, then x is the unique fixed point of Φ in $\mathbb{B}_\alpha(\bar{x})$.

When the mapping F is metrically regular at $\bar{x}$ for $-f(\bar{x})$ we are able to prove the quadratic convergence of our iterative scheme (3) as it is shown below.

Theorem 3.2. *Assume that the mapping F is metrically regular at $\bar{x}$ for $-f(\bar{x})$ with a growth constant κ such that $\kappa l < 1/2$. Then one can find $\delta > 0$ such that for every distinct initial guesses $x_0, x_1 \in B_\delta(\bar{x})$, there exists a sequence x_n, defined by (3), which satisfies*

$$\| x_{n+1} - \bar{x} \| \le \|x_n - \bar{x}\|^2. \tag{6}$$

Proof. Define the set-valued mapping $G : X \rightrightarrows Y$ by $G(x) = f(\bar{x}) + F(x)$ and for any positive integer n, set

$$\varepsilon_n(x) = f(\bar{x}) - f(x) - (\,[x_{n-1}, x, f] - [x_{n-1}, x_n, f]\,)(x - x_n),$$

where the mapping ε_n measures the error made in approximating the function f at $\bar{x}$ at step n. Then, we introduce the mapping $\Phi_n : X \rightrightarrows Y$ defined by $\Phi_n(x) = G^{-1}(\varepsilon_n(x))$ for $n = 1, 2, \ldots$ The idea is to prove that for two given iterates

x_{n-1} and x_n the mapping Φ_n admits a fixed point x_{n+1} that is a point satisfying $x_{n+1} \in \Phi(x_{n+1})$; the latter being equivalent to (3).

Let a, b and κ be positive constants such that the mapping F is metrically regular at $\bar{x}$ for $-f(\bar{x})$ with constant κ and neighborhoods $I\!B_a(\bar{x})$ and $I\!B_b(f(\bar{x}))$. Taking a smaller if necessary one may assume that $a < 1/8$. From the extended Lyusternik-Graves theorem (see Theorem 2.2) the metric regularity of the mapping F at $\bar{x}$ for $-f(\bar{x})$ yields the metric regularity of the mapping G at $\bar{x}$ for 0, *i.e.,* we have

$$d(x, G^{-1}(y)) \leq \kappa d(y, G(x)) \text{ for all } x \in I\!B_a(\bar{x}),\ y \in I\!B_b(0). \tag{7}$$

Take two distinct points x_0 and x_1 in $I\!B_a(\bar{x})$, then

$$\|\varepsilon_1(\bar{x})\| = \|(\,[x_0, \bar{x}, f] - [x_0, x_1, f]\,)(\bar{x} - x_1)\|.$$

Hence,

$$\|\varepsilon_1(\bar{x})\| \leq l\|x_1 - \bar{x}\|^2, \tag{8}$$

which yields $\|\varepsilon_1(\bar{x})\| \leq a^2 l$ and adjusting a if necessary we get,

$$\|\varepsilon_1(\bar{x})\| \leq b. \tag{9}$$

Moreover, from the metric regularity of the mapping G at $\bar{x}$ for 0, together with relation (9), we get

$$d(\bar{x}, \Phi_1(\bar{x})) = d(\,\bar{x}, G^{-1}(\varepsilon_1(\bar{x}))\,) \leq \kappa d(\varepsilon_1(\bar{x}), G(\bar{x})) \leq \kappa\|\varepsilon_1(\bar{x})\|.$$

And thanks to relation (8) we infer

$$d(\bar{x}, \Phi_1(\bar{x})) \leq \kappa l\|x_1 - \bar{x}\|^2 \leq (1 - \kappa l)\|x_1 - \bar{x}\|^2 = \alpha_1(1 - \kappa l), \tag{10}$$

where $\alpha_1 := \|x_1 - \bar{x}\|^2$. Thus, $\alpha_1 < a$. Further, take u and v in $I\!B_{\alpha_1}(\bar{x}) \subset I\!B_a(\bar{x})$. We have,

$$\begin{aligned}
\|\varepsilon_1(v)\| &= \|f(\bar{x}) - f(v) - (\,[x_0, v, f] - [x_0, x_1, f]\,)(v - x_1)\| \\
&\leq \|f(\bar{x}) - f(v)\| + \|(\,[x_0, v, f] - [x_0, x_1, f]\,)(v - x_1)\| \\
&\leq l_0\|v - \bar{x}\| + l\|v - x_1\|^2 \leq a(l_0 + 4al).
\end{aligned}$$

Taking a smaller if necessary we get $\|\varepsilon_1(v)\| \leq b$ and by the metric regularity of G we have,

$$\begin{aligned}
e(\,\Phi_1(u) \cap I\!B_{r_1}(\bar{x}), \Phi_1(v)\,) &= \sup_{x \in G^{-1}(\varepsilon_1(u)) \cap \boldsymbol{B}_{r_1}(\bar{x})} d(x, G^{-1}(\varepsilon_1(v))) \\
&\leq \sup_{x \in G^{-1}(\varepsilon_1(u)) \cap \boldsymbol{B}_{r_1}(\bar{x})} \kappa d(\varepsilon_1(v), G(x)) \\
&\leq \kappa\|\varepsilon_1(u) - \varepsilon_1(v)\| \\
&\leq \kappa(\,\|f(v) - f(u)\| + \|([x_0, u, f] - [x_0, x_1, f])(u - v)\| \\
&\quad + \|([x_0, v, f] - [x_0, u, f])(v - x_1)\|\,) \\
&\leq \kappa(\,l_0 + l\|u - x_1\| + l\|v - x_1\|\,)\|u - v\| \\
&\leq \kappa(l_0 + 4al)\|u - v\|.
\end{aligned}$$

Since $l_0 < l/2$ and $a < 1/8$ we get

$$e(\,\Phi_1(u) \cap I\!B_{r_1}(\bar{x}), \Phi_1(v)\,) \leq \kappa l \|u - v\|. \tag{11}$$

Thanks to relations (10) and (11) we are able to apply Theorem 3.1 to the mapping Φ_1, hence, there exists a point $x_2 \in I\!B_{\alpha_1}(\bar{x})$ such that $x_2 \in \Phi_1(x_2)$, *i.e.*,

$$f(x_2) + (\,[x_0, x_2, f] - [x_0, x_1, f]\,)(x_2 - x_1) + F(x_2) \ni 0.$$

The induction step is already clear, starting with two points x_{n-1} and x_n in $I\!B_a(\bar{x})$ we prove that the mapping Φ_n admits a fixed point $x_{n+1} \in I\!B_{\alpha_n}(\bar{x})$ with $\alpha_n = \|x_n - \bar{x}\|^2$, that is, a point satisfying both of the following assertions:

$$f(x_{n+1}) + (\,[x_{n-1}, x_{n+1}, f] - [x_{n-1}, x_n, f]\,)(x_{n+1} - x_n) + F(x_{n+1}) \ni 0, \tag{12}$$

and

$$\|x_{n+1} - \bar{x}\| \leq \|x_n - \bar{x}\|^2. \tag{13}$$

□

4. **Generalized equations with isolated solutions.** In this section we examine the case where the inclusion (1) admits an isolated solution $\bar{x}$. To this end, we will consider subsequently two regularity properties for the mapping F namely, the strong regularity and the strong subregularity. The following statement asserts that, whenever the mapping F is strongly metrically regular at $\bar{x}$ for $-f(\bar{x})$, the sequence whose existence x_n is proven in Theorem 3.2 is unique.

Theorem 4.1. *Assume that the mapping F is strongly metrically regular at $\bar{x}$ for $-f(\bar{x})$ with a growth constant κ such that $\kappa l < 1/2$. Then the condition of Theorem 3.2 holds and, in addition, the sequence x_n is unique.*

Proof. Since F is strongly metrically regular at $\bar{x}$ for $-f(\bar{x})$ the mapping $G := f(\bar{x}) + F$ is strongly metrically regular at $\bar{x}$ for 0 (this is a consequence of [12, Theorem 4.3] where a result similar to the Lyusternik-Graves for strong metric regularity is stated). Thus, there are positive constants α and β such that the mapping

$$y \mapsto G^{-1}(y) \cap I\!B_\alpha(\bar{x}) \text{ is single-valued on } I\!B_\beta(0). \tag{14}$$

Moreover the strong metric regularity forces the metric regularity hence by repeating the proof of the Theorem 3.2, using $\alpha/3$ instead of a we get the existence of a sequence x_n whose elements are in $I\!B_{\alpha/3}(\bar{x})$ and satisfy (3) with the same properties as in Theorem 3.2. To complete the proof it remains to show that the sequence x_n is unique. Assume that, for given x_{n-1} and x_n, there exist two points x_{n+1} and z_{n+1} which are obtained by (3) and such that

$$x_{n+1} \in G^{-1}(\varepsilon_n(x_{n+1})) \text{ and } z_{n+1} \in G^{-1}(\varepsilon_n(z_{n+1})), \tag{15}$$

where

$$\varepsilon_n(x) = f(\bar{x}) - f(x) - (\,[x_{n-1}, x, f] - [x_{n-1}, x_n, f]\,)(x - x_n).$$

From the proof of Theorem 3.2 we know that, taking α smaller if necessary, for all $u \in I\!B_{\alpha/3}(\bar{x})$ and all integer n we have $\varepsilon_n(u) \in I\!B_b(0)$; then from the single-valuedness of the graphical localization of G^{-1} we obtain

$$z_{n+1} = G^{-1}(\varepsilon_n(z_{n+1})) \cap I\!B_\alpha(\bar{x}).$$

Morever $z_{n+1} \in I\!B_{2\alpha/3}(x_{n+1}) \subset I\!B_\alpha(\bar{x})$ then $z_{n+1} = G^{-1}(\varepsilon_n(z_{n+1})) \cap I\!B_{2\alpha/3}(x_{n+1})$ and we have

$$\begin{aligned}\|x_{n+1} - z_{n+1}\| &= d\big(x_{n+1}, G^{-1}(\varepsilon_n(z_{n+1})\big) \cap I\!B_{2\alpha/3}(x_{n+1}))\\ &= d\big(x_{n+1}, G^{-1}(\varepsilon_n(z_{n+1}))\big)\\ &\le \kappa d(G(x_{n+1}), \varepsilon_n(z_{n+1})).\end{aligned}$$

Since $\varepsilon_n(x_{n+1}) \in G(x_{n+1})$ it follows that

$$\|x_{n+1} - z_{n+1}\| \le \kappa \|\varepsilon_n(x_{n+1}) - \varepsilon_n(z_{n+1})\| \tag{16}$$

Set $\Delta_n := \|\varepsilon_n(x_{n+1}) - \varepsilon_n(z_{n+1})\|$, we have

$$\begin{aligned}\Delta_n &\le \|f(z_{n+1}) - f(x_{n+1})\| + \| - ([x_{n-1}, x_{n+1}, f] - [x_{n-1}, x_n, f])(x_{n+1} - x_n)\\ &\qquad + \ ([x_{n-1}, z_{n+1}, f] - [x_{n-1}, x_n, f])(z_{n+1} - x_{n+1} + x_{n+1} - x_n)\|\\ &\le \|f(z_{n+1}) - f(x_{n+1})\| + \|([x_{n-1}, z_{n+1}, f] - [x_{n-1}, x_n, f])(z_{n+1} - x_{n+1})\|\\ &\qquad + \|([x_{n-1}, z_{n+1}, f] - [x_{n-1}, x_{n+1}, f])(x_{n+1} - x_n)\|\\ &\le l_0\|z_{n+1} - x_{n+1}\| + l\|z_{n+1} - x_{n+1}\|(\|z_{n+1} - x_n\| + \|x_{n+1} - x_n\|)\\ &\le (l_0 + \frac{4\alpha l}{3})\|z_{n+1} - x_{n+1}\|.\end{aligned}$$

Finally, thanks to (16) we get

$$\|x_{n+1} - z_{n+1}\| \le \kappa(l_0 + \frac{4\alpha l}{3})\|x_{n+1} - z_{n+1}\|. \tag{17}$$

Since $\kappa l < 1/2$ and $l_0 < 2l$ we have $\kappa l_0 < 1$ then adjusting α if necessary we get $\kappa(l_0 + \frac{4\alpha l}{3}) < 1$ thus relation (17) is absurd and the proof is complete. □

Now, we investigate the case when the mapping T is strongly metrically subregular at $\bar{x}$ for $-f(\bar{x})$.

Theorem 4.2. *Assume that the mapping F is strongly metrically subregular at $\bar{x}$ for $-f(\bar{x})$ with a growth constant κ such that $\kappa(l_0 + 2l) < 1$. Then there exists a neighborhood Ω of $\bar{x}$ such that every sequence x_n generated by (3) and whose elements are in Ω converges linearly to $\bar{x}$.*

Proof. Let a and κ be such that the mapping F is strongly metrically subregular at $\bar{x}$ for $-f(\bar{x})$ with a constant κ and a neighborhood $I\!B_a(\bar{x})$. Then, by the definition of the strong subregularity, we have

$$\|x - \bar{x}\| \le \kappa\, d(-f(\bar{x}), F(x)) \text{ for all } x \in I\!B_a(\bar{x}). \tag{18}$$

Adjusting a if necessary one may assume that $a < 1/2$. Now, suppose that (3) generates a sequence x_n such that $x_n \in I\!B_a(\bar{x})$ for all n. Then

$$-f(x_{n+1}) - (\,[x_{n-1}, x_{n+1}, f] - [x_{n-1}, x_n, f]\,)(x_{n+1} - x_n) \in F(x_{n+1}), \tag{19}$$

and if we set $A_n = \|f(\bar{x}) - f(x_{n+1}) - (\,[x_{n-1}, x_{n+1}, f] - [x_{n-1}, x_n, f]\,)(x_{n+1} - x_n)\|$, we have

$$A_n \le l_0\|x_{n+1} - \bar{x}\| + l\|x_n - x_{n+1}\|^2 \le l_0\|x_{n+1} - \bar{x}\| + l\|x_n - x_{n+1}\|.$$

Using (18) and (19) we obtain

$$\begin{aligned}
\|x_{n+1}-\bar{x}\| &\le \kappa(l_0\|x_{n+1}-\bar{x}\| + l\|x_n - x_{n+1}\|)\\
&\le \kappa(l_0\|x_{n+1}-\bar{x}\| + l\|x_n - \bar{x} + \bar{x} - x_{n+1}\|)\\
&\le \kappa(l_0+l)\|x_{n+1}-\bar{x}\| + \kappa l\|x_n-\bar{x}\|\\
&\le \frac{\kappa l}{1-\kappa(l_0+l)}\|x_n-\bar{x}\|.
\end{aligned}$$

□

By slightly modifying the iteration (3) we are able to accelerate the convergence in Theorem 4.2. More precisely we replace (3) with

$$f(x_{n+1}) + \lambda_n(\,[x_{n-1},x_{n+1},f] - [x_{n-1},x_n,f]\,)(x_{n+1}-x_n) + F(x_{n+1}) \ni 0, \quad (20)$$

where λ_n is a sequence of positive reals having to go to zero. Then we obtain the following improvement of Theorem 4.2 which may be proved in much the same way as Theorem 4.2.

Theorem 4.3. *Assume that the mapping F is strongly metrically subregular at $\bar{x}$ for $-f(\bar{x})$ with a growth constant κ such that $\kappa l_0 < 1$. Then there exists a neighborhood Ω of $\bar{x}$ such that every sequence x_n generated by (20) and whose elements are in Ω converges superlinearly to $\bar{x}$.*

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Received July 2008; revised February 2009.

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