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POSITIVE SOLUTIONS OF NONLOCAL BOUNDARY VALUE PROBLEMS WITH SINGULARITIES

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Abstract. In this note we discuss the existence of positive solutions for some nonlocal boundary value problem where the boundary conditions involve linear functionals on the space $C[0,1]$ and the involved nonlinearity might be singular. Our main ingredient is the theory of fixed point index.

1. **Introduction.** Second order boundary value problems (BVPs) where the involved nonlinearity has a singularity either in the dependent variable or in the independent variable, have been studied by several authors, see for example [1, 2, 4, 18, 20, 21, 22, 24, 29, 35] and references therein. Here we establish new results on the existence of positive solutions for the second order differential equation

$$-u''(t) = g(t)f(t, u(t)),\ t \in (0,1), \tag{1}$$

where g is a non-negative L^1-function and f is a non-negative function, allowed to be singular in its second variable, subject to the nonlocal boundary conditions (BCs)

$$u'(0) + \alpha[u] = 0,\ \sigma u'(1) + u(\eta) = \beta[u],\ \eta \in [0,1]. \tag{2}$$

Here $\alpha[u],\ \beta[u]$ are bounded linear functionals on $C[0,1]$ given by

$$\alpha[u] = \int_0^1 u(s)\, dA(s),\quad \beta[u] = \int_0^1 u(s)\, dB(s), \tag{3}$$

involving Stieltjes integrals with *signed* measures, that is, A, B are functions of bounded variation. This type of BCs are quite general and include, as special cases, the well-known multi-point and integral BCs, that have been studied by several authors, see for example [7, 8, 9, 13, 15, 16, 17, 23, 25, 26, 27, 32, 34, 36, 37] and the references therein.

One motivation for studying (1)-(2) is that this type of BVP arises in some heat flow problems. Infante and Webb [14, 30, 31], motivated by an earlier work of Guidotti and Merino [5], studied the BCs

$$u'(0) = 0,\ \sigma u'(1) + u(\eta) = 0,$$

that model a heated bar of length 1 with a thermostat, where a controller at $t = 1$ adds or removes heat according to the temperature detected by a sensor at $t = \eta$.

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Infante and Webb [10, 11, 15] and Palamides, Infante and Pietramala [28] studied the existence of solutions that are positive (or sign-changing) under the more general BCs

$$u'(0) + \alpha[u] = 0, \ \sigma u'(1) + u(\eta) = 0.$$

A common feature in [10, 11, 14, 15, 30, 31] is that classical fixed point index theory is used, whereas in [28] Sperner's Lemma is the main ingredient.

The case of nonzero terms $\alpha[u]$ and $\beta[u]$, also has a physical interpretation, for example when $\alpha[u] = \alpha u(\xi)$ and $\beta[u] = \beta u(\tau)$, the five point BCs

$$u'(0) + \alpha u(\xi) = 0, \ \sigma u'(1) + u(\eta) = \beta u(\tau), \ \xi, \eta, \tau \in [0, 1], \tag{4}$$

can be seen as a model for a heated bar, this time with two controllers at $t = 0$ and $t = 1$, and three sensors at $t = \xi$, $t = \eta$ and $t = \tau$.

Here we prove the existence of multiple positive solutions of (1)-(2) under suitable oscillatory behavior of f. Our approach is to rewrite the BVP (1)-(2) as a perturbed Hammerstein integral equation of the form

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t, s)g(s)f(s, u(s))\, ds \tag{5}$$

where $\alpha[u]$ and $\beta[u]$ are positive functionals, γ and δ are positive continuous functions and f is a positive function that is allowed to have singularities. This type of integral equation, with f non singular, has been studied recently by Infante and Webb [15, 33, 34]. Lan [22] studied the Hammerstein case, that corresponds to $\gamma(t)\alpha[u] = \delta(t)\beta[u] = 0$, but this time with a singular f. Infante [10], using results from [15, 22], studied the singular case under one perturbation only (i.e. $\delta(t)\beta[u] = 0$), where $\alpha[u]$ is an affine functional of the form

$$\alpha[u] = A_0 + \int_0^1 u(s)\, dA(s)$$

and $dA(s)$ is a positive measure. The cone used in [10] is

$$K_0 = \big\{u \in C[0, 1] : \min_{t\in[0,1]} u(t) \geq c\|u\|\big\}, \tag{6}$$

a type of cone first used by Krasnosel'skiĭ, see e.g. [19], and D. Guo, see e.g. [6].

Our main ingredient is the classical fixed point index and we work in a cone smaller than K_0, that takes into account the two functionals involved in the BCs.

We make use of results and ideas from the papers [10, 22, 33].

2. **Positive solutions of perturbed Hammerstein integral equations with singularities.** We are interested in finding positive solutions of the integral equation

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t, s)g(s)f(s, u(s))\, ds := Tu(t). \tag{7}$$

We achieve this by seeking fixed points of an auxiliary operator $\tilde{T}$ in a suitable cone in $C[0, 1]$. Our main assumptions on the terms occurring in (7) are the following:

(C_1) There exist constants $0 \leq r_1 < r_2$ such that $f : [0, 1] \times [r_1, r_2] \to [0, \infty)$ is continuous.

(C_2) $k : [0, 1] \times [0, 1] \to [0, \infty)$ is continuous.

(C_3) There exist a measurable function $\Phi : [0, 1] \to [0, \infty)$ and a constant $c_1 \in (0, 1]$ such that

$$c_1\Phi(s) \leq k(t, s) \leq \Phi(s) \text{ for } t \in [0, 1] \text{ and almost every } s \in [0, 1].$$

(C_4) $g\,\Phi \in L^1[0,1]$, $g \geq 0$ almost everywhere, and $\int_0^1 \Phi(s)g(s)\,ds > 0$.
(C_5) A, B are of bounded variation and $\mathcal{K}_A(s),\ \mathcal{K}_B(s) \geq 0$ for almost every s, where

$$\mathcal{K}_A(s) := \int_0^1 k(t,s)\,dA(t) \text{ and } \mathcal{K}_B(s) := \int_0^1 k(t,s)\,dB(t).$$

(C_6) $\gamma \in C[0,1],\ \gamma(t) \geq 0,\ \ 0 \leq \alpha[\gamma] < 1,\ \ \beta[\gamma] \geq 0$.
There exists $c_2 \in (0,1]$ such that $\gamma(t) \geq c_2\|\gamma\|$ for $t \in [0,1]$.
(C_7) $\delta \in C[0,1],\ \delta(t) \geq 0,\ \ 0 \leq \beta[\delta] < 1,\ \ \alpha[\delta] \geq 0$.
There exists $c_3 \in (0,1]$ such that $\delta(t) \geq c_3\|\delta\|$ for $t \in [0,1]$.
(C_8) $D := (1-\alpha[\gamma])(1-\beta[\delta]) - \alpha[\delta]\beta[\gamma] > 0$.

We set $c = \min\{c_1, c_2, c_3\}$ and use classical fixed point index for compact maps (see for example [3] or [6]) on the cone, introduced in [33],

$$K = \big\{u \in C[0,1] : \min_{t\in[0,1]} u(t) \geq c\|u\|,\ \alpha[u] \geq 0,\ \beta[u] \geq 0\big\}. \tag{8}$$

In order to use the results of [33], we extend f to all of $[0,1]\times[0,\infty)$ in a similar way to that of Lan [22]. We define $\tilde{f} : [0,1]\times[0,\infty) \to [0,\infty)$ as

$$\tilde{f}(t,u) := \begin{cases} f(t,r_1) & \text{if } 0 \leq u \leq r_1,\\ f(t,u) & \text{if } r_1 \leq u \leq r_2,\\ f(t,r_2) & \text{if } r_2 \leq u < \infty,\end{cases}$$

and consider the operator

$$\tilde{T}u(t) := \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t,s)g(s)\tilde{f}(s,u(s))\,ds. \tag{9}$$

By construction

$$Tu = \tilde{T}u \text{ for } u \in K(r_1,r_2),$$

where

$$K(r_1,r_2) = \{u \in K : r_1 \leq u(t) \leq r_2 \text{ for } t \in [0,1]\}.$$

We now look for fixed points of $\tilde{T}$ in $K(r_1,r_2)$ to find solutions of (7). First of all we note that $\tilde{T} : K \to K$ is compact, that is, $\tilde{T}$ is continuous and maps bounded sets in precompact sets.

Theorem 2.1. [33] *Assume that (C_1)-(C_8) hold. Then $\tilde{T}$ maps K into K and is compact.*

Definition 2.2. We write $K_r = \{u \in K : \|u\| < r\}$, $\overline{K}_r = \{u \in K : \|u\| \leq r\}$ and make use of the open set

$$V_\rho = \big\{u \in K : \min_{t\in[0,1]} u(t) < \rho\big\}.$$

The set V_ρ was introduced in [15] and is equal to the set called $\Omega_{\rho/c}$ in [21]. One advantage of using V_ρ is that it makes clearer that choosing c as large as possible provides a weaker condition to be satisfied by the nonlinearity $\tilde{f}$ in Lemma 2.3.

The following two Lemmas are special cases of Theorem 3.3 of [33]. The first one gives conditions which imply that the fixed point index is 0.

Lemma 2.3. *Assume that there exist* $\rho > 0$ *such that*

$$\tilde{f}_{\rho,\rho/c}\Big(\big(\frac{c_2\|\gamma\|}{D}(1-\beta[\delta])+\frac{c_3\|\delta\|}{D}\beta[\gamma]\big)\int_0^1 \mathcal{K}_A(s)g(s)\,ds + \big(\frac{c_2\|\gamma\|}{D}\alpha[\delta]+\frac{c_3\|\delta\|}{D}(1-\alpha[\gamma])\big)\int_0^1 \mathcal{K}_B(s)g(s)\,ds+\frac{1}{M}\Big)>1, \quad (10)$$

where

$$\tilde{f}_{\rho,\rho/c}=\inf\Big\{\frac{\tilde{f}(t,u)}{\rho}:\ (t,u)\in[0,1]\times[\rho,\rho/c]\Big\} \text{ and } \frac{1}{M}=\inf_{t\in[0,1]}\int_0^1 k(t,s)g(s)\,ds. \quad (11)$$

Then the fixed point index, $i_K(\tilde{T},V_\rho)$, *is* 0.

The second result implies that the index is 1.

Lemma 2.4. *Assume that there exists* $\rho > 0$ *such that*

$$\tilde{f}^{0,\rho}\Big(\big(\frac{\|\gamma\|}{D}(1-\beta[\delta])+\frac{\|\delta\|}{D}\beta[\gamma]\big)\int_0^1 \mathcal{K}_A(s)g(s)\,ds + \big(\frac{\|\gamma\|}{D}\alpha[\delta]+\frac{\|\delta\|}{D}(1-\alpha[\gamma])\big)\int_0^1 \mathcal{K}_B(s)g(s)\,ds+\frac{1}{m}\Big)<1, \quad (12)$$

where

$$\tilde{f}^{0,\rho}=\sup\Big\{\frac{\tilde{f}(t,u)}{\rho}:\ (t,u)\in[0,1]\times[0,\rho]\Big\} \text{ and } \frac{1}{m}=\sup_{t\in[0,1]}\int_0^1 k(t,s)g(s)\,ds. \quad (13)$$

Then we have $i_K(\tilde{T},K_\rho)=1$.

The above results valid for $\tilde{T}$ lead to the following new result on existence of multiple positive solutions for Eq. (7).

Theorem 2.5. *Eq. (7) has one positive solution in* $K(r_1,r_2)$ *if either of the following conditions hold.*

(H_1) *There exist* ρ_1,ρ_2 *with* $r_1\le c\rho_1<\rho_1<\rho_2\le cr_2$ *such that* (12) *is satisfied for* ρ_1 *and* (10) *is satisfied for* ρ_2.

(H_2) *There exist* ρ_1,ρ_2 *with* $r_1\le\rho_1<c\rho_2<\rho_2\le r_2$ *such that* (10) *is satisfied for* ρ_1 *and* (12) *is satisfied for* ρ_2.

Eq. (7) has two positive solutions in $K(r_1,r_2)$ *if one of the following conditions hold.*

(S_1) *There exist* ρ_1,ρ_2,ρ_3 *with* $r_1\le c\rho_1<\rho_1<\rho_2<c\rho_3<\rho_3\le r_2$ *such that* (12) *is satisfied for* ρ_1, (10) *is satisfied for* ρ_2 *and* (12) *is satisfied for* ρ_3.

(S_2) *There exist* ρ_1,ρ_2,ρ_3 *with* $r_1\le\rho_1<c\rho_2<c\rho_3<\rho_3\le cr_2$ *such that* (10) *is satisfied for* ρ_1, (12) *is satisfied for* ρ_2 *and* (10) *is satisfied for* ρ_3.

Proof. Suppose (H_1) holds. Then it follows from properties of fixed point index (see for example [12, 20]), that the map $\tilde{T}$ has a fixed point u in $V_{\rho_2}\backslash K_{\rho_1}$. Then $\rho_1\le\|u\|\le\rho_2/c$ and $c\rho_1\le u(t)\le\rho_2/c$ for $t\in[0,1]$, hence $u\in K(r_1,r_2)$. Since $Tu=\tilde{T}u$ for $u\in K(r_1,r_2)$, Eq. (7) has one positive solution in $K(r_1,r_2)$. The other assertions are proved similarly. □

Remark 1. It is possible to state, by similar arguments, results for three or more positive solutions, we refer the reader to [21] for the type of results that may be stated.

3. **The Boundary Value Problem.** We now consider the BVP

$$-u''(t) = g(t)f(t,u(t)),\ \text{a.e. on } [0,1], \tag{14}$$

with boundary conditions

$$u'(0) + \alpha[u] = 0,\ \sigma u'(1) + u(\eta) = \beta[u],\ \eta \in [0,1]. \tag{15}$$

The solution of $-u'' = y$ under these BCs can be written

$$u(t) = (\sigma + \eta - t)\alpha[u] + \beta[u] + \sigma \int_0^1 y(s)ds + \int_0^\eta (\eta - s)y(s)ds - \int_0^t (t-s)y(s)ds.$$

By a solution of the BVP (14)-(15) we mean a solution $u \in C[0,1]$ of the corresponding integral equation

$$u(t) = (\sigma + \eta - t)\alpha[u] + \beta[u] + \int_0^1 k(t,s)g(s)f(s,u(s))ds,$$

where

$$k(t,s) = \sigma + \begin{cases} \eta - s, & s \le \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t - s, & s \le t \\ 0, & s > t. \end{cases} \tag{16}$$

Note that $k(t,s)$ in (16) is the Green's function associated to the BCs

$$u'(0) = 0,\ \sigma u'(1) + u(\eta) = 0,$$

studied by Infante and Webb in [14]. Here we discuss the case $\sigma + \eta > 1$, that leads to the existence of *positive solutions*.
Upper and lower bounds for $k(t,s)$ and $\gamma(t)$ were given in [10, 14] as follows:

$$\|\gamma\| = \sigma + \eta, \quad \Phi(s) = \begin{cases} \sigma, & s > \eta, \\ \sigma + \eta - s, & s \le \eta, \end{cases} \quad c_1 = c_2 = 1 - \frac{1}{\sigma + \eta}. \tag{17}$$

Note that $\delta(t) \equiv 1$, so $\|\delta\| = 1$ and one may take $c_3 = 1$.
This leads to the choice

$$c = 1 - \frac{1}{\sigma + \eta}. \tag{18}$$

Hence we work on the cone

$$K = \{u \in C[0,1] : \min_{t \in [0,1]} u(t) \ge c\|u\|,\ \alpha[u] \ge 0,\ \beta[u] \ge 0\},$$

with c as in (18).

We state a result for the existence of one positive solution, of course there are more general results, analogous to Theorem 2.5 and Remark 1.

Theorem 3.1. *Let $c_1 - c_3$ as above, m as in (13) and M as in (11). Then the BVP (14)-(15) has at least one positive solution, if either (H_1) or (H_2) of Theorem 2.5 hold.*

Remark 2. Consider the BVP

$$-u''(t) = f(t,u(t)),\ \text{a.e. on } [0,1], \tag{19}$$

with BCs (4), that is,

$$u'(0) + \alpha u(\xi) = 0,\ \sigma u'(1) + u(\eta) = \beta u(\tau), \tag{20}$$

where $\xi, \eta, \tau \in [0,1]$ and $\xi < \eta < \tau$. In this case $g \equiv 1$ and we may take $dA(s)$ the Dirac measure of weight $\alpha > 0$ at ξ and $dB(s)$ the Dirac measure of weight $\beta > 0$ at τ.

A direct calculation gives

$$\frac{1}{m} = \sigma + \eta^2/2 \quad \text{and} \quad \frac{1}{M} = (2\sigma - 1 + \eta^2)/2. \tag{21}$$

We use this specific value of the constant M, rather than the one given by Webb in [30], since here, for our technique to work, we have fixed $[a, b] = [0, 1]$. For the same reason, in [10] the correct M to use is the one in (21).

To verify (C_5) we need

$$\mathcal{K}_A(s) := \int_0^1 k(t, s)dA(t) \geq 0 \;\text{ and }\; \mathcal{K}_B(s) := \int_0^1 k(t, s)dB(t) \geq 0,$$

that is

$$\alpha k(\xi, s) \geq 0 \;\text{ and }\; \beta k(\tau, s) \geq 0.$$

So A, B should satisfy

$$\begin{array}{lll} \alpha(\sigma+\eta-\xi) \geq 0, & s \leq \xi, \\ \alpha(\sigma+\eta-s) \geq 0, & \xi < s \leq \eta, \\ \alpha\sigma \geq 0, & s > \eta, \end{array} \quad \text{and} \quad \begin{array}{ll} \beta(\sigma+\eta-\tau) \geq 0, & s \leq \eta, \\ \beta(\sigma-\tau+s) \geq 0, & \eta < s \leq \tau, \\ \beta\sigma \geq 0, & s > \tau. \end{array}$$

Thus, we require $\alpha, \beta \geq 0$. We also need

$$0 \leq \alpha[\gamma] < 1, \;\; \alpha[\delta] \geq 0, \;\; 0 \leq \beta[\delta] < 1, \;\; \beta[\gamma] \geq 0, \quad \text{and} \;\; D > 0.$$

For our BVP we have

$$\alpha[\gamma] = \alpha(\sigma + \eta - \xi), \; \alpha[\delta] = \alpha, \; \beta[\gamma] = \beta(\sigma + \eta - \tau), \; \beta[\delta] = \beta,$$

and therefore the total requirement is

$$\begin{aligned} &0 \leq \alpha, \;\; \alpha(\sigma + \eta - \xi) < 1, \;\; 0 \leq \beta < 1, \\ &0 < 1 + \alpha\beta(\tau - \xi) - [\beta + \alpha(\sigma + \eta - \xi)]. \end{aligned}$$

Furthermore, we have

$$\int_0^1 \mathcal{K}_A(s)g(s)\,ds = \frac{1}{2}\alpha(2\sigma + \eta^2 - \xi^2), \;\text{ and }\; \int_0^1 \mathcal{K}_B(s)g(s)\,ds = \frac{1}{2}\beta(2\sigma + \eta^2 - \tau^2).$$

Thus all the ingredients that appear in (10) and (12), needed for our index calculations, can be computed. In the next example we provide explicit constants.

Example 1. Fix $\xi = 1/3$, $\eta = 1/2$, $\tau = \sigma = 2/3$ and $\alpha = \beta = 1/4$. This gives $c_2 = 1/7$, $m = 24/19$ and $M = 24/7$. Then $i_K(\tilde{T}, V_\rho) = 0$ condition needs $\tilde{f}_{\rho,\rho/c} \geq 243/142$ and $i_K(\tilde{T}, K_\rho) = 1$ requires $\tilde{f}^{0,\rho} \leq 486/677$.

Note that, as observed in [10], with the same technique we can handle nonlinearities with more than one singularity.

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