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STABILITY, BIFURCATION ANALYSIS IN A NEURAL NETWORK MODEL WITH DELAY AND DIFFUSION

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Abstract. We consider a delayed neural network model with diffusion. By analyzing the distributions of the eigenvalues of the system and applying the center manifold theory and normal form computation, we show that, regarding the connection coefficients as the perturbation parameter, the system, with the different boundary conditions, undergoes some bifurcations including transcritical bifurcation, Hopf bifurcation and Hopf-zero bifurcation. The normal forms are given to determine the stabilities of the bifurcated solutions.

1. **Introduction and preliminary.** In this paper, incorporating the effect of diffusion and time delay, we consider a model including a pair of neurons with time-delayed connections between the neurons and time delayed feedback from each neuron to itself,

$$\begin{aligned} \partial_t u &= d_1\partial_x^2 u - u(t) + af(u(t-\tau)) + bf(v(t-\tau)) \\ \partial_t v &= d_2\partial_x^2 v - v(t) + af(v(t-\tau)) + bf(u(t-\tau)), \end{aligned} \tag{1}$$

where a, b denotes the feedback and connection strength respectively, τ is the time delay, d_1 and d_2 are diffusion coefficients, the nonlinear feedback function $f : \mathbb{R} \to \mathbb{R}$ is smooth enough with $f(0) = 0$ and without loss of generality, $f'(0) = 1,\ f''(0) \neq 0$. This model is based on the one considered in [5]. In their work, without involving the effect of diffusion, Campbell and Shayer gave conditions for the stability of the trivial solution and analyzed possible bifurcations that may occur at trivial equilibrium point. For more study of dynamics of delayed systems, see ([8, 7]) and the references therein. Our purpose is to investigate the stabilities and bifurcations in (1) under two different boundary conditions by computing the normal forms. The explicit normal forms for (1) are given based on the work of Faria ([1], [2]).

The article is organized as follows. In Section 2, we consider (1) with Neumann boundary condition and discuss the transcritical bifurcation and Hopf bifurcation at critical values by considering the associated functional differential equation (FDE) introduced in [1] and [2]. Furthermore, the interaction of Hopf bifurcation and transcritical bifurcation is studied. (1) with Dirichlet boundary condition is investigated in Section 3. In this case, although we can not profit directly from the normal form of the associated FDE, some useful information about the bifurcations still can be obtained. Conclusions and remarks are given in the final section.

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Throughout the paper, $\Omega \subset \mathbb{R}$ is open, X is a Hilbert space of functions from $\overline{\Omega}$ to $\mathbb{R}^2$ with inner product $\langle \cdot, \cdot \rangle$, $\mathcal{C} = C([-\tau, 0]; X)$ $(\tau > 0)$ is the Banach space of continuous maps from $[-\tau, 0]$ to X with the supremum norm and $u_t(\theta) = u(t+\theta) \in \mathcal{C}$, $(-\tau \le \theta \le 0)$.

In [2], a general partial functional differential equation (PFDE) is considered which in abstract form with an equilibrium point at the origin as

$$du/dt = d\Delta u(t) + L(u_t) + F(u_t) \tag{2}$$

where $d > 0, \mathrm{dom}(\Delta) \subset X$, $L : \mathcal{C} \to X$ is a linear operator and $F : \mathcal{C} \to X$ is a C^k function $(k \ge 2)$ such that $F(0) = 0$ and $DF(0) = 0$. The author presented an approach to obtain the explicit normal forms for reaction-diffusion equations with delays on the center manifold, and showed that, up to a certain order of terms, the normal form of the PFDE coincides with that of an FDE associated to the given PFDE, under the following hypothesis (see [6] as well),

(H1) $d\Delta$ generates a C_0 semigroup $T(t)_{t\ge 0}$ on X with $|T(t)| \le Me^{\omega t}$ for some $M \ge 1$, $\omega \in \mathbb{R}$ and any $t \ge 0$, $T(t)$ is a compact operator for $t > 0$;

(H2') let $\delta_k^{i_k} : k \in \mathbb{N}, i_k = 1, \cdots p_k$ be the eigenvalue of $d\Delta$ and $\beta_k^{i_k}$ be eigenfunctions corresponding to $\{\delta_k^{i_k}\}$, such that $\{\beta_k^{i_k} : k \in \mathbb{N}, i_k = 1, \cdots, p_k\}$ form an orthonormal basis for X;

(H3') the subspaces $\mathcal{B}'_k \subset \mathcal{C}$, $\mathcal{B}'_k$:=span$\{\langle v(\cdot), \beta_k^{i_k}\rangle \beta_k^{i_k} | v \in \mathcal{C}, i_k = 1, \cdots, p_k\}$ satisfies $L(\mathcal{B}'_k) \subset \mathrm{span}\{\beta_k^1, \cdots \beta_k^{p_k}\}$;

(H4) L can be extended to a bounded linear operator from $B\mathcal{C}$=$\{\psi : [-r, 0] \to X | \psi$ is continuous on$[-r, 0), \exists \lim_{\theta \to 0^-} \psi(\theta) \in X\}$ to X with the supremum norm.

Under these hypothesis, the characteristic equation of the linearized system of PFDE (2), $\triangle(\lambda)y := \lambda y - d\Delta y - L(e^{\lambda \cdot} y) = 0$ for some nonzero $y \in \mathrm{dom}(\Delta)$, is equivalent to the sequence of equations $\det\triangle_k(\lambda) = 0$. Here $\triangle_k(\lambda) := \lambda I - M_k - L_k(e^{\lambda \cdot} I)$ $(k \in \mathbb{N})$ with $M_k = \mathrm{diag}(\delta_k^1, \cdots, \delta_k^{p_k})$ and $L_k(\varphi) = (L_k^1(\varphi), \cdots, L_k^{p_k}(\varphi))$ satisfying $L(\varphi_1 \beta_k^1, \cdots, \varphi_{p_k} \beta_k^{p_k}) = \sum_{i_k=1}^{p_k} L_k^{i_k}(\varphi)\beta_k^{i_k}$, for $\varphi = (\varphi_1, \cdots, \varphi_{p_k}) \in C_{p_k} = C([-\tau, 0], \mathbb{R}^{p_k})$.

On $\mathcal{B}'_k$, the linearized equation $\frac{d}{dt}u(t) = d\Delta u(t) + L(u_t)$ is equivalent to the FDE $\dot{z}(t) = M_k z(t) + L_k z_t$ on C_{p_k}. Let $\Lambda_k = \{\lambda \in \mathbb{C} : \lambda$ is a solution of $\det\triangle_k(\lambda) = 0$ with $\mathrm{Re}\lambda = 0\}$ and $\Lambda = \bigcup_{k=1}^N \Lambda_k$, for some $N \in \mathbb{N}$. We assume $\Lambda \ne \emptyset$. Then $C_{p_k} = P_k \bigoplus Q_k$, where $P_k = \mathrm{span}\{\Phi_k\}$ and Φ_k is the eigenfunction space of the FDE on C_{p_k} corresponding to Λ_k. Thus, we can decompose the phase space of PFDE (2) by a projection $\pi : \mathcal{C} \to \mathcal{P}$, $\mathcal{P} = \mathrm{Im}\pi$, $\mathcal{Q} = \mathrm{Ker}\pi$ and for $\hat{\varphi} \in \mathcal{C}$, $\pi(\hat{\varphi}) = \sum_{k=1}^N \sum_{i_k=1}^{p_k} c_k^{i_k}(\hat{\varphi})\beta_k^{i_k}$ with $(c_k^{i_k}(\hat{\varphi}))_{i_k=1}^{p_k} = \Phi_k(\Psi_k, (\langle \hat{\varphi}(\cdot), \beta_k^{i_k}\rangle)_{i_k=1}^{p_k})_k$, $(\Psi_k, \Phi_k)_k = I$, $(\cdot, \cdot)_k$ being the bilinear form (see [4]).

According to [1, Theorem 4.1], if another hypothesis (H5') holds:

$$\text{(H5')} \quad \langle DF_2(u)(\varphi \beta_j^{i_j}), \beta_n^{i_n} \rangle = 0, \forall u \in \mathcal{P}, \ \forall \varphi \in C([-\tau, 0]; \mathbb{R})$$

for $1 \le n \le N$, $\ 1 \le i_n \le p_n \ \ j > N$ and $1 \le i_j \le p_j$, then the normal forms of the PFDE (2) and its associated FDE are the same, up to at least the third order terms on the center manifolds. The associated FDE is defined as

$$\dot{x}(t) = R(x_t) + G(x_t) \tag{3}$$

where $x(t) = (x_k(t))_{k=1}^N$ with $x_k \in \mathbb{R}^{p_k}$, and $R, G : C_J \to \mathbb{R}^J$ with $J = \sum_{k=1}^N p_k$ are

$$R(\varphi) = (M_k \varphi_k(0) + L_k(\varphi_k))_{k=1}^N, \ G(\varphi) = (\langle F(\sum_{k=1}^N (\beta_k^1, \cdots, \beta_k^{p_k}) \varphi_k^T), \beta_n^{i_n}\rangle_{i_n=1}^{p_n})_{n=1}^N$$

for $\varphi=(\varphi_1,\cdots,\varphi_N)^T\in C_J, \varphi_k=(\varphi_k^1,\cdots,\varphi_k^{p_k})\in C_{p_k}$, $k=1,\cdots,N$. In the following, for convenience we denote $\varphi=(\varphi_1,\varphi_2)^T\in C([-\tau,0],\mathbb{R}^2)$, $\widehat{\varphi}=(\widehat{\varphi}_1,\widehat{\varphi}_2)^T\in\mathcal{C}$, $k\in\mathbb{N}$, $n\in\mathbb{N}\cup\{0\}=\mathbb{N}_0$, $V_1=(1,1)^T$, $V_2=(1,-1)^T$ and $\Upsilon_t^s=\begin{pmatrix} s & t\\ t & s\end{pmatrix}$, fix the value of the delay τ and choose the diffusion coefficients $d_1=d_2=1$.

2. **With Neumann boundary condition.** First, we consider (1) with Neumann boundary condition in $X=\{(u,v): u,v\in W^{2,2}(0,\pi), du/dx=dv/dx=0$ at $x=0,\pi\}$, with the inner product $\langle\cdot,\cdot\rangle$ induced by that of the Sobolev space $W^{2,2}(0,\pi)$. Setting $W(t)=(u(t),v(t))^T$ and using Taylor expansion at the trivial equilibrium point, (1) can be given, in abstract form in $\mathcal{C}=C([-\tau,0];X)$ as

$$dW/dt = \Delta W(t)+L(W)+F(W), \tag{4}$$

with $L(\widehat{\varphi})=-\widehat{\varphi}(0)+\Upsilon_b^a\widehat{\varphi}(-\tau)$ and $F(\widehat{\varphi})=\Upsilon_b^a\sum_{j\geq 2}\widehat{\varphi}_t^j(-\tau)f^{(j)}(0)/j!$.

Eigenvalues of Laplacian on X are $\delta_k^{i_k}=-(k-1)^2=:\delta_k, i_k=1,2$ with eigenfunctions $\beta_k^1=(\gamma_k,0)^T$ and $\beta_k^2=(0,\gamma_k)^T$ respectively for $\gamma_k(x)=\cos((k-1)x)/\|\cos((k-1)x)\|_{2,2}$. (H1)-(H4) hold with $p_k=2$, since the linear part $L(\cdot)$ of (4) satisfies

$$L(\varphi_1\beta_k^1+\varphi_2\beta_k^2)=(-\varphi_1(0)+a\varphi_1(-\tau)+b\varphi_2(-\tau))\beta_k^1+(-\varphi_2(0)+a\varphi_2(-\tau)+b\varphi_1(-\tau))\beta_k^2.$$

2.1. **Local stability.** The characteristic equation of the linearized equation of (4) is equivalent to

$$det\triangle_k(\lambda)=[\lambda+(k-1)^2+1-(a-b)e^{-\lambda\tau}][\lambda+(k-1)^2+1-(a+b)e^{-\lambda\tau}]=0.\ (k\in\mathbb{N})$$

$\lambda\in\mathbb{C}$ is an eigenvalue if and only if $P_C^k(\lambda)=\lambda+(k-1)^2+1-Ce^{-\lambda\tau}=0$ for some k with $C=a-b$ or $C=a+b$. We first analyze the distribution of zeros of P_C^k with zero real parts. Let $\lambda=it, t\in\mathbb{R}$. By comparing the imaginary and real parts of $P_C^k(it)$, we get a parametric system as $C(t)=(1+(k-1)^2)\cos(t\tau)-t\sin(t\tau)$ and $\mathrm{Im}P_{C(t)}^k(it)=0$. To solve this system, we consider its corresponding curve Γ_k determined by $S(t)=C(t)$ and $T(t)=(1+(k-1)^2)\sin(t\tau)+t\cos(t\tau)$. Let $\theta(t)=T(t)/S(t)$. Then $\theta'(t)>0$ for all $t\in\mathbb{R}$ satisfying $S(t)\neq 0$. Thus Γ_k moves counterclockwise around the origin on (S,T)-plane. It is easy to see that at a sequence of critical values $\{t_n^k\}_{n=0}^\infty$ with $t_0^k=0, t_n^k\in((2n-1)\pi/(2\tau),n\pi/\tau)$, Γ_k intersects with S-axis at $(C_n^k,0)$. Since $S^2(t)+T^2(t)=(1+(k-1)^2)^2+t^2$, $C_n^k=(-1)^n\sqrt{(1+(k-1)^2)^2+(t_n^k)^2}$ and $\{|C_n^k|\}_{n\in\mathbb{N}_0}$ is an increasing sequence. Obviously $C_0^k=1+(k-1)^2$ and $(-1)^nC_n^k>0$, hence the following result holds.

Lemma 2.1. *(see Figure 1) For some k, (i) $P_C^k(\lambda)$ has a simple pair of purely imaginary roots $\pm it_n^k$ if and only if $C=C_n^k$ for $n\neq 0$; $P_C^k(\lambda)$ has a simple zero root $\lambda=0$ if and only if $C=C_0^k$;*
(ii) $P_C^k(\lambda)$ only has roots with negative real parts if $C_1^k<C<C_0^k$; $2(l+1)$ roots with positive real parts if $C_{2l+3}^k\leq C<C_{2l+1}^k$; $2l+1$ roots with positive real parts if $C_{2l}^k\leq C<C_{2l+2}^k$, $l\in\mathbb{N}_0$.

Proof. (i) From $P_C^k(\lambda)=0$ and $(P_C^k(\lambda))'=1+C\tau e^{-\lambda\tau}$, $P'(\pm it_n^k)\neq 0$ and $P'(0)\neq 0$, (i) is obvious from the process to form C_n^k.

(ii) First, λ is a continuous function of C according to the implicit function theorem. If $C=0$, $P_C^k(\lambda)=0$ has only one root $\lambda=-(1+(k-1)^2)<0$. Moreover, differentiating $P_C^k(\lambda)=0$ with respect to C, we have $d\lambda/dC=e^{-\lambda\tau}/(1+C\tau e^{-\lambda\tau})$. By computation, $sign(Re\frac{d\lambda}{dC}|_{C=C_n^k})=sign(C_n^k)$. Hence, as C increases to $C_0^k=1+(k-1)^2>0$, only one root of $P_C^k=0$ is zero while others have negative real parts; when C lies between C_0^k and C_2^k, P_C^k has one zero with positive real part while others have negative real parts. As C reaches C_2^k, a pair of complex roots of $P_C^k = 0$ have zero real part and one has positive real part while others have negative real parts; when C crosses C_2^k, P_C^k has three zeros with positive real parts while others have negative real parts. Similarly, we can finish the remaining proof. □

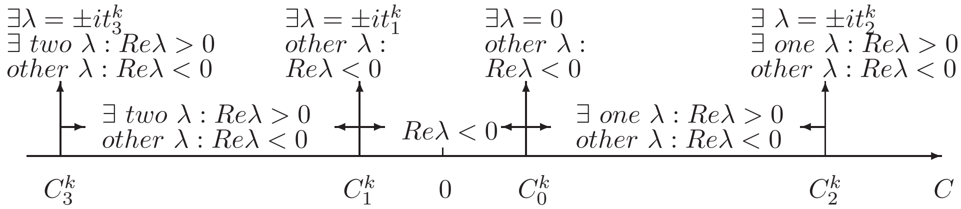

Figure 1. If $C_1^k < C < C_0^k$, all the roots of $P_C^k(\lambda) = 0$ have negative real parts; at $C = C_1^k$, there exist pure imaginary roots $\pm it_1^k$ and other roots have negative real parts; at $C = C_0^k$, except a zero root, all the others have negative real parts. If $C \in \mathbb{R} - [C_1^k, C_0^k]$, there is, at least one root with positive real part.

In order to study the dynamical behavior in (4), we need to discuss the distribution of roots in $\det\triangle_k(\lambda) = 0$. From $P_C^k(it_n^k) = 0$ we have $-t_n^k/(1 + (k-1)^2) = tan(t_n^k\tau)$, then $t_n^k < t_n^{k+1}$, and $|C_n^k| < |C_n^{k+1}|$. Thus we have,

Theorem 2.2. *(See Figure 2.) For the characteristic equation of the linearized equation of (4) with Neumann boundary condition,*
(i) all eigenvalues have negative real parts if and only if $C_1^1 < a + b < C_0^1 = 1$ and $C_1^1 < a - b < 1$, which implies that, when $(a, b) \in \{(a, b) : C_1^1 < a + b < 1,\ C_1^1 < a - b < 1\}$, the trivial solution of system (1) is asymptotically stable;
(ii) if $a + b = C_1^1$ and $C_1^1 < a - b < 1$ or $a - b = C_1^1$ and $C_1^1 < a + b < 1$, then all eigenvalues but $\lambda = \pm it_1^1$ have strictly negative real parts, where $\pm it_1^1$ is a pair of purely imaginary roots of $det\triangle_1(\lambda) = 0$ when $C = C_1^1$;
(iii) if $a + b = C_1^1, a - b = 1$ or $a + b = 1, a - b = C_1^1$ then all eigenvalues, except $\lambda = \pm it_1^1$ and $\lambda = 0$, have strictly negative real parts.

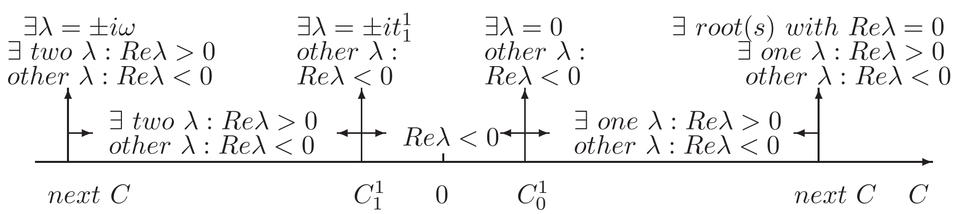

Figure 2

Combining the results in Lemma 2.1 and Theorem 2.2, we know that with the parameter C is beyond $[C_1^1, C_0^1]$, there exists at least one eigenvalue with positive real part and the trivial solution will loss its stability and bifurcation may occur.

2.2. **Bifurcation.** The occurrence of bifurcation implies a qualitative change in the solutions. The study of such changes is important, especially when the system exists only center manifold and stable manifold near the trivial solution, we are able to determine the whole dynamical behavior of the system. In this subsection, we will study all the possible bifurcations at trivial solution of (1) with Neumann boundary condition. We are only interested in the bifurcations at the boundary critical values, implying only $\det\triangle_1 = 0$ has eigenvalues with zero real part, so $N = 1$ in (3). More precisely, the potential bifurcations include steady-state bifurcation with simple zero eigenvalue at $C = C_0^1 = 1$, Hopf bifurcation with a simple pair of purely imaginary eigenvalues $\pm it_1^1$ at $C = C_1^1$ and Hopf-zero bifurcation, the interaction of the two Codimension-one bifurcations.

To discuss the Codimension-one bifurcation, we fix b and perturb the parameter a at the critical value a_0 as $a = a_0 + \mu, \mu \in \mathbb{R}$. Then in (4) $L(\widehat{\varphi}) = -\widehat{\varphi}(0) + \Upsilon_b^{a_0}\widehat{\varphi}(-\tau)$ and $F(\widehat{\varphi}) = \Upsilon_0^{\mu}\widehat{\varphi}(-\tau) + \Upsilon_b^a \sum_{j\geq 2} \widehat{\varphi}^j(-\tau) f^{(j)}(0)/j!$. In (3), $R(\varphi) = L_1(\varphi)$ since $\delta_1 = 0$ and $M_1 = 0$,

$$L_1(\varphi) = -\left(\varphi_1(0),\ \varphi_2(0)\right)^T + \Upsilon_b^{a_0}\left(\varphi_1(-\tau),\ \varphi_2(-\tau)\right)^T \tag{5}$$

satisfying $L(\varphi_1\beta_1^1 + \varphi_2\beta_1^2) = \left(\beta_1^1, \beta_1^2\right) L_1(\varphi)$. Let $f^{(i)}(0)\langle\gamma_1^i, \gamma_1\rangle = \varsigma_i$, then

$$G(\varphi) = \Upsilon_0^{\mu}\varphi(-\tau) + \sum\nolimits_{j\geq 2} \Upsilon_b^a \varphi^j(-\tau)\varsigma_j/j! \tag{6}$$

where $\varphi^j(-\tau) = (\varphi_1^j(-\tau), \varphi_2^j(-\tau))^T$. Therefore, with $\langle\gamma_1^j, \gamma_1\rangle = (1/\sqrt{\pi})^{j-1}$, the FDE associated with (4) by Λ at the trivial equilibrium point is

$$\dot{x}(t) = -x(t) + (\Upsilon_b^{a_0} + \Upsilon_0^{\mu})x_t(-\tau) + \sum\nolimits_{j\geq 2} \Upsilon_b^a \varsigma_j x_t^j(-\tau)/j! \tag{7}$$

where $x(t) = (x_1(t), x_2(t))^T \in C([-\tau, 0]; \mathbb{R}^2)$, $x^j(t) = (x_1^j(t), x_2^j(t))^T$. Denote $\frac{F_2(\widehat{\varphi},\mu)}{2}$ be the second order term of the nonlinear terms in this associated FDE, we have

$$F_2(\widehat{\varphi}, \mu)/2 = \Upsilon_0^{\mu}\widehat{\varphi}(-\tau) + \Upsilon_b^{a_0} f''(0)\widehat{\varphi}^2(-\tau)/2. \tag{8}$$

Case 1. Transcritical bifurcation We first consider the simplest bifurcation occurring in (4). When the critical value a_0 satisfies (i) $a_0 + b = C_0^1 = 1$ and $a_0 - b \in (C_1^1, 1)$, or (ii) $a_0 - b = C_0^1$ and $a_0 + b \in (C_1^1, 1)$, Theorem 2.2 implies $\Lambda = \{0\}$. It suffices to discuss the case (i). The phase space $\mathcal{C}$ of the linearized equation of (4) can be decomposed as $\mathcal{C} = \mathcal{P} \bigoplus \mathcal{Q}$ with respect to $\pi : \mathcal{C} \to \mathcal{P}$,

$$\pi(\widehat{\varphi}) = \left(\beta_1^1, \beta_1^2\right)\left[\Phi(\Psi, (\langle\widehat{\varphi}(\cdot), \beta_1^i\rangle)_{i=1}^2)\right], \tag{9}$$

with $\Phi = V_1, \Psi = V_1^T(2 + 2C_0^1\tau)^{-1} \coloneqq V_1^T D_2, \beta_1^1 = (1/\sqrt{\pi}, 0)^T$ and $\beta_1^2 = (0, 1/\sqrt{\pi})^T$.

Following Theorem 4.1 in [2], the normal forms of PFDE and its associated FDE are the same for the first and second order terms. By computation, we can obtain the normal form of the associated FDE (7) up to the second-order, with respect to $\Lambda = \{0\}$ as

$$\dot{z} = 2D_2(\mu z + C_0^1\varsigma_2 z^2/2) + h.o.t. \tag{10}$$

Thus the bifurcation at a_0 is transcritical since $\varsigma_2 = f''(0)/\sqrt{\pi} \neq 0$.

Case 2. Hopf bifurcation If a_0 satisfies (i) $a_0 + b = C_1^1$ and $a_0 - b \in (C_1^1, 1)$, or (ii) $a_0 - b = C_1^1$ and $a_0 + b \in (C_1^1, 1)$, the system undergoes a Hopf bifurcation at $a = a_0$ since the transversality condition is confirmed in the proof of Lemma2.1, $\Lambda = \{-it_1^1, it_1^1\}$. We consider (i) only, (ii) can be treated in a similar way. The phase space of the linearized system of (4) $\mathcal{C} = \mathcal{P} \bigoplus \mathcal{Q}$ with respect to π defined by (9) for

$$\Phi = [e^{it_1^1\theta}V_1, e^{-it_1^1\theta}V_1] := (\phi_1, \phi_2), \quad \Psi = \mathrm{col}(V_1^T D_1 e^{-it_1^1 s}, V_1^T \overline{D}_1 e^{it_1^1 s}), \tag{11}$$

with $-\tau \leq \theta \leq 0 \leq s \leq \tau$ and $D_1 = [2 + 2\tau C_1^1 e^{-it_1^1\tau}]^{-1}$. For any $u \in \mathcal{P} = \mathrm{span}\{(\beta_1^1, \beta_1^2)\phi_1,\ (\beta_1^1, \beta_1^2)\phi_2\}$, $u = p(\beta_1^1, \beta_1^2)\phi_1 + q(\beta_1^1, \beta_1^2)\phi_2 := (\varphi_1, \varphi_2)^T/\sqrt{\pi}$ for $p, q \in \mathbb{R}$. (H5') holds since from (8), for $k \geq 2$, $\psi_1, \psi_2 \in C([-\tau, 0]; \mathbb{R})$ and $u \in \mathcal{P}$

$$\begin{aligned}&\frac{1}{2}D_1F_2(u,\mu)(\psi_1\beta_k^1 + \psi_2\beta_k^2)\\ =&\{\mu\psi_1(-\tau) + \varsigma_2[a_0\varphi_1(-\tau)\psi_1(-\tau) + b\varphi_2(-\tau)\psi_2(-\tau)]\}\beta_k^1\\ &+ \{\mu\psi_2(-\tau) + \varsigma_2[b\varphi_1(-\tau)\psi_1(-\tau) + a_0\varphi_2(-\tau)\psi_2(-\tau)]\}\beta_k^2.\end{aligned} \tag{12}$$

Hence we can derive the normal form of (7) in polar coordinate, up to the third-order, as

$$\begin{cases} \dot{\rho} = Re(e^{-it_1^1\tau}D_1)\mu\rho + Re(K_1)\rho^3 + h.o.t \\ \dot{\xi} = -t_1^1 + h.o.t \end{cases} \tag{13}$$

where with $\varsigma_3 = f'''(0)/\pi$

$$\begin{aligned} K_1 &= \varsigma_2 C_1^1 D_1[\varsigma_2 C_1^1 \frac{2i}{t_1^1}(D_1 e^{-i2t_1^1\tau} - \frac{7}{3}\overline{D}_1) + e^{it_1^1\tau}h_1 + e^{-it_1^1\tau}h_2] + \varsigma_3 C_1^1 e^{-it_1^1\tau}D_1, \\ h_1 &= 2C_1^1\varsigma_2 e^{-2it_1^1\tau}\{-\frac{e^{it_1^1\theta}}{it_1^1}(D_1 + \frac{e^{-2it_1^1\theta}}{3}\overline{D}_1) + (2it_1^1 + 1 - C_1^1 e^{-2it_1^1\tau})^{-1}e^{2it_1^1\theta} \\ &\qquad \times[\frac{1}{2} - 2Re(D_1) + \frac{2it_1^1+1}{it_1^1}(D_1 + \frac{\overline{D}_1}{3}) - \frac{C_1^1}{it_1^1}e^{-it_1^1\tau}(D_1 + \frac{e^{2it_1^1\tau}\overline{D}_1}{3})]\}, \\ h_2 &= 4C_1^1\varsigma_2\{[\frac{1}{2} + 2Re(D_1(-1 - \frac{1}{it_1^1} + \frac{C_1^1 e^{-it_1^1\tau}}{it_1^1}))](1 - C_1^1)^{-1} + 2Re(\frac{D_1 e^{it_1^1\theta}}{it_1^1})\}. \end{aligned} \tag{14}$$

Case 3. Hopf-zero bifurcation To discuss the codimension-two bifurcation, we perturb the parameters a, b at the critical values a_0 and b_0 as $a = a_0 + \mu$, $b = b_0 + \nu$, $\mu, \nu \in \mathbb{R}$. Then in (4), $L(\widehat{\varphi}) = -\widehat{\varphi}(0) + \Upsilon_{b_0}^{a_0}\widehat{\varphi}(-\tau)$ and $F(\widehat{\varphi}) = \Upsilon_\nu^\mu\widehat{\varphi}(-\tau) + \Upsilon_b^a\sum_{j\geq 2} f^{(j)}(0)\widehat{\varphi}^j(-\tau)/j!$. When the critical values a_0, b_0 satisfy (i) $a_0 - b_0 = C_0^1$ and $a_0 + b_0 = C_1^1$, or (ii) $a_0 - b_0 = C_1^1$ and $a_0 + b_0 = C_0^1$, $\Lambda = \{\pm it_1^1, 0\}$. It is sufficient to consider the case (i). The phase space $\mathcal{C}$ related to (4) can be decomposed by π similarly as (9) with $\Phi = [\phi_1, \phi_2, \phi_3]$ and Ψ satisfying

$$\Phi(\theta) = [e^{it_1^1\theta}V_1, e^{-it_1^1\theta}V_1, V_2], \ \Psi(s) = \mathrm{col}(V_1^T D_1 e^{-it_1^1 s}, V_1^T\overline{D}_1 e^{it_1^1 s}, V_2^T D_2), \tag{15}$$

where $-\tau \leq \theta \leq 0 \leq s \leq \tau$, D_1 and D_2 are defined in (9) and (11) respectively.

The FDE associated with (4) by Λ has a similar form as (7) with $\Upsilon_b^{a_0}$ and Υ_0^μ replaced by $\Upsilon_{b_0}^{a_0}$ and Υ_ν^μ respectively. It is easy to verify that (H5') holds. Since in the normal form of the associated FDE the coefficients of the second-order terms are zero due to the structure in (1), the higher-order terms have qualitative effects and we need to compute the normal form up to the third order. Therefore, the normal form of (4), up to the third-order, can be obtained in cylindrical coordinate as

$$\begin{cases} \dot{\rho} = (\mu + \nu)Re(D_1 e^{-it_1^1\tau})\rho + Re(K_1)\rho^3 + Re(K_3)\rho z^2 + h.o.t \\ \dot{\theta} = -t_1^1 + h.o.t \\ \dot{z} = 2D_2(\mu - \nu)z + K_2 z\rho^2 + K_4 z^3 + h.o.t, \end{cases} \tag{16}$$

with K_1 and h_2 given in (14), and
$K_2 = 2D_2\varsigma_2 C_0^1 Re(4i\varsigma_2 C_1^1 e^{-it_1^1\tau}D_1/t_1^1 + e^{it_1^1\tau}h_3 + h_2)/2 + 2\varsigma_3 D_2 C_0^1$,
$K_3 = D_1\varsigma_2 C_1^1\{2i\varsigma_2[C_1^1(D_1 e^{-2it_1^1\tau} - \overline{D}_1) - 2e^{-it_1^1\tau}D_2 C_0^1]/t_1^1 + h_3\} + \varsigma_3 D_1 C_1^1 e^{-it_1^1\tau}$,
$K_4 = 2\, D_2\, \varsigma_2\, C_0^1\, Re(2i\, \varsigma_2\, C_1^1\, e^{-it_1^1\, \tau}\, D_1\, /\, t_1^1) + \varsigma_3\, D_2\, C_0^1\, /\, 3$,
$h_3 = 4D_2 C_0^1\varsigma_2 e^{-it_1^1\tau}\{i/t_1^1 - e^{it_1^1\theta}[1/2 + C_1^1 i/t_1^1](it_1^1 + 1 - C_1^1 e^{-it_1^1\tau})^{-1}\}$.

3. **With Dirichlet boundary condition.** In this section we study (1) with Dirichlet boundary condition $u(t,0) = v(t,0) = u(t,\pi) = v(t,\pi) = 0$. Under this condition, (H5') does not hold and we will set up the relationship between the normal forms of the PFDE and its associated FDE. Define $X = \{(u,v) : u, v \in L^2(0,\pi) : u(0) = v(0) = u(\pi) = v(\pi) = 0\}$ with the inner product $\langle\cdot,\cdot\rangle$ induced by that of $L^2(0,\pi)$. It is easy to see that zero is no longer an eigenvalue of Laplacian in this case. In fact, eigenvalues in X of Δ are $\delta_k^{i_k} = -k^2 =: \delta_k$, $i_k = 1, 2$ with

corresponding normalized eigenfunctions $\beta_k^1 = (\gamma_k, 0)^T, \beta_k^2 = (0, \gamma_k)^T$ respectively, $\gamma_k(x) = \sin(kx)\sqrt{2/\pi}$. Similarly (H1)-(H4) hold with $p_k = 2$ and at the trivial equilibrium point, (1) can be transformed into (4) in $\mathcal{C} = C([-\tau, 0]; X)$. Denote

$$det\triangle_k(\lambda) = [\lambda + k^2 + 1 - (a-b)e^{-\lambda\tau}][\lambda + k^2 + 1 - (a+b)e^{-\lambda\tau}] = 0,$$

$P_C^k = \lambda + k^2 + 1 - Ce^{-\lambda\tau}$, $C_n^k = (-1)^n\sqrt{(1+k^2)^2 + (t_n^k)^2}$. Lemma 2.1 holds and the distribution of the eigenvalues of the characteristic equation is the same as that described in Theorem 2.2, so $N = 1$.

For the same reason as that in the previous section, we only need to consider the region $\Omega_0 = \{(a, b) : C_1^1 \leq C = a \pm b \leq C_0^1\}$ where $C_0^1 = 2$ and $C_1^1 = -\sqrt{4 + (t_1^1)^2}$. At the boundary critical values of Ω_0, the system has possible bifurcations including steady-state (simple zero) at $C = C_0^1$, Hopf bifurcation at $C = C_1^1$ and the interaction of these two one-codimension bifurcations.

To discuss the Codimension-one bifurcation, we fix b and let $a = a_0 + \mu, \mu \in \mathbb{R}$. Then in (4) $L(\widehat{\varphi}) = -\widehat{\varphi}(0) + \Upsilon_b^{a_0}\widehat{\varphi}(-\tau)$ and $F(\widehat{\varphi}) = \Upsilon_0^{\mu}\widehat{\varphi}(-\tau) + \Upsilon_b^{a}\sum_{j\geq 2}\widehat{\varphi}^j(-\tau)f^{(j)}(0)/j!$. Corresponding to (3), $\delta_1 = -1$, $M_1 = diag(-1, -1)$. In the associated FDE of (4), $G(\varphi)$ is defined in (6) with our choice of β_1^1 and β_1^2, $R(\varphi) = M_1\varphi(0) + L_1(\varphi)$ where $L_1(\cdot)$ is defined in (5). Parallel to the discussion in Section 2.2, we have

Case 1. Transcritical bifurcation Let a_0 satisfy $a_0 + b = C_0^1 = 2$ and $a_0 - b \in (C_1^1, 2)$, then $\Lambda = \{0\}$. The phase space $\mathcal{C}$ can be decomposed similarly with respect to π as (9) with $\Phi = V_1, \Psi = V_1^T D_2$, The normal form of the associated FDE with respect to Λ has the same form as (10) with $C_0^1 = 2$ and $\varsigma_2 = 4(2/\pi)^{3/2}f''(0)/3$. It is clear that the bifurcation at $a_0 = 2 - b$ is transcritical bifurcation.

Case 2. Hopf bifurcation Let a_0 satisfy $a_0 + b = C_1^1$ and $a_0 - b \in (C_1^1, C_0^1)$, then $\Lambda = \{\pm it_1^1\}$. The phase space $\mathcal{C}$ can be decomposed as before by π as (9) associated with Λ, and Φ and Ψ have the same form as that in (11). But (H5') fails. In fact, for all $u \in \mathcal{P}$, $u = \gamma_1(\varphi_1, \varphi_2)^T$, for $k \geq 2$, $\forall \psi_1,\ \psi_2 \in C([-\tau, 0]; \mathbb{R})$, $1/2D_1F_2(u, \mu)(\psi_1\beta_k^1 + \psi_2\beta_k^2)$ has the same form as (12), whereas

$$\langle (1/2)D_1F_2(u,\mu)(\psi_1\beta_k^1 + \psi_2\beta_k^2), \beta_1^i\rangle_{i=1}^2 = \Upsilon_b^a(\varphi_1(-\tau)\psi_1(-\tau), \varphi_2(-\tau)\psi_2(-\tau))f''(0)\alpha_k$$

with $\alpha_k := \langle \gamma_k\gamma_1, \gamma_1\rangle = 0$ if k is even, or $-4(2/\pi)^{3/2}/[k(k^2-4)]$ if k is odd.

Since (H5') does not hold, we can not obtain the information directly from the normal form of the associated FDE. However, we can still make use of the relationship between the normal forms of PFDE (4) and its associated FDE to study the Hopf bifurcation. By the decomposition of $\mathcal{C}$, (4) can be transformed to

$$\dot{z} = Bz + \sum\nolimits_{j\geq 2} f_j^1(z, y)/j! \qquad \text{and} \qquad dy/dt = A_1y + \sum\nolimits_{j\geq 2} f_j^2(z, y)/j!$$

where $B = \text{diag}\{it_1^1, -it_1^1\}$, $z = (z_1, z_2) \in \mathbb{C}^2$, $y \in \mathcal{C}_0^1 \bigcap \mathcal{Q}$, $\mathcal{C}_0^1 := \{\widehat{\varphi} \in \mathcal{C} : \dot{\widehat{\varphi}} \in \mathcal{C}, \widehat{\varphi}(0) \in \text{dom}(\Delta)\}$, $A_1\widehat{\varphi} = \dot{\widehat{\varphi}} + X_0[L(\widehat{\varphi}) + \Delta\widehat{\varphi}(0) - \dot{\widehat{\varphi}}(0)]$, for $\widehat{\varphi} \in \mathcal{C}_0^1 \bigcap \mathcal{Q}$, $f_j^1(z, y) = \Psi(0)\left(\langle F_j\left(\left(\beta_1^1, \beta_1^2\right)[\Phi z] + y\right), \beta_1^i\rangle\right)_{i=1}^2$ and $f_j^2(z, y) = (I - \pi)X_0F_j\left(\left(\beta_1^1, \beta_1^2\right)[\Phi z] + y\right)$.

Since the characteristic equation of the associated FDE, $\det\triangle_1 = 0$, only has a pair of eigenvalues with zero real parts, i.e. $\pm t_1^1 i$ which correspond to the eigenfunction space Φ, then we can decompose the phase space $C_2 = C([-\tau, 0]; \mathbb{R}^2)$ of the associated FDE as $C_2 = \text{span}\{\Phi\} \bigoplus Q$. Let $x_t = \Phi z(t) + y$, with $z(t) \in \mathbb{C}^2$ and here, different from that in PFDE, $y \in Q \bigcap \text{dom}(A_{01})$, $A_{01}\varphi = \dot{\varphi} + X_0[R(\varphi) - \dot{\varphi}(0)]$, $\varphi \in \text{dom}(A_{01}) \subset C_2$. Denote $f_{0,j}^1(z, y) = \Psi(0)\left(\langle F_j\left(\left(\beta_1^1, \beta_1^2\right)[\Phi z + y]\right), \beta_1^i\rangle\right)_{i=1}^2$, $f_{0,j}^2(z, y) = (I - \pi)X_0\left(\langle F_j\left(\left(\beta_1^1, \beta_1^2\right)[\Phi z + y]\right), \beta_p^i\rangle\right)_{i=1}^2$, the associated FDE becomes

$$\dot{z} = Bz + \sum_{j\geq 2} f_{0j}^1(z, y)/j! \qquad \text{and} \qquad dy/dt = A_{01}y + \sum_{j\geq 2} f_{0j}^2(z, y)/j!.$$

Let $\dot z = Bz + g_2^1(z,0,\mu)/2 + \cdots$ and $\dot z = Bz + g_{0,2}^1(z,0,\mu)/2 + \cdots$ be normal forms in complex coordinates on the center manifold at zero for PFDE (4) and its associated FDE respectively, we have the follows:

Theorem 3.1. *The normal form of PFDE (4) is*

$$\begin{aligned}\dot z &= Bz + g_2^1(z,0,\mu)/2! + g_3^1(z,0,\mu)/3! + h.o.t\\ &= Bz + g_{0,2}^1(z,0,\mu)/2! + g_{0,3}^1(z,0,\mu)/3! + (K_5\, z_1^2\, z_2, \overline{K}_5 z_2^2 z_1)^T + h.o.t\end{aligned}$$

where $K_5 = \widetilde{\alpha}_k \sum_{k>1}(C_{0,k}e^{-it_1^1\tau} + C_{1,k}e^{it_1^1\tau})D_1,\ C_{0,k} = 2\widetilde{\alpha}_k/[k^2+1-C_1^1],\ C_{1,k} = \widetilde{\alpha}_k/[(2t_1^1 i + k^2+1)e^{2t_1^1\tau i} - C_1^1]$ *with* $\widetilde{\alpha}_k = f''(0)C_1^1\alpha_k$.

Proof. From the proof of [2, Theorem 4.1] and because of the occurrence of Hopf bifurcation in associated FDE, $g_{0,2}^1(z,0) = g_2^1(z,0) = 0$ and for $y=0$

$$\overline{f}_3^1(z,0,\mu) = \overline{f}_{0,3}^1(z,0,\mu) + \frac{3}{2}[D_y f_2^1(z,y,\mu)|_{y=0}U_2^2(z,\mu) - D_y f_{0,2}^1(z,y,\mu)|_{y=0}U_{0,2}^2(z,\mu)]$$

$$= \overline{f}_{0,3}^1(z,0,\mu) + 3\Psi(0)\Big(\frac{1}{2}\langle D_1F_2((\beta_1^1,\beta_1^2)[\Phi z],\mu)\Big(\sum_{k>1}(h_k^1\beta_k^1 + h_k^2\beta_k^2)(z,\mu)\Big),\beta_1^i\rangle\Big)_{i=1}^2 \tag{17}$$

where $h(z,\mu) := U_2^2(z,\mu) = \sum_{k\geq 1}(h_k^1\beta_k^1 + h_k^2\beta_k^2)$ is the unique solution of $(M_2^2 h)(z,\mu) = D_z h(z,\mu)Bz - A_1(h(z,\mu)) = f_2^2(z,0,\mu)$ (see [3]). For $h_k^i(z) := h_k^i(z,0)$ $(i=1,2)$,

$$\begin{aligned}&(\langle \tfrac{1}{2}DF_2((\beta_1^1,\beta_1^2)[\Phi z])(\textstyle\sum_{k>1}(h_k^1\beta_k^1 + h_k^2\beta_k^2))(z),\beta_1^i\rangle)_{i=1}^2\\ &= \Big(\langle \Upsilon_b^{a_0} f''(0)(e^{it_1^1\theta}z_1 + e^{-it_1^1\theta}z_2)\gamma_1 \textstyle\sum_{k>1}\gamma_k(h_k^1(z),h_k^2(z))^T,\beta_1^i\rangle\Big)_{i=1}^2\\ &= \textstyle\sum_{k>1}(e^{-it_1^1\tau}z_1 + e^{it_1^1\tau}z_2)\alpha_k f''(0)\Upsilon_b^{a_0}(h_k^1(z)(-\tau),h_k^2(z)(-\tau))^T.\end{aligned}$$

Now, we need to compute $h_k^1(z),\ h_k^2(z)$ in (17) by solving $(M_2^2 h)(z,0) = f_2^2(z,0,0)$,

$$f_2^2(z,0,0) = X_0F_2\left((\beta_1^1,\beta_1^2)[\Phi z]\right) - (\beta_1^1,\beta_1^2)\left[\Phi\Psi(0)\left(\langle F_2\left((\beta_1^1,\beta_1^2)[\Phi z],0\right),\beta_1^i\rangle\right)_{i=1}^2\right].$$

On the other hand, since $(M_2^2h)(z,0) = D_zh(z)Bz - A_1h(z)$, then for $k>1, i=1,2$,

$$\begin{cases} D_z h_k^i Bz - \dot h_k^i = 0\\ -L_k^i(h_k^1,h_k^2) + k^2 h_k^i(0) + \dot h_k^i(0) = (e^{-it_1^1\tau}z_1 + e^{it_1^1\tau}z_2)^2\widetilde{\alpha}_k\end{cases} \tag{18}$$

where $\widetilde{\alpha}_k = C_1^1\alpha_k f''(0)$, $\alpha_k = \langle\gamma_1^2,\gamma_k\rangle = \langle\gamma_1\gamma_k,\gamma_1\rangle$ and

$$L_k(h_k^1,h_k^2) = \Upsilon_b^{a_0}(h_k^1(-\tau),h_k^1(-\tau))^T - (h_k^1(0),h_k^1(0))^T = (L_k^1(h_k^1,h_k^2),L_k^2(h_k^1,h_k^2))^T,$$

$\dot h_k^i(z)(0) = \frac{d}{d\theta}h_k^i(z)(\theta)|_{\theta=0}$. Starting from the lowest order, we set $h_k^i(z) = h_{20,k}^i z_1^2 + h_{11,k}^i z_1z_2 + h_{02,k}^i z_2^2$. Solving (18), we have $h_{11,k}^i = 2\widetilde{\alpha}_k/[k^2+1-C_1^1] := C_{0,k}$, $h_{20,k}^i = e^{2t_1^1 i\theta}\widetilde{\alpha}_k/[e^{2t_1^1\tau i}(2t_1^1 i + k^2 + 1) - C_1^1] := C_{1,k}e^{2t_1^1 i\theta}$ and $h_{02,k}^i = \overline{h}_{20,k}^i$. After obtaining $h_k^i (i=1,2)$ and substituting $h(z,\mu) = \sum_{k\geq 1}(h_k^1\beta_k^1 + h_k^2\beta_k^2)$ into (17), then

$$\begin{aligned}&\overline{f}_3^1(z,0)\\ &= \overline{f}_{0,3}^1(z,0) + 3\Psi(0)\left[\sum_{k>1} f''(0)\alpha_k\Upsilon_b^{a_0}(h_k^1,h_k^2)^T(e^{it_1^1\theta}z_1 + e^{-it_1^1\theta}z_2)(-\tau)\right]\\ &= \overline{f}_{03}^1(z,0) + 6[\sum_{k>1} f''(0)\alpha_k C_1^1((C_{0,k}e^{-it_1^1\tau} + C_{1,k}e^{it_1^1\tau})z_1^2z_2\\ &\quad + C_{0,k}e^{it_1^1\tau} + \overline{C}_{1,k}e^{-it_1^1\tau})z_2^2z_1)](D_1,\overline{D}_1)^T + h.o.t.\end{aligned} \tag{19}$$

Thus, with $K_5 = \widetilde{\alpha}_k\sum_{k>1}(C_{0,k}e^{-it_1^1\tau}\langle\gamma_1^3,\gamma_1\rangle + C_{1,k}e^{it_1^1\tau})D_1,\ g_3^1(z,0,0) = g_{03}^1(z,0,0) + (6K_5z_1^2z_2, 6\overline{K}_5z_2^2z_1)^T$, and we completed the proof. ∎

In fact, the normal form of the associated FDE in polar coordinate has the same form as (13) with corresponding value of t_1^1. Then the normal form of PFDE in polar coordinate is

$$\begin{cases} \dot{\rho} = Re(e^{-it_1^1\tau}D_1)\mu\rho + Re(K_1+K_5)\rho^3 + h.o.t \\ \dot{\xi} = -t_1^1 + h.o.t. \end{cases}$$

Case 3. Hopf-zero bifurcation Let $a = a_0+\mu$, $b = b_0+\nu$, a_0, b_0 satisfy $a_0 + b_0 = C_1^1$ and $a_0 - b_0 = C_0^1$. Then $\Lambda = \{\pm it_1^1, 0\}$ and in (4), $L(\widehat{\varphi}) = -\widehat{\varphi}(0) + \Upsilon_{b_0}^{a_0}\widehat{\varphi}(-\tau)$ and $F(\widehat{\varphi}) = \Upsilon_\nu^\mu\widehat{\varphi}(-\tau) + \Upsilon_b^a\sum_{j\geq 2} f^{(j)}(0)\widehat{\varphi}^j(-\tau)/j!$.

In this case, the associated FDE is in the form of (3) in which $R(\varphi) = M_1\varphi(0) + L_1(\varphi)$ with $L_1(\cdot)$ defined in (5), $M_1 = -I$ and $G(\varphi)$ is defined in (6) with Υ_0^μ replaced by Υ_ν^μ. With the same procedure as that in Case 2, it is easy to verify that (H5') fails. As for the relationship between the normal forms of PFDE (4) and its associated FDE, similar to Theorem 3.1, we have the following result, the proof is similar to that of Theorem 3.1 and we omit it here.

Theorem 3.2. *Let $\dot{z} = Bz + g_2^1(z,0,\mu,\nu)/2 + \cdots$ and $\dot{z} = Bz + g_{0,2}^1(z,0,\mu,\nu)/2 + \cdots$, where $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, $B = diag\{it_1^1, -it_1^1, 0\}$, be normal forms in complex coordinates on the center manifold at zero for PFDE (4) and its associated FDE respectively. Then, the normal form of PFDE is:*

$$\dot{z} = Bz + \frac{1}{2}g_{0,2}^1(z,0,\mu,\nu) + \frac{1}{3!}g_{0,3}^1(z,0,0,0) + \begin{pmatrix} K_5 z_1^2 z_2 + K_6 z_3^2 z_1 \\ \overline{K}_5 z_2^2 z_1 + \overline{K}_6 z_3^2 z_2 \\ K_7 z_3^3 + K_8 z_1 z_2 z_3 \end{pmatrix} + h.o.t,$$

where K_5, $C_{0,k}$ and $C_{1,k}$ are given in Theorem 3.1, and

$$K_6 = \sum_{k>1}\widetilde{\alpha}_k D_1(C_{0,k}e^{-it_1^1\tau}/2 + C_{2,k}), \qquad K_7 = \frac{1}{2}\sum_{k>1} f''(0)\alpha_k D_2 C_0^1 C_{0,k},$$

$$K_8 = \sum_{k>1} f''(0)\alpha_k D_2 C_0^1(2Re(C_{2,k}) + C_{0,k}), \ C_{2,k} = 2f''(0)\alpha_k/[(t_1^1 i + k^2 + 1)e^{t_1^1\tau i}/C_0^1 - 1].$$

According to the result in Section 2, the normal form of associated FDE has the same form as (16) in cylindrical coordinates. Then the normal form of PFDE in cylindrical coordinates is

$$\begin{cases} \dot{\rho} = (\mu+\nu)Re(D_1 e^{-it_1^1\tau})\rho + Re(K_1+K_5)\rho^3 + Re(K_3+K_6)z^2\rho + h.o.t \\ \dot{\theta} = -t_1^1 + h.o.t \\ \dot{z} = 2D_2(\mu-\nu)z + (K_2+K_8)z\rho^2 + (K_4+K_7)z^3 + h.o.t. \end{cases}$$

4. **Conclusion and remarks.** In this paper, sufficient conditions for ascertaining local stability around the trivial equilibrium have been derived through a detailed analysis of the mathematical properties of the system of PFDE (1) under two boundary conditions. The stability depends on the connection of the coefficients of the nonlinear feedback functions, $a+b$ and $a-b$. When the coefficients varies, the qualitative property of solutions changes and bifurcation occur. The discussion of the bifurcations of (1) is related to the discussion of its associated FDE. Under Neumann boundary condition, the normal form of (1) is the same as that of its associated FDE at least up to the third order term while under Dirichlet boundary condition, the norm form of (1) is different from that of its associated FDE from the third order term.

It is worth to consider a certain region of parameters such that there are only center manifold and stable manifold near the trivial solution since beyond this region, the system is unstable. At the critical points we obtain three kinds of bifurcations under both boundary conditions. Similar results and more codimension-two bifurcations were obtained in [5] by numerical simulations and choosing different time delays. We can see that the effect of diffusion in this model is not strong enough to change the qualitative properties of solutions while the effect of parameters is still notable. We have to mention that since $N = 1$ in the considered region under each boundary condition, it means that only one of the equations $\det\triangle_k(\lambda) = 0$ has simple eigenvalues with zero real parts. Moreover, since the set of simple eigenvalues with zero real parts of $\det\triangle_k(\lambda) = 0$ has the same structure as that in the system without diffusion, it implies that the impact of diffusion is weak and it may only change the dynamical properties quantitatively (such as in Theorem 3.1 and Theorem 3.2, only the coefficients are changed) instead of qualitative change. When $N > 1$, the effect of diffusion on the system may be strengthened and more dynamical phenomenon are expected.

REFERENCES

[1] T.Faria, *Bifurcation aspects for some delayed population models with diffusion*, Fields Instit. Commun. **21** (1999), 143–158.

[2] T.Faria, *Normal form and Hopf bifurcation for partial differential equations with delays*, Trans. Am. Math. Soc. **352** (2000), 2217–2238.

[3] T.Faria and L.T.Magalhaes, *Normal forms for retarded functional differential equations with parameters and applications to Hopf bifurcation*, J.Differential Equations, **122** (1995), 181–200.

[4] J.K.Hale. and S.M.Verduyn-Lunel, "Introduction to Functional Differential Equations," Springer-Verlag, New York, 1993.

[5] L.P. Shayer and S.A.Campbell, *Stability, bifurcation, and multistability in a system of two coupled neurons with multiple time delays*, SIAM J. Applied Math., **61** (2000), 673–700.

[6] J.Wu, "Theory and Applications of Partial Functional Differential Equations," Springer-Verlag, New York, 1996.

[7] Y. Yuan and S.A.Campbell, *Stability and synchronization of a ring of identical cells with delayed coupling*, J. Dynam. Differential Equations. **16** (2004), 709–744.

[8] Y. Yuan and J. Wei, *Multiple bifurcation analysis in a neural network model with delays*, Inter. J. Bifur. Chaos Appl. Sci. Engrg. **16** (2006), 2903–2913.

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