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CLASSIFICATION OF NONOSCILLATORY SOLUTIONS OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. Nonoscillatory solutions of a general class of second order functional neutral differential equations of the form

$$\left(r\left(t\right)\left(x\left(t\right)+p\left(t\right)x\left(t-\tau\right)\right)'\right)'+f\left(t,x\left(\sigma_1\left(t\right)\right),x\left(\sigma_2\left(t\right)\right),\ldots,x\left(\sigma_n\left(t\right)\right)\right)=0$$

have been classified in accordance with their asymptotic behavior.

1. **Introduction.** This paper is a continuation of our recent research in [3, 4] regarding asymptotic behavior of nonoscillatory solutions of certain classes of nonlinear neutral differential equations. We are concerned with a second order nonlinear neutral differential equation

$$\left(r\left(t\right)\left(x\left(t\right)+p\left(t\right)x\left(t-\tau\right)\right)'\right)'+f\left(t,x\left(\sigma_1\left(t\right)\right),x\left(\sigma_2\left(t\right)\right),\ldots,x\left(\sigma_n\left(t\right)\right)\right)=0, \quad (1)$$

where $t\in\mathbb{I}\overset{\text{def}}{=}[t_0,+\infty)$, $t_0\in\mathbb{R}$, $\tau>0$, $r\in C^1\left(\mathbb{I},(0,+\infty)\right)$, $p,\sigma_i\in C\left(\mathbb{I},\mathbb{R}\right)$ for all $i\in\Delta\overset{\text{def}}{=}\{1,2,\ldots,n\}$, and $f\in C\left(\mathbb{I}\times\mathbb{R}^n,\mathbb{R}\right)$. By a solution of (1) we mean a continuous function $x\left(t\right)$, defined on some interval $[t_x,T_x)$, such that $r\left(t\right)\left(x\left(t\right)+p\left(t\right)x\left(t-\tau\right)\right)'$ is continuously differentiable and $x\left(t\right)$ satisfies Eq. (1) for all $t\in[t_x,T_x)$. To concentrate attention only on the asymptotic behavior of solutions, we tacitly assume that solutions of Eq. (1) always exist and can be indefinitely continued to the right beyond t_0.

In addition to numerous papers where oscillatory behavior of solutions for various classes of neutral differential equations has been considered, many authors were interested in existence of nonoscillatory solutions, in particular positive solutions, as well as in the asymptotic behavior of nonoscillatory solutions and their classification, see, for instance, [1] - [15] and the references cited therein. For a second order neutral differential equation

$$\left(x\left(t\right)+px\left(t-\tau\right)\right)''+f\left(t,x\left(t\right),x'\left(t\right)\right)=0,$$

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Džurina [1] established conditions under which all nonoscillatory solutions behave like linear functions $at+b$ as $t\to+\infty$ for some $a,b\in\mathbb{R}$. Further results in this direction were obtained by the authors for a second order equation

$$(x(t)+p(t)x(t-\tau))''+f(t,x(t),x(\rho(t)),x'(t),x'(\sigma(t)))=0$$

in [3], and for an n-th order equation

$$(x(t)+p(t)x(t-\tau))^{(n)}+f(t,x(t),x(\rho(t)),x'(t),x'(\sigma(t)))=0$$

in [4]. Related theorems for higher order nonlinear neutral differential equations

$$\frac{d^n}{dt^n}\left[x(t)+\lambda x(t-\tau)\right]+\sigma F(t,x(g(t)))=0$$

and

$$\frac{d^n}{dt^n}\left[x(t)-h(t)x(\tau(t))\right]+f(t,x(g(t)))=0$$

can be found, respectively, in the papers by M. Naito [11] and Y. Naito [12].

Classification of nonoscillatory solutions for second order neutral differential equations

$$(x(t)-p(t)x(t-\tau))''+f(t,x(t-\delta))=0$$

and

$$\left(r(t)(x(t)-p(t)x(t-\tau))'\right)'+f(t,x(t-\delta))=0 \tag{2}$$

was suggested in the papers by Lu [10] and Li [7].

For higher order neutral differential equations, Kong et al. [5] proposed classification of nonoscillatory solutions of an odd order linear neutral differential equation

$$(x(t)-x(t-\tau))^{(n)}+p(t)x(t-\sigma)=0$$

and established conditions for the existence of each type of nonoscillatory solutions. Recently, Zhou [15] and Li and Fei [9] addressed existence of positive solutions of

$$\left(r(t)\left(x(t)-p(t)x(t-\tau)\right)^{(n-1)}\right)'+f(t,x(t-\tau))=0$$

and

$$\left(r(t)x^{(n-1)}(t)\right)'+f(t,x(t-\tau))=0.$$

Ouyang et al. [13] discussed existence and classification of positive solutions of

$$\left(r(t)\left(x(t)-\sum_{i=1}^{m}P_i(t)x(t-\tau_i)^{(n-1)}\right)\right)'+f(t,x(t-\sigma_1),\ldots,x(t-\sigma_l))=0,$$

whereas Li [8] explored asymptotic behavior of nonoscillatory solutions of

$$\left(x(t)-\sum_{i=1}^{m}P_i(t)x(t-\tau_i)\right)^{(n)}+\delta\sum_{j=1}^{m}Q_j(t)f_j(x(h_j(t)))=0.$$

Finally, we mention that sufficient conditions for existence of solutions with the "weak" properties A and B for a neutral differential equation

$$(x(t)+\mu(t)x(\rho(t)))^{(n)}+f(t,x(\tau_1(t)),x(\tau_2(t)),\ldots,x(\tau_m(t)))=0$$

have been suggested by Grammatikopoulos and Koplatadze [2].

In what follows, it is supposed that the following conditions hold:

(A_1) for all $t\geq t_0$ and $i\in\Delta$, $\sigma_i(t)\leq t$ and $\lim_{t\to+\infty}\sigma_i(t)=+\infty$;
(A_2) $0\leq p(t)\leq p_*<1$ and $\lim_{t\to+\infty}p(t)=p_0$;
(A_3) $x_1 f(t,x_1,x_2,\ldots,x_n)>0$ for $x_1x_i>0$, $i\in\Delta$.

Asymptotic behavior of nonoscillatory solutions to (1) differs depending on whether the improper integral $\int_{t_0}^{+\infty} 1/(r(u))\,du$ converges or not. Therefore, we consider separately two cases when

$$\int_{t_0}^{+\infty} \frac{1}{r(u)}du < +\infty \tag{3}$$

and

$$\int_{t_0}^{+\infty} \frac{1}{r(u)}du = +\infty. \tag{4}$$

Developing and refining the ideas used by Li [7] for Eq. (2), in this paper we only discuss classification of nonoscillatory solutions of (1). In contrast to the cited paper, we remove a restrictive monotonicity condition on the nonlinearity,

$$f(t,x_1) \geq f(t,x_2) \qquad \text{for} \quad x_1 \geq x_2 > 0 \quad \text{or} \quad x_1 \leq x_2 < 0, \quad t \geq t_0,$$

imposed by Li in addition to the standard sign condition (A_3). Furthermore, our proofs of the two classification results are based on four lemmas and thus are simpler compared to more intricate proofs in Li's paper [7] relying on seven auxiliary lemmas. Main theorems are illustrated with examples of neutral equations that possess exact solutions with certain asymptotic behavior. Conditions that guarantee existence of solutions which belong to the classes considered in the sequel will form the subject of one of the forthcoming publications.

2. **Preliminary lemma.** The following auxiliary result helps us to study the asymptotic behavior of nonoscillatory solutions of Eq. (1).

Lemma 2.1 (cf. [3, Lemma 1])**.** *Let $x(t) > 0$ (or $x(t) < 0$) eventually, $\tau > 0$, and let $p(t)$ satisfy (A_2). For $t \geq t_0$, define*

$$z(t) = x(t) + p(t)\,x(t-\tau). \tag{5}$$

If there exists a finite limit $\lim_{t\to+\infty} z(t) = c$, then

$$\lim_{t\to+\infty} x(t) = \frac{c}{1+p_0}. \tag{6}$$

Proof. Let $x(t)$ be eventually positive. Then (5) implies that $c \geq 0$ and, by (6), one has

$$\liminf_{t\to+\infty} x(t) \leq \frac{c}{1+p_0} \leq \limsup_{t\to+\infty} x(t).$$

Suppose that there exist $\alpha_1, \alpha_2 \geq 0$ and two sequences μ_n, ν_n diverging to $+\infty$ such that

$$\limsup_{t\to+\infty} x(t) = \lim_{t\to+\infty} x(\mu_n) = \frac{c+\alpha_1}{1+p_0},$$
$$\liminf_{t\to+\infty} x(t) = \lim_{t\to+\infty} x(\nu_n) = \frac{c-\alpha_2}{1+p_0}.$$

Case 1. Assume that $\alpha_1 > 0$ and $\alpha_1 \geq \alpha_2 \geq 0$. It follows from (5) that, for any $\varepsilon > 0$,

$$z(t) \geq x(t) + p(t)\frac{c-\alpha_2-\varepsilon}{1+p_0}. \tag{7}$$

Letting in (7) $t = \mu_n$ and passing to the limit as $n \to +\infty$, we obtain

$$c \geq \frac{c+\alpha_1}{1+p_0} + p_0\frac{c-\alpha_2-\varepsilon}{1+p_0},$$

or, equivalently,

$$\alpha_1 \le p_0 (\alpha_2 + \varepsilon) . \tag{8}$$

Let $\varepsilon = (2p_0)^{-1} (1 - p_0) \alpha_2$. Since $p_0 < 1$, (8) yields

$$\alpha_1 \le \frac{1}{2} \alpha_2 (p_0 + 1) < \alpha_2,$$

which contradicts our initial assumption that $\alpha_1 \ge \alpha_2$.

Case 2. Assume now that $\alpha_2 > 0$ and $\alpha_2 \ge \alpha_1 \ge 0$. Similarly to Case 1, (5) implies that, for any $\varepsilon > 0$,

$$z(t) \le x(t) + p(t) \frac{c + \alpha_1 + \varepsilon}{1 + p_0},$$

and one has

$$\alpha_2 \le p_0 (\alpha_1 + \varepsilon) . \tag{9}$$

Choosing $\varepsilon = (2p_0)^{-1} (1 - p_0) \alpha_1$ and proceeding as earlier, we obtain the desired contradiction. The proof is complete now. □

3. The case when (3) holds.

Lemma 3.1. *Let $x(t)$ be a nonoscillatory solution of (1) and $z(t)$ be defined by (5). Then $z(t)$ is eventually increasing or decreasing and there exists a finite limit $\lim_{t\to+\infty} z(t) = L$.*

Proof. Assume that both $x(t) > 0$ and $x(\sigma_i(t)) > 0$, for all $t \ge T_0 \ge t_0$. By (13) and (14), for all $t > s \ge T_0$, one has

$$z(t) < z(s) + r(s) z'(s) \int_s^t \frac{1}{r(u)} du. \tag{10}$$

(i) If there exists a $T_1 \ge T_0$ such that $z'(T_1) < 0$, then it follows form (10) that $z(t) < z(s)$ for $t > s > T_1$. Since $z(t)$ is positive and decreasing, there should exist a finite limit $\lim_{t\to+\infty} z(t) = L$.

(ii) On the other hand, if there exists no $s \ge T_1$ such that $z'(s) < 0$, then $z'(s) \ge 0$ for all $s \ge T_1$. By virtue of (3), it follows from (10) that $z(t)$ is bounded above. Therefore, there exists a finite limit $\lim_{t\to+\infty} z(t) = L$. The proof is complete now. □

Lemma 3.2. *Let $x(t)$ be a nonoscillatory solution of Eq. (1) and $z(t)$ be defined by (5). Then there exist two positive constants K_1, K_2 and a number $t_* \ge t_0$ such that, for all $t \ge t_*$, either*

$$K_1 \int_t^{+\infty} \frac{1}{r(u)} du \le z(t) \le K_2,$$

or

$$-K_2 \le z(t) \le -K_1 \int_t^{+\infty} \frac{1}{r(u)} du.$$

Proof. Assume again that $x(t) > 0$ and $x(\sigma_i(t)) > 0$ for $t \ge T_0 \ge t_0$, then (13) holds. We consider two cases.

(i) Let $z'(t) > 0$ eventually, then (10) holds, for all $t \ge s \ge t_0$. Lemma 3.1 asserts that $z(t)$ is bounded, that is, there exists a positive constant K_2 such that

$$z(t) \le K_2.$$

By virtue of (3),

$$\lim_{t\to+\infty}\int_t^{+\infty}\frac{1}{r(u)}du=0.$$

Since $z(t)$ is positive, one can choose $t_*\geq T_0$ large enough so that, for some positive constant K_1,

$$K_1\int_t^{+\infty}\frac{1}{r(u)}du\leq z(t).$$

(ii) Suppose now that $z(t)$ is eventually decreasing, then (14) yields, for any $t>s\geq t_0$

$$z(s)>z(t)-r(s)z'(s)\int_s^t\frac{1}{r(u)}du.$$

Since by Lemma 3.1 there exists a finite limit $\lim_{t\to+\infty}z(t)=L$, passing in the latter inequality to the limit as $t\to+\infty$, we conclude that

$$z(s)\geq L-r(s)z'(s)\int_s^{+\infty}\frac{1}{r(u)}du. \tag{11}$$

Taking into account the fact that $r(t)z'(t)$ is decreasing, one can choose a $T_1\geq T_0$ large enough to ensure that $r(T_1)z'(T_1)<0$. Then, for all $s\geq T_1$, we have $r(s)z'(s)\leq r(T_1)z'(T_1)=-K_1$, where K_1 is a positive constant. Therefore, it follows from (11) that

$$z(t)\geq K_1\int_t^{+\infty}\frac{1}{r(u)}du,$$

for all $t\geq T_1$. The proof is complete. □

Now we are in a position to state the first classification theorem.

Theorem 3.3. *Assume that conditions (3) and (A_1)-(A_3) hold. Then any nonoscillatory solution of Eq. (1) belongs to one of the following four classes:*

Γ_1 : $\lim_{t\to+\infty}x(t)=\frac{c}{1+p_0}\neq 0,\quad \lim_{t\to+\infty}z(t)=c\neq 0,\quad \lim_{t\to+\infty}r(t)z'(t)=\mu;$

Γ_2 : $\lim_{t\to+\infty}x(t)=\frac{c}{1+p_0}\neq 0,\quad \lim_{t\to+\infty}z(t)=c\neq 0,$
$\lim_{t\to+\infty}r(t)z'(t)=\pm\infty;$

Γ_3 : $\lim_{t\to+\infty}x(t)=\lim_{t\to+\infty}z(t)=0,\quad \lim_{t\to+\infty}r(t)z'(t)=\mu\neq 0;$

Γ_4 : $\lim_{t\to+\infty}x(t)=\lim_{t\to+\infty}z(t)=0,\quad \lim_{t\to+\infty}r(t)z'(t)=\pm\infty,$

where c and μ are some constants.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). As above, without loss of generality, we may assume that $x(t)$ is eventually positive for all $t\geq T_0\geq t_0$. Then (13) holds and, by Lemma 3.1, there exists a finite limit $\lim_{t\to+\infty}z(t)=c$. Obviously, we have either

$$\lim_{t\to+\infty}z(t)=c\neq 0,$$

or

$$\lim_{t\to+\infty}z(t)=0,$$

which, by Lemma 2.1, yields either

$$\lim_{t\to+\infty}x(t)=\frac{c}{1+p_0},$$

or

$$\lim_{t\to+\infty} x(t) = 0.$$

Taking into account that, by (A_3), $r(t)\,z'(t)$ is decreasing, there are two possibilities:

$$\lim_{t\to+\infty} r(t)\,z'(t) = \mu,$$

where $\mu \in \mathbb{R}$, or,

$$\lim_{t\to+\infty} r(t)\,z'(t) = -\infty.$$

Repeating similar reasoning for an eventually negative nonoscillatory solution $x(t)$ of Eq. (1), it is not difficult to see that all nonoscillatory solutions fall into one of the classes described in the theorem.

It remains only to prove that, for solutions that belong to Γ_3, one has $\mu \neq 0$. To this end, assume that $x(t)$ is an eventually positive nonoscillatory solution of Eq. (1) that belongs to Γ_3. Observe that assumption (3) and positivity of $r(t)$ yield, respectively,

$$\lim_{t\to+\infty}\int_t^{+\infty}\frac{1}{r(u)}du = 0, \qquad \frac{d}{dt}\left(\int_t^{+\infty}\frac{1}{r(u)}du\right) < 0.$$

Applying l'Hôpital's rule, we conclude that

$$\lim_{t\to+\infty}\frac{z(t)}{\int_t^{+\infty}\frac{1}{r(u)}du} = \lim_{t\to+\infty}\frac{\frac{d}{dt}(z(t))}{\frac{d}{dt}\left(\int_t^{+\infty}\frac{1}{r(u)}du\right)} = \lim_{t\to+\infty}\left(-r(t)\,z'(t)\right) = -\mu.$$

By virtue of Lemma 3.2, there exist two positive constants K_1 and K_2 such that

$$K_1\int_t^{+\infty}\frac{1}{r(u)}du \le z(t) \le K_2,$$

which implies, in particular, that

$$\frac{z(t)}{\int_t^{+\infty}\frac{1}{r(u)}du} \ge K_1,$$

and thus, $\mu \neq 0$. This completes the proof. □

Example 1. *For $t \ge 2$, consider the neutral differential equation*

$$\left(r(t)\left(x(t) + p(t)\,x(t-1)\right)'\right)' + a(t)\,x^2(\sigma(t)) = 0, \tag{12}$$

where

$$r(t) = t^2, \qquad p(t) = \frac{t}{2t+1}, \qquad \sigma(t) = \frac{t}{2},$$

$$a(t) = \frac{2t^3\left(4t^3 + 6t - 1\right)}{2\,(t+1)^2\left(t - 2t^2 + 1\right)^3}.$$

Observe that

$$\int_1^{+\infty}\frac{1}{r(u)}du = \int_1^{+\infty}\frac{1}{u^2}du = 1 < +\infty.$$

An exact solution to Eq. (12) is

$$x(t) = \frac{1}{t} + 2.$$

It is easy to see that

$$\lim_{t\to+\infty} x(t) = 2 = \frac{3}{1+1/2},$$

$$\lim_{t\to+\infty} z(t) = \lim_{t\to+\infty}\left(\frac{1}{t} + 2 + \frac{t}{2t+1}\left(\frac{1}{t-1} + 2\right)\right) = 3,$$

and

$$\lim_{t\to+\infty} r(t)\, z'(t) = -\lim_{t\to+\infty} \frac{4t^4 - 4t^2 + 2t + 1}{(t - 2t^2 + 1)^2} = -1.$$

Thus, $x(t)$ belongs to the class Γ_1.

4. The case when (4) holds.

Lemma 4.1. *Let $x(t)$ be an eventually positive (negative) solution of Eq. (1), $z(t)$ be defined by (5), and assume that (4) holds. Then $z'(t) \geq 0$ ($z'(t) \leq 0$) eventually.*

Proof. Without loss of generality, we assume that there exists a $T_0 \geq t_0$ such that $x(t) > 0$ and $x(\sigma_i(t)) > 0$, for all $t \geq T_0 \geq t_0$ and all $i \in \Delta$. The case when $x(t)$ is eventually negative is considered similarly. Then, one obviously has

$$z(t) \geq x(t) > 0, \qquad \text{for all } t \geq T_0, \tag{13}$$

and since

$$(r(t)\, z'(t))' = -f(t, x(\sigma_1(t)), x(\sigma_2(t)), \ldots, x(\sigma_n(t))) < 0, \tag{14}$$

we observe that $r(t)\, z'(t)$ is decreasing. Suppose, contrary to our assertion, that there exits a $T_1 \geq T_0$ such that $z'(T_1) < 0$. Then, $r(T_1)\, z'(T_1) < 0$, and, by virtue of (14), for all $t \geq T_1$,

$$r(t)\, z'(t) \leq r(T_1)\, z'(T_1) < 0. \tag{15}$$

Integrating (15) from T_1 to t, one has

$$z(t) < z(T_1) + r(T_1)\, z'(T_1) \int_{T_1}^{t} \frac{1}{r(u)} du. \tag{16}$$

Passing in (16) to the limit as $t \to +\infty$, we conclude that $z(t) \to -\infty$, which contradicts the fact that $z(t) > 0$ eventually. Hence, z is eventually nondecreasing, and the proof is complete. □

Theorem 4.2. *Assume that conditions (4) and (A_1)-(A_3) hold. Then any nonoscillatory solution of Eq. (1) belongs to one of the following five classes:*

Ω_1: $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} z(t) = 0, \quad \lim_{t\to+\infty} r(t)\, z'(t) = 0;$

Ω_2: $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} z(t) = 0, \quad \lim_{t\to+\infty} r(t)\, z'(t) = \mu \neq 0;$

Ω_3: $\lim_{t\to+\infty} x(t) = \dfrac{c}{1+p_0} \neq 0, \quad \lim_{t\to+\infty} z(t) = c \neq 0, \quad \lim_{t\to+\infty} r(t)\, z'(t) = 0;$

Ω_4: $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} z(t) = +\infty, \quad \lim_{t\to+\infty} r(t)\, z'(t) = 0;$

Ω_5: $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} z(t) = +\infty, \quad \lim_{t\to+\infty} r(t)\, z'(t) = \mu \neq 0.$

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). Without loss of generality, we may assume that $x(t)$ is positive for all $t \geq T_0 \geq t_0$, and thus (13) holds. There are three possibilities for the behavior of $z(t)$: either $\lim_{t\to+\infty} z(t) = 0$, or $\lim_{t\to+\infty} z(t) = c \neq 0$, or $\lim_{t\to+\infty} z(t) = +\infty$, and, by Lemma 2.1, $x(t)$ behaves correspondingly. Note that Lemma 4.1 implies that, for sufficiently large t, one has $z'(t) \geq 0$, and it follows from the inequality (14) that either

$$\lim_{t\to+\infty} r(t)\, z'(t) = 0,$$

or

$$\lim_{t\to+\infty} r(t)\, z'(t) = \mu > 0.$$

On the other hand, $z'(t) \geq 0$, and $r(t)\, z'(t)$ is nonnegative and decreasing. Therefore,

$$\lim_{t\to+\infty} z(t) = c, \qquad 0 < c < +\infty, \qquad \text{or} \qquad \lim_{t\to+\infty} z(t) = +\infty$$

and

$$\lim_{t\to+\infty} r(t)\, z'(t) = \mu \geq 0. \tag{17}$$

Decreasing nature of $r(t)\, z'(t)$ and (17) yield, for t large enough,

$$r(t)\, z'(t) \geq \mu,$$

that is,

$$z'(t) \geq \frac{\mu}{r(t)}.$$

Integration of the latter inequality from s to t implies that, for all $t > s \geq t_0$,

$$z(t) \geq z(s) + \mu \int_s^t \frac{1}{r(u)} du.$$

If $\mu > 0$, then, by (4),

$$\lim_{t\to+\infty} z(t) = +\infty,$$

and thus, $\lim_{t\to+\infty} x(t) = +\infty$. If $\mu = 0$ and $\lim_{t\to+\infty} z(t) = c \neq 0$ (respectively, $+\infty$), then by Lemma 2.1,

$$\lim_{t\to+\infty} x(t) = \frac{c}{1+p_0} \neq 0$$

(respectively, $+\infty$). Completing the proof by considering in a similar manner eventually negative nonoscillatory solutions, we conclude that any nonoscillatory solution $x(t)$ of Eq. (1) should belong to one of the classes described in the statement of the theorem. □

Example 2. *For $t \geq 2$, consider the second order neutral differential equation*

$$\left(r(t)\left(x(t) + p(t)\, x(t-1)\right)'\right)' + b(t)\, \frac{x^2(\sigma_1(t))}{x^2(\sigma_2(t)) + 1} = 0, \tag{18}$$

where

$$r(t) = \frac{1}{t}, \qquad p(t) = \frac{1}{3t+1}, \qquad \sigma_1(t) = t-1, \qquad \sigma_2(t) = \frac{t}{2},$$

and

$$b(t) = \frac{\left(27t^8 - 54t^7 - 36t^6 + 14t^5 + 120t^4 - 126t^3 + 2t^2 + 18t + 3\right)}{4t^6\left(t^2 - 2t + 2\right)^2\left(3t^4 - 2t - 1\right)^3} \times (t-1)^2\left(\left(t^2+4\right)^2 + 4t^2\right).$$

An exact nonoscillatory solution to (18) is given by

$$x(t) = \frac{1}{t} + t.$$

Clearly,

$$\lim_{t\to+\infty} x(t) = +\infty,$$

$$\lim_{t\to+\infty} z(t) = \lim_{t\to+\infty}\left(\frac{1}{t} + t - \frac{1}{3t+1} + \frac{1}{(3t+1)(t-1)} + \frac{t}{3t+1}\right) = +\infty,$$

and

$$\lim_{t\to+\infty} r(t) z'(t) = \lim_{t\to+\infty} \frac{9t^6 - 12t^5 - 7t^4 + 2t^3 + 9t^2 - 4t - 1}{t^3\left(2t - 3t^2 + 1\right)^2} = 0.$$

Hence, $x(t)$ belongs to the class Ω_4.

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