DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **322–332**

DIMENSION SPLITTING FOR TIME DEPENDENT OPERATORS

Eskil Hansen and Alexander Ostermann

Institut für Mathematik, Universität Innsbruck
Technikerstraße 13, A-6020 Innsbruck, Austria

Abstract. In this paper we are concerned with the convergence analysis of splitting methods for nonautonomous abstract evolution equations. We introduce a framework that allows us to analyze the popular Lie, Peaceman–Rachford and Strang splittings for time dependent operators. Our framework is in particular suited for analyzing dimension splittings. The influence of boundary conditions is discussed.

1. **Introduction.** Splitting methods form a class of highly effective time discretizations for evolution equations. In this paper, we are concerned with the analysis of splitting methods for linear nonautonomous problems of the form

$$u'(t) = L(t)u(t) = \big(A(t) + B(t)\big)u(t). \tag{1}$$

The central idea behind splitting is that solutions of the evolution equations related to $A(t)$ and $B(t)$, respectively, can be computed much faster than the solution of the whole problem (1). This can indeed be the case, if the splitting into $A(t)$ and $B(t)$ is performed appropriately. For a profound introduction to the subject, we recommend the textbook [4] and the survey article [6].

The literature on splitting methods is vast. As some of the central results have already been discussed in the introductions to our papers [2, 3], it is needless to repeat this summary on the recent literature here.

The purpose of this short note is to show that the techniques, introduced in [2] for analyzing splitting methods for linear time-invariant operators, can also be used for nonautonomous problems. The basic idea of this extension is the following. Let S_n denote the discrete evolution operator of the splitting method at time t_n. In order to estimate the local error $S_n u(t_n) - u(t_{n+1})$, we introduce an auxiliary method with evolution operator E_n and estimate the right hand side of

$$S_n u(t_n) - u(t_{n+1}) = S_n u(t_n) - E_n u(t_n) + E_n u(t_n) - u(t_{n+1})$$

with the triangle inequality. The key point is to build up the propagators S_n and E_n by the operators A and B, evaluated at the *same* time points. Then the difference $S_n u(t_n) - E_n u(t_n)$ can be analyzed as in the time-invariant case. On the other hand, the operator E_n has to be chosen such that the local error of the auxiliary method $E_n u(t_n) - u(t_{n+1})$ has the right order and can be estimated easily.

The paper is organized as follows: In Section 2, we describe the analytic framework and we give an example of a parabolic problem and its dimension splitting,

2000 *Mathematics Subject Classification.* Primary: 65M12; Secondary: 65L05, 65J10.

Key words and phrases. Nonautonomous linear problems, dimension splitting, Lie splitting, exponential splitting, Peaceman–Rachford splitting, stability, convergence, variable time steps.

The first author was financially supported by the Austrian Science Fund under grant M961-N13.

which fits into our framework. In Section 3, we prove first order convergence of the Lie splitting and in Section 4, we derive the corresponding result for the exponential Lie splitting. The Peaceman–Rachford splitting is analyzed in Section 5. Here, under an additional compatibility condition which is satisfied by smooth solutions in case of periodic boundary conditions, e.g., the method is shown to be second order convergent. Finally, we show in Section 6 how these results can be generalized to Banach spaces and to variable step sizes.

2. Analytic framework.

Throughout the paper, we consider the nonautonomous evolution equation

$$u'(t) = L(t)u(t) = \big(A(t) + B(t)\big)u(t), \qquad u(0) = u_0 \tag{2}$$

with linear unbounded operators on an arbitrary (complex) Hilbert space $(H, \|\cdot\|)$. The related operator norm will also be referred to as $\|\cdot\|$. The evolution system of this linear equation will be denoted by $U(t,\tau)$, i.e.,

$$u(t) = U(t,\tau)u(\tau) \qquad \text{for } t \geq \tau.$$

Furthermore, C denotes a generic constant that assumes different values at different occurrences.

As we will conduct an analysis including parabolic problems, the following definition is adequate.

Definition 2.1. A one-parameter family of operators $\{F(t)\ ;\ 0 \leq t \leq T\}$ is called *S-regular*, if it satisfies the following properties:

(a) For each t, the operator $F(t)$ is maximal dissipative on H.
(b) The domain $\mathcal{D} = \mathcal{D}\big(F(t)\big)$ does not depend on t.
(c) The map $t \mapsto F(t)$ is m times continuously differentiable with $\mathcal{D}\big(F^{(k)}(t)\big) \supseteq \mathcal{D}$, and there exists a constant C_{D} such that

$$\big\|F^{(k)}(t)\big(I - F(s)\big)^{-1}\big\| \leq C_{\mathrm{D}}, \qquad 0 \leq k \leq m, \quad 0 \leq s,t, \leq T.$$

In our analysis below, we require $m = 1$ or $m = 2$.

Obviously, S-regular operators fulfill the Lipschitz bound

$$\big\|\big(F(t) - F(s)\big)\big(I - F(s)\big)^{-1}\big\| \leq C_{\mathrm{L}}|t - s|$$

with a uniform constant C_{L} for $0 \leq t, s \leq T$.

We are now prepared to state the main hypothesis on the operators.

Assumption 1. *The families of operators $A(t)$, $B(t)$ and $L(t)$ are S-regular.*

Assumption 1 implies that the operators generate C_0 semigroups of contractions and the related resolvents are all nonexpansive on H. Moreover, we infer from (2) the inclusion

$$\mathcal{D}\big(A(t)\big) \cap \mathcal{D}\big(B(t)\big) \subseteq \mathcal{D}\big(L(t)\big).$$

We refer to [7, 10] for a general introduction to the linear semigroup theory and to evolution systems.

Our convergence analysis below requires that the solution $u(t)$ of (2) is sufficiently smooth in time. We will exploit this assumption here for later use. Differentiating (2) a second time gives

$$u''(t) = L'(t)u(t) + L(t)u'(t) \ \in\ H. \tag{3}$$

Since the operator $L'(t)\big(I - L(t)\big)^{-1}$ is bounded on H, we infer from this identity

$$u'(t) \in \mathcal{D}\big(L(t)\big), \qquad u(t) \in \mathcal{D}\big(L(t)^2\big). \tag{4}$$

Differentiating (3) once more gives

$$u'''(t) = L''(t)u(t) + 2L'(t)u'(t) + L(t)u''(t) \ \in \ H$$

which implies by (4) the relation

$$u''(t) \in \mathcal{D}\big(L(t)\big). \tag{5}$$

In general, however, we *cannot* deduce from these relations that the solution $u(t)$ belongs to $\mathcal{D}\big(L(t)^3\big)$.

The following example describes a standard situation where dimension splitting can be of advantage.

Example 2.2. Let $\Omega \subset \mathbf{R}^d$ be an open d-dimensional rectangle. We consider the parabolic initial-boundary value problem

$$\frac{\partial u}{\partial t}(x,t) = L(x,t)u(x,t), \qquad x \in \Omega, \quad 0 < t \le T \tag{6}$$

with initial value $u(x,0) = u_0(x)$, $x \in \Omega$, and homogeneous Dirichlet boundary conditions. Here $L(x,t)$ denotes the second order uniformly strongly elliptic differential operator given by

$$L(x,t)w = \sum_{i=1}^{d} \partial_i \big(a_i(x,t)\partial_i w\big) \tag{7}$$

with smooth coefficients $a_i(x,t)$. We consider it as an unbounded operator $L(t) = L(x,t)$ on the Hilbert space $L^2(\Omega)$ with dense domain $\mathcal{D}\big(L(t)\big) = H^2(\Omega) \cap H_0^1(\Omega)$. The dimension splitting $L(t) = A(t) + B(t)$ is defined by

$$A(t)w = \sum_{i=1}^{s} \partial_i \big(a_i(\cdot,t)\partial_i w\big), \quad B(t)w = \sum_{i=s+1}^{d} \partial_i \big(a_i(\cdot,t)\partial_i w\big).$$

It is well-known that the operators $L(t)$, $A(t)$ and $B(t)$ satisfy Assumption 1, see [2, Section 5] and [7, 10].

We remark that Assumption 1 is equally satisfied for periodic conditions. For Neumann boundary conditions, however, the domains $\mathcal{D}\big(L(t)\big)$ in general depend on time through the boundary conditions. Since this is in conflict with the assumed S-regularity, such problems are not covered by our framework.

We close this section with some notation. For the generator F of a C_0 semigroup on H and a real number $h \ge 0$, we define the bounded operators $\varphi_0(hF) = \mathrm{e}^{hF}$ and

$$\varphi_k(hF) = \int_0^1 \mathrm{e}^{(1-\tau)hF} \frac{\tau^{k-1}}{(k-1)!}\,\mathrm{d}\tau, \qquad \text{for } k \ge 1.$$

These operators satisfy the recurrence relation

$$\varphi_k(hF) = \tfrac{1}{k!}I + hF\varphi_{k+1}(hF), \qquad k \ge 0. \tag{8}$$

For a time dependent linear operator $F(t)$ and a fixed time $t = t_j$, we henceforth denote $F_j = F(t_j)$ for short.

3. **Lie splitting.** For a first order time discretization of (2) we consider the so-called *Lie splitting*. This is a one step method that, given a numerical approximation u_n to the exact solution $u(t)$ at time $t = t_n$, defines an approximation u_{n+1} at time $t_{n+1} = t_n + h$ by

$$\begin{aligned}\widetilde{u}_{n+1} &= u_n + hA_n\widetilde{u}_{n+1},\\ u_{n+1} &= \widetilde{u}_{n+1} + hB_n u_{n+1}.\end{aligned} \tag{9}$$

For its analysis, it is convenient to rewrite this scheme in a more compact form as

$$u_{n+1} = S_n u_n \tag{10a}$$

with

$$S_n = (I - hB_n)^{-1}(I - hA_n)^{-1}, \qquad n \geq 0. \tag{10b}$$

The composition of these split step operators defines the discrete evolution system, denoted here by

$$S_{n,j} = \begin{cases} \prod_{k=j}^{n-1} S_k = S_{n-1}\cdots S_j, & 0 \leq j < n,\\ I & j = n.\end{cases} \tag{11}$$

We note for later use that the stability $\|S_{n,j}\| \leq 1$ of the splitting trivially holds, as the resolvents in (10b) are both nonexpansive on H.

The convergence analysis further requires some relations between the operators at hand. In particular, we need some compatibility of the domains.

Assumption 2. *There exists a constant C such that for all $t \in [0,T]$*

$$\mathcal{D}\big(L(t)^2\big) \subseteq \mathcal{D}\big(A(t)B(t)\big) \cap \mathcal{D}\big(A(t)\big) \quad \textit{and} \quad \big\|A(t)B(t)\big(I - L(t)\big)^{-2}\big\| \leq C.$$

For a verification of Assumption 2 in the context of Example 2.2, see [2, Example 5.1]. We are now in the position to formulate our first convergence result.

Theorem 3.1. *Let Assumptions 1 and 2 hold, and let the solution of (2) be twice differentiable with $u'' \in L^\infty(0,T;H)$. Then the Lie splitting is first order convergent,*

$$\Big\|\big(S_{n,0} - U(t_n,0)\big)u_0\Big\| \;\leq\; Ch\Big(\max_{0\leq j\leq n}\|u(t_j)\| + \max_{0\leq j\leq n}\|u'(t_j)\| + \max_{0\leq \tau\leq t_n}\|u''(\tau)\|\Big)$$

for $0 \leq t_n \leq T$. The positive constant C can be chosen uniformly on $[0,T]$ and, in particular, independent of n and h.

Proof. Our estimate relies on the telescopic identity

$$\big(S_{n,0} - U(t_n,0)\big)u_0 = \sum_{j=0}^{n-1} S_{n,j+1}\big(S_j - U(t_{j+1},t_j)\big)u(t_j). \tag{12}$$

Due to the stability bound $\|S_{n,j+1}\| \leq 1$, it remains to bound the terms

$$\big(S_j - U(t_{j+1},t_j)\big)u(t_j), \qquad j \geq 0 \tag{13}$$

which are the local errors of the Lie splitting. For this purpose, let

$$E_j = (I - hL_j)^{-1}$$

denote the propagator of an Euler-type method. By Assumption 1 this is again a bounded operator. The main idea of the proof consists in using the identity

$$\big(S_j - U(t_{j+1},t_j)\big)u(t_j) = \big(S_j - E_j\big)u(t_j) + \big(E_j - U(t_{j+1},t_j)\big)u(t_j). \tag{14}$$

For estimating the second term on the right hand side, we consider one step of the auxiliary method

$$v_{j+1} = u(t_j) + hL_j v_{j+1} \tag{15}$$

with initial value on the exact solution. Inserting the exact solution into (15) gives

$$u(t_{j+1}) = u(t_j) + hL_j u(t_{j+1}) + \delta_{j+1} \tag{16}$$

with defects

$$\delta_{j+1} = h(L_{j+1} - L_j)u(t_{j+1}) - \int_{t_j}^{t_{j+1}} (\tau - t_j)\, u''(\tau)\mathrm{d}\tau.$$

Subtracting then (15) from (16) shows at once the bound

$$\begin{aligned}\big\|\big(E_j - U(t_{j+1}, t_j)\big)u(t_j)\big\| \ &\leq \\ &Ch^2\Big(\|u(t_{j+1})\| + \|L_{j+1}u(t_{j+1})\| \ + \max_{t_j \leq \tau \leq t_{j+1}} \|u''(\tau)\|\Big).\end{aligned}$$

In view of (14) it thus remains to bound

$$\big(S_j - E_j\big)u(t_j). \tag{17}$$

As all operators in S_j and E_j are evaluated at time t_j, we are locally in the situation of constant operators and thus can apply the techniques of our previous paper [2].

To simplify notation we introduce the abbreviations

$$\begin{gathered} a = hA_j, \quad b = hB_j, \quad \ell = hL_j, \\ \alpha = (I - a)^{-1}, \quad \beta = (I - b)^{-1}, \quad \lambda = E_j = (I - \ell)^{-1}, \end{gathered}$$

and observe the relations

$$\begin{gathered} a\alpha = \alpha a, \qquad \alpha = I + a\alpha, \\ \beta b\alpha = \beta\alpha b + \beta b a\alpha - \beta\alpha a b \qquad \text{on } \mathcal{D}(B_j). \end{gathered} \tag{18}$$

Thus, we get the following equalities on $\mathcal{D}(L_j^2)$

$$\begin{aligned} S_j - E_j &= \beta\alpha - \lambda \\ &= \beta\alpha - (\beta - b\beta)(\alpha - a\alpha)\lambda \\ &= \beta\alpha(I - \lambda) + (\beta\alpha a + \beta b\alpha - \beta b a\alpha)\lambda \\ &= -\beta\alpha\ell\lambda + \big(\beta\alpha a + (\beta\alpha b + \beta b a\alpha - \beta\alpha a b) - \beta b a\alpha\big)\lambda \\ &= -\beta\alpha a b\lambda \\ &= -h^2 S_j A_j B_j (I - L_j)^{-2} E_j (I - L_j)^2. \end{aligned}$$

By inserting this into (17) the latter can be bounded in the desired way under the made assumptions. ☐

4. **Exponential Lie splitting.** Another first order time discretization of (2) is achieved by the *exponential Lie splitting*, where the numerical approximation u_{n+1} at time t_{n+1} is defined by

$$u_{n+1} = S_n u_n \quad \text{with} \quad S_n = \mathrm{e}^{hB_n}\mathrm{e}^{hA_n}. \tag{19}$$

The composition $S_{n,j}$ of these split step operators is again defined by (11). The stability $\|S_{n,j}\| \leq 1$ of the splitting holds trivially, as the resolvents are both non-expansive on H. For this splitting, we need the following compatibility of domains.

Assumption 3. *There exists a constant C such that for all $t \in [0,T]$*

$$\mathcal{D}\big(L(t)^2\big) \subseteq \mathcal{D}\big(A(t)^2\big) \quad \textit{and} \quad \big\|A(t)^2\big(I - L(t)\big)^{-2}\big\| \leq C$$

as well as

$$\mathcal{D}\big(L(t)^2\big) \subseteq \mathcal{D}\big([A(t), B(t)]\big) \quad \textit{and} \quad \big\|[A(t), B(t)]\big(I - L(t)\big)^{-2}\big\| \leq C.$$

Here, $[A(t), B(t)] = A(t)B(t) - B(t)A(t)$ denotes the commutator. For the exponential Lie splitting, the following convergence result holds.

Theorem 4.1. *Let Assumptions 1 and 3 hold, and let the solution of (2) be twice differentiable with $u'' \in L^\infty(0,T;H)$. Then the exponential Lie splitting (19) is first order convergent.*

Proof. The proof can be carried out along the lines of the proof of Theorem 3.1. As auxiliary scheme, however, we use an exponential integrator and choose $E_j = \mathrm{e}^{hL_j}$.

For estimating the local error of the frozen semiflow

$$\big(E_j - U(t_{j+1}, t_j)\big)u(t_j),$$

we start from the identity

$$\mathrm{e}^{hL_j}u(t_j) = u(t_j) + hL_j u(t_j) + h^2\varphi_2(hL_j)L_j^2\, u(t_j) \tag{20}$$

which is a consequence of (8). On the other hand, Taylor series expansion of the exact solution gives

$$u(t_{j+1}) = u(t_j) + hL_j u(t_j) + \delta_{j+1} \tag{21}$$

with defects

$$\delta_{j+1} = \int_{t_j}^{t_{j+1}} (t_{j+1} - \tau)\, u''(\tau)\mathrm{d}\tau.$$

Subtracting (20) from (21) shows at once the desired bound.

In order to bound the difference (17), we start with

$$\begin{aligned}\mathrm{e}^{hL_j}u(t_j) &= \mathrm{e}^{hB_j}u(t_j) + \int_0^h \mathrm{e}^{sB_j}A_j\mathrm{e}^{(h-s)L_j}u(t_j)\mathrm{d}s \\ &= \mathrm{e}^{hB_j}u(t_j) + h\mathrm{e}^{hB_j}A_j u(t_j) \\ &\qquad - \int_0^h s\,\mathrm{e}^{sB_j}(B_jA_j - A_jL_j)\mathrm{e}^{(h-s)L_j}u(t_j)\,\mathrm{d}s,\end{aligned} \tag{22}$$

and use once more

$$\mathrm{e}^{hA_j} = I + hA_j + h^2\varphi_2(hA_j)A_j^2,$$

to obtain

$$\mathrm{e}^{hB_j}u(t_j) = \mathrm{e}^{hB_j}\mathrm{e}^{hA_j}u(t_j) - h\mathrm{e}^{hB_j}A_ju(t_j) - h^2\mathrm{e}^{hB_j}\varphi_2(hA_j)A_j^2u(t_j).$$

Inserting this back into (22), the difference (17) can now be bounded as desired. □

5. **Peaceman–Rachford splitting.** The convergence result for the Lie splitting can be generalized to higher order methods. As an example, we consider here the popular *Peaceman–Rachford splitting.* A single time step of this method is given as

$$u_{n+1} = S_{n+\frac12}u_n \tag{23a}$$

with

$$S_{n+\frac12} = \Big(I - \tfrac{h}{2}B_{n+\frac12}\Big)^{-1}\Big(I + \tfrac{h}{2}A_{n+\frac12}\Big)\Big(I - \tfrac{h}{2}A_{n+\frac12}\Big)^{-1}\Big(I + \tfrac{h}{2}B_{n+\frac12}\Big). \tag{23b}$$

The operators (23b) define the discrete evolution system

$$S_{n,j} = \begin{cases} \prod_{k=j}^{n-1} S_{k+\frac12} = S_{n-\frac12}\cdots S_{j+\frac12}, & 0 \le j < n, \\ I & j = n. \end{cases}$$

For this system, we have the following stability bound.

Lemma 5.1. *Under Assumption 1, the stability bound*

$$\left\| S_{n,j} \left(I - \tfrac{h}{2} B_{j+\frac{1}{2}} \right)^{-1} \right\| \;\le\; C_{\mathrm{S}}, \qquad 0 \le j \le n,$$

holds with a constant C_{S} that can be chosen uniformly on bounded time intervals $0 \le nh \le T$ and, in particular, independent of j, n and h.

Proof. It is well known that operators of the form

$$\left(I + \tfrac{h}{2} A_{j+\frac{1}{2}} \right) \left(I - \tfrac{h}{2} A_{j+\frac{1}{2}} \right)^{-1}$$

are nonexpansive under Assumption 1. In order to bound the transition operator

$$\begin{aligned}\left(I + \tfrac{h}{2} B_{j+\frac{1}{2}} \right) \left(I - \tfrac{h}{2} B_{j-\frac{1}{2}} \right)^{-1} \;=\; &\left(I + \tfrac{h}{2} B_{j-\frac{1}{2}} \right) \left(I - \tfrac{h}{2} B_{j-\frac{1}{2}} \right)^{-1} \\ &+ \tfrac{h}{2} \left(B_{j+\frac{1}{2}} - B_{j-\frac{1}{2}} \right) \left(I - \tfrac{h}{2} B_{j-\frac{1}{2}} \right)^{-1}\end{aligned}$$

that arises between two consecutive steps, we employ once more Assumption 1. Altogether, we get the estimate

$$\left\| S_{n,j} \left(I - \tfrac{h}{2} B_{j+\frac{1}{2}} \right)^{-1} \right\| \;\le\; \prod_{k=j}^{n-1} \bigl(1 + C_{\mathrm{L}} h \bigr) \;\le\; \mathrm{e}^{C_{\mathrm{L}} T},$$

which is the desired stability bound. □

For a second order convergence result, we require the additional compatibility condition

$$\mathcal{D}\bigl(L(t)\bigr) \subseteq \mathcal{D}\bigl(B(t)\bigr) \quad \text{and} \quad \bigl\| B(t) \bigl(I - L(t) \bigr)^{-1} \bigr\| \le C \tag{24}$$

as well as

$$u(t) \in \mathcal{D}\bigl(L(t)^3\bigr). \tag{25}$$

The latter condition is satisfied for smooth solutions of the parabolic problem in Example 2.2, if the problem is equipped with *periodic* boundary conditions. We note, however, that temporal smoothness of the solution might not be sufficient to guarantee (25) in general.

For time-invariant operators, stability and convergence of the Peaceman–Rachford splitting have already been analyzed in [2, 9].

Theorem 5.2. *Let Assumptions 1 and 2, and conditions (24), (25) hold, and let the solution of (2) be three times differentiable with $u''' \in L^\infty(0, T; H)$. Then the Peaceman–Rachford splitting (23) is second order convergent,*

$$\left\| \bigl(S_{n,0} - U(t_n, 0) \bigr) u_0 \right\| \;\le\; C h^2, \qquad 0 \le t_n \le T.$$

The positive constant C can be chosen uniformly on $[0, T]$. It depends on the solution and its derivatives (as specified in the proof), but it is independent of n and h.

Proof. Our estimate relies again on the telescopic identity (12). In order to bound the local error

$$\bigl(S_{j+\frac{1}{2}} - U(t_{j+1}, t_j) \bigr) u(t_j), \tag{26}$$

of the Peaceman–Rachford splitting, we consider as auxiliary method this time the implicit midpoint rule with propagator

$$G_{j+\frac{1}{2}} = \left(I + \tfrac{h}{2} L_{j+\frac{1}{2}} \right) \left(I - \tfrac{h}{2} L_{j+\frac{1}{2}} \right)^{-1}.$$

(a) Starting from the exact solution at time t_j, the implicit midpoint rule is given by the two stage formula

$$\begin{aligned} u_{j+\frac12} &= u(t_j) + \frac{h}{2} L_{j+\frac12} u_{j+\frac12}, \\ u_{j+1} &= u(t_j) + h L_{j+\frac12} u_{j+\frac12}. \end{aligned} \tag{27}$$

In order to analyze its local error, we insert the exact solution into the numerical scheme to obtain

$$\begin{aligned} u(t_{j+\frac12}) &= u(t_j) + \frac{h}{2} L_{j+\frac12} u(t_{j+\frac12}) + \delta_{j+\frac12}, \\ u(t_{j+1}) &= u(t_j) + h L_{j+\frac12} u(t_{j+\frac12}) + \delta_{j+1} \end{aligned} \tag{28}$$

with defects

$$\begin{aligned} \delta_{j+\frac12} &= -\int_{t_j}^{t_j+\frac{h}{2}} (\tau - t_j)\, u''(\tau) \mathrm{d}\tau, \\ \delta_{j+1} &= \int_{t_j}^{t_{j+1}} \int_{t_j+\frac{h}{2}}^{\tau} (\tau-\sigma)\, u'''(\sigma)\, \mathrm{d}\sigma\, \mathrm{d}\tau \;=\; \int_{t_j}^{t_{j+1}} \int_{t_j+\frac{h}{2}}^{\tau} u''(\sigma)\, \mathrm{d}\sigma\, \mathrm{d}\tau. \end{aligned}$$

Subtracting (27) from (28) shows at once the bound

$$\begin{aligned} &\Big\| \Big(I + \frac{h}{2} L_{j+\frac12} \Big) \; \big(G_{j+\frac12} - U(t_{j+1}, t_j) \big)\, u(t_j) \Big\| \;\le \\ &\qquad Ch^3 \Big(\max_{t_j \le \tau \le t_{j+1}} \big\| L_{j+\frac12} u''(\tau) \big\| \; + \max_{t_j \le \tau \le t_{j+1}} \big\| u'''(\tau) \big\| \Big). \end{aligned}$$

(b) In view of (26) it thus remains to bound

$$\big(S_{j+\frac12} - G_{j+\frac12} \big) u(t_j).$$

An appropriate bound is again achieved with the techniques introduced in our paper [2]. There, we considered time-invariant operators only. For sake of simplicity we introduce the notation

$$\begin{aligned} a &= \tfrac{h}{2} A_{j+\frac12}, \quad b = \tfrac{h}{2} B_{j+\frac12}, \quad \ell = h L_{j+\frac12}, \\ \alpha &= (I-a)^{-1}, \quad \beta = (I-b)^{-1}, \qquad \lambda_0 = G_{j+\frac12} \end{aligned}$$

and observe the following relations

$$\begin{aligned} a\alpha &= \alpha a, \qquad \alpha = I + a\alpha, \\ \beta b \alpha &= \beta\alpha b + \beta b a \alpha - \beta \alpha a b \qquad \text{on } \mathcal{D}(B_{j+\frac12}). \end{aligned} \tag{29}$$

For the resolvent $\lambda_1 = \big(I - \frac{h}{2} L_{j+\frac12} \big)^{-1}$ one derives at once the following identities

$$\lambda_0 = I + \ell\lambda_1, \qquad \frac12 I - \lambda_1 + \frac12 \lambda_0 = 0.$$

Thus, we get the following equalities on $\mathcal{D}(L^3_{j+\frac12})$

$$\begin{aligned} S_{j+\frac12} - G_{j+\frac12} &= \beta(I+a)\alpha(I+b) - \lambda_0 \\ &= \beta\alpha(I+a)(I+b) - (\beta - b\beta)(\alpha - a\alpha)\lambda_0 \\ &= \beta\alpha(I - \lambda_0 + \tfrac12 \ell) + \beta\alpha a b + (\beta\alpha a + \beta b \alpha - \beta b a \alpha)\lambda_0 \\ &= \beta\alpha\ell\big(\tfrac12 I - \lambda_1\big) + \beta\alpha a b + \big(\beta\alpha a + (\beta\alpha b + \beta b a\alpha - \beta\alpha a b) - \beta b a \alpha\big)\lambda_0 \\ &= \beta\alpha\ell\big(\tfrac12 I - \lambda_1 + \tfrac12 \lambda_0\big) + \beta\alpha a b (I - \lambda_0) \\ &= -\beta\alpha a b \ell \lambda_1. \end{aligned}$$

The leading operator β in this expression is combined with $S_{n,j}$. Recall that such a structure is required for our stability bound in Lemma 5.1.

(c) It remains to bound the term $\alpha ab\ell\lambda_1 u(t_j)$. We encounter here a small technical problem as the arising operators and the solution are evaluated at different times. Therefore, we expand

$$\begin{aligned} L(t_{j+\frac{1}{2}}) &= L(t_j) + hR_1 \\ &= L(t_j) + \frac{h}{2}L'(t_j) + h^2 R_2 \end{aligned} \tag{30}$$

with remainders

$$R_1 = \frac{1}{h}\int_{t_j}^{t_j+\frac{h}{2}} L'(\tau)\mathrm{d}\tau, \qquad R_2 = \int_{t_j}^{t_j+\frac{h}{2}} \frac{t_j+\frac{h}{2}-\tau}{h^2} L''(\tau)\mathrm{d}\tau.$$

Due to Assumption 1 these remainders satisfy the bound

$$\left\| R_k (I - L_j)^{-1} \right\| \le C, \qquad k = 1, 2. \tag{31}$$

Using expansion (30), we obtain

$$\begin{aligned} \alpha ab\ell\lambda_1 u(t_j) &= \alpha ab\ell\lambda_1 \big(I - L_{j+\frac{1}{2}}\big)^{-1}\big(I - L_j - \tfrac{h}{2}L'_j - h^2 R_2\big)u(t_j) \\ &= h\alpha ab\lambda_1 L_{j+\frac{1}{2}}\big(I - L_{j+\frac{1}{2}}\big)^{-1}\big(I - L_j\big)u(t_j) && \text{(32a)} \\ &\quad - \frac{h^2}{2}\alpha ab\lambda_1 L_{j+\frac{1}{2}}\big(I - L_{j+\frac{1}{2}}\big)^{-1} L'_j u(t_j) && \text{(32b)} \\ &\quad - h^3 \alpha ab\lambda_1 L_{j+\frac{1}{2}}\big(I - L_{j+\frac{1}{2}}\big)^{-1} R_2 u(t_j). && \text{(32c)} \end{aligned}$$

The term (32c) has already the desired form. In order to bound (32b), we write

$$L'_j u(t_j) = \big(I - L_{j+\frac{1}{2}}\big)^{-1}\big(I - L_{j+\frac{1}{2}}\big)\big(I - L_j\big)^{-1}\big(I - L_j\big)L'_j u(t_j)$$

and employ Assumption 1 and (25). Finally, in order to bound (32a), we rewrite

$$\begin{aligned} (I - L_j)u(t_j) &= \big(I - L_{j+\frac{1}{2}}\big)^{-1}(I - L_j)^2 u(t_j) && \text{(33a)} \\ &\quad - h\big(I - L_{j+\frac{1}{2}}\big)^{-1} R_1 (I - L_j)u(t_j). && \text{(33b)} \end{aligned}$$

The term (32a) with (33b) is bounded at once by using (31), whereas the expression (33a) is rewritten as

$$\big(I - L_{j+\frac{1}{2}}\big)^{-2}\big(I - L_{j+\frac{1}{2}}\big)(I - L_j)^{-1}(I - L_j)^3 u(t_j)$$

and inserted back into (32a). The arising term is then easily bounded with the help of (25). This concludes our proof. □

6. **Generalizations.** In this final section we sketch how the above results can be extended to more general situations. In particular, we discuss the Strang splitting and we indicate extensions to Banach spaces and to variable step sizes.

6.1. **Strang splitting.** The exponential splitting

$$S_{n+\frac{1}{2}} = \mathrm{e}^{\frac{1}{2}h_n A(t_{n+1/2})}\mathrm{e}^{h_n B(t_{n+1/2})}\mathrm{e}^{\frac{1}{2}h_n A(t_{n+1/2})},$$

better known as Strang splitting, is very popular in applications. The stability of the discrete evolution system

$$S_{n,j} = \begin{cases} \prod_{k=j}^{n-1} S_{k+\frac{1}{2}} = S_{n-\frac{1}{2}} \cdots S_{j+\frac{1}{2}}, & 0 \le j < n, \\ I & j = n \end{cases}$$

is again straightforward under Assumption 1.

Second order convergence (for problems with periodic boundary conditions) can be shown along the lines of the proof of Theorem 5.2 or, alternatively, as in the proof of Theorem 4.1. For the corresponding time-invariant computations, see also [5]. In order to bound the local error

$$\big(S_{j+\frac{1}{2}} - U(t_{j+1}, t_j)\big)u(t_j), \tag{34}$$

of the Strang splitting, we this time consider a Magnus integrator [1] with propagator

$$M_{j+\frac{1}{2}} = \mathrm{e}^{h_n L(t_{n+1/2})}$$

as the auxiliary method. The local error of this integrator is analyzed in the same way as the local error of the exponential integrator, see the proof of Theorem 4.1.

6.2. **Banach spaces.** Our techniques can also be used in the case when H is only a Banach space. However, one has to be careful with the stability bounds. For the Lie splitting, a sufficient hypothesis is that the resolvents $(I - hA_j)^{-1}$ and $(I - hB_j)^{-1}$ are nonexpansive or at least bounded by $1 + Ch$. Such a hypothesis, however, is in general not sufficient for the Peaceman–Rachford splitting, see [8, Appendix to Section 2] for an example.

For the exponential splittings, the following growth condition on the semigroups

$$\big\|\mathrm{e}^{sA(t)}\big\| \le \mathrm{e}^{\omega s}, \quad \big\|\mathrm{e}^{sB(t)}\big\| \le \mathrm{e}^{\omega s}, \qquad t \in [0, T], \quad s \ge 0$$

is sufficient for the stability of the splitting methods. Here, ω denotes an arbitrary but fixed real number.

6.3. **Variable step sizes.** For notational simplicity, we have restricted our analysis so far to constant step sizes. Note, however, that the step size h only plays the role of a parameter in the consistency proofs. Moreover, the stability lemma for the Peaceman–Rachford splitting (Lemma 5.1) holds for variable step sizes, as well. Therefore, all convergence results of this paper extend at once to variable step sizes.

REFERENCES

[1] C. González, A. Ostermann, M. Thalhammer, A second-order Magnus-type integrator for nonautonomous parabolic problems. J. Comput. Appl. Math., **189** (2006), 142–156.

[2] E. Hansen, A. Ostermann, Dimension splitting for evolution equations. Numer. Math., **108** (2008), 557–570.

3. E. Hansen, A. Ostermann, Exponential splitting for unbounded operators. Math. Comp., **78** (2009), 1485–1496.

[4] W. Hundsdorfer, J. Verwer, Numerical solution of time-dependent advection-diffusion-reaction equations. Springer-Verlag, Berlin, 2003.

[5] T. Jahnke, C. Lubich, Error bounds for exponential operator splittings. BIT, **40** (2000), 735–744.

[6] R.I. McLachlan, G.R.W. Quispel, Splitting methods. Acta Numer., **11** (2002), 341–434.

[7] A. Pazy, Semigroups of linear operators and applications to partial differential equations. Springer-Verlag, New York, 1983.

8. S. Rasmussen, Non-linear semi-groups, evolution equations and productintegral representations. Various publications series **20**, Aarhus University, 1972.

[9] M. Schatzman, Stability of the Peaceman-Rachford approximation. J. Funct. Anal., **162** (1999), 219–255.

[10] H. Tanabe, Equations of Evolution, Pitman, London, 1979.

Received August 2008; revised April 2009.

E-mail address: eskil@maths.lth.se
E-mail address: alexander.ostermann@uibk.ac.at