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EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR A NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract. We give sufficient conditions on the value $\tau \in (0,T]$ such that the nonlinear fractional boundary value problem

$$D^{\alpha}_{0^+}u(t) + f(t,u(t)) = 0, \quad t \in (0,\tau),$$
$$I^{\gamma}u(0^+) = 0, \quad I^{\beta}u(\tau) = 0,$$

where $1-\alpha < \gamma \leq 2-\alpha$, $2-\alpha < \beta < 0$, D^{α}_{0+} is the Riemann-Liouville differential operator of order α, and $f \in C([0,T]\times\mathbb{R})$ is nonnegative, has a positive solution. We also present a nonexistence result.

1. **Introduction.** While much attention has focused on the Cauchy problem for fractional differential equations for both the Reimann-Liouville and Caputo differential operators, see [5, 4, 7, 11, 12, 13, 15] and references therein, there are few papers devoted to the study of fractional order boundary value problems, see for example [1, 2, 3, 6, 8, 10, 14, 16, 17]. The paper by Kosmatov [10] demonstrates an eloquent means of inverting an fractional order boundary value problem that we mimic in this manuscript.

The history, definitions, theory, and applications of fractional calculus are well laid out in the books by Miller and Ross [11], Oldham and Spanier [12], Podlubny [13], and Samko, Kilbas, and Marichev [15]. In particular, the book by Oldham and Spanier [12] has a chronological listing on major works in the study of fractional calculus starting with the correspondence between Leibnitz and L'Hospital in the late seventeenth century and continuing to 1974. Additionally, the web site http://people.tuke.sk/igor.podlubny/fc.html, authored by I. Podlubny, is a very useful resource for those studying fractional calculus and its application.

For $u \in L^p[0,T], 1 \leq p < \infty$, the Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$I^{\alpha}_{0^+}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds.$$

If $n-1 \leq \alpha < n$ then the Riemann-Liouville fractional derivative of order α is defined by

$$D^{\alpha}_{0^+}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)\,ds.$$

For $\alpha < 0$, we will sometimes use the notation $I^{\alpha}_{0^+} = D^{-\alpha}_{0^+}$.

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Let $T > 0$ and let $\alpha \in (2,3)$. We seek $\tau \in (0,T)$ such that there exists a positive solution $u \in C[0,\tau]$ of

$$D^{\alpha}_{0+}u(t) + f(t,u(t)) = 0, \quad t \in (0,\tau), \tag{1}$$

satisfying

$$I^{\gamma}u(0^+) = 0, \quad I^{\beta}u(\tau) = 0, \tag{2}$$

where $1-\alpha < \gamma \le 2-\alpha$ and $2-\alpha < \beta < 0$. Throughout the paper we will assume the following condition holds.

(H1) $f \in C([0,T]\times\mathbb{R})$ is nonnegative and that $f(t,0)$ does not vanish identically on any compact subinterval of $[0,T]$.

In the remainder of this section we present some fundamental results of fractional calculus that will be used later in the paper. For more information on fractional calculus we refer the reader to the monographs [11, 12, 13, 15].

It is well known that if $n-1 \le \alpha < n$, then $D^{\alpha}_{0+}t^{\alpha-k} = 0$, $k = 1,2,\ldots,n$. Furthermore, if $u \in L^1[0,T]$ and $\alpha > 0$, then

$$D^{\alpha}_{0+}I^{\alpha}_{0+}u(t) = u(t). \tag{3}$$

Of particular importance to us is the semi-group property

$$I^{\delta}_{0+}I^{\alpha}_{0+}u(t) = I^{\delta+\alpha}_{0+}u(t), \quad \text{if } \delta+\alpha > 0. \tag{4}$$

If $D^{\alpha}_{0+}u \in L^1[0,T]$, then

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) - \sum_{j=1}^{k}\left[D^{\alpha-j}_{0+}u(t)\right]_{t=0}\frac{t^{\alpha-j}}{\Gamma(\alpha-j+1)}$$

Also, if $u \in C[0,T]$ and $n-1 \le \alpha < n$, then

$$\begin{aligned}\lim_{t\to 0^+}\left(D^{\alpha-n}_{0+}u\right)(t) &= \lim_{t\to 0^+}\left(I^{n-\alpha}_{0+}u\right)(t)\\ &= \lim_{t\to 0^+}\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-1-\alpha}u(s)\,ds\\ &= 0.\end{aligned}$$

and consequently if $u \in C^m[0,T]$, then $D^{\alpha-n+m}_{0+}u(0) = 0$.

In Section 2, we use the above properties to find an equivalent integral operator to (1), (2). We also state the fixed point theorems that we employ to find solutions. In Section 3, we present our main results.

2. **Preliminaries.**

Let u be integrable and let $\alpha \in (2,3)$. Then,

$$\begin{aligned}I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) =& u(t) - D^{\alpha-1}_{0+}u(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)} - D^{\alpha-2}_{0+}u(0)\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\\ &- D^{\alpha-3}_{0+}u(0)\frac{t^{\alpha-3}}{\Gamma(\alpha-2)}.\end{aligned} \tag{5}$$

For $u \in C[0,\tau]$, we have $D^{\alpha-3}_{0+}u(0) = 0$. In this case, (5) becomes

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) - D^{\alpha-1}_{0+}u(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)} - D^{\alpha-2}_{0+}u(0)\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}. \tag{6}$$

Let $g(t) = D^{\alpha}_{0+}u(t) \in C[0,\tau]$, and rewrite (6) as

$$u(t) = D^{\alpha-1}_{0+}u(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)} + D^{\alpha-2}_{0+}u(0)\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} + I^{\alpha}_{0+}g(t)$$

Since $1-\alpha<\gamma\leq 2-\alpha$, we have by the semi-group property (4),

$$I_{0^+}^{\gamma}u(t)=D_{0^+}^{\alpha-1}u(0)\frac{t^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+D_{0^+}^{\alpha-2}u(0)\frac{t^{\alpha+\gamma-2}}{\Gamma(\alpha+\gamma-1)}+I_{0^+}^{\gamma+\alpha}g(t).$$

Note that

$$\lim_{t\to 0^+}I_{0^+}^{\gamma+\alpha}g(t)=\lim_{t\to 0^+}\frac{1}{\Gamma(\alpha+\gamma)}\int_0^t(t-s)^{\alpha+\gamma-1}g(s)\,ds=0.$$

Since $\lim_{t\to 0^+}I_{0^+}^{\gamma}u(t)=0$, then $D_{0^+}^{\alpha-2}u(0)=0$.

Equation (5) simplifies to

$$u(t)=D_{0^+}^{\alpha-1}u(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)}+I_{0^+}^{\alpha}g(t).$$

Applying the operator $I_{0^+}^{\beta}$ to both sides, we obtain

$$I_{0^+}^{\beta}u(t)=D_{0^+}^{\alpha-1}u(0)\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+I_{0^+}^{\alpha+\beta}g(t).$$

Since $I_{0^+}^{\beta}u(\tau)=0$,

$$D_{0^+}^{\alpha-1}u(0)=-\frac{\Gamma(\alpha+\beta)}{\tau^{\alpha+\beta-1}}I_{0^+}^{\alpha+\beta}g(\tau),$$

and so

$$u(t)=-\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{t^{\alpha-1}}{\tau^{\alpha+\beta-1}}I_{0^+}^{\alpha+\beta}g(\tau)+I_{0^+}^{\alpha}g(t). \tag{7}$$

Thus, if $u\in C[0,\tau]$ is a solution of (1), (2), then $u\in C[0,\tau]$ is a solution of (7). The converse is also true in view of (3). We summarize this result below.

Lemma 2.1. *Let $1-\alpha<\gamma\leq 2-\alpha$ and $2-\alpha<\beta<0$. Assume (H1) holds. Then $u\in C[0,\tau]$ is a solution of the boundary value problem (1), (2) if and only if $u\in C[0,\tau]$ is a solution of*

$$\begin{aligned} u(t)=&\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{t^{\alpha-1}}{\tau^{\alpha+\beta-1}}\int_0^{\tau}(\tau-s)^{\alpha+\beta-1}f(s,u(s))\,ds\\ &-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))\,ds. \end{aligned} \tag{8}$$

Define $K(t,s;\beta,\tau)$ by

$$K(t,s;\beta,\tau)=\frac{1}{\Gamma(\alpha)}\begin{cases} k_1(t,s;\beta,\tau), & 0\leq s\leq t,\\ k_2(t,s;\beta,\tau), & t<s\leq\tau.\end{cases}$$

where

$$k_1(t,s;\beta,\tau)=\Gamma(\alpha+\beta)t^{\alpha-1}\left(1-\frac{s}{\tau}\right)^{\alpha+\beta-1}-(t-s)^{\alpha-1},$$

$$k_2(t,s;\beta,\tau)=\Gamma(\alpha+\beta)t^{\alpha-1}\left(1-\frac{s}{\tau}\right)^{\alpha+\beta-1}.$$

Lemma 2.2. *Suppose that $2-\alpha<\beta<0$; then the kernel $K(t,s;\beta,\tau)$ satisfies $K(t,s;\beta,\tau)>0$, for all $(t,s)\in(0,\tau)\times(0,\tau)$.*

Proof. It is clear that $k_2(t,s;\beta,\tau) > 0$. We will show that $k_1(t,s;\beta,\tau) > 0$.

$$\begin{aligned} k_1(t,s;\beta,\tau) &= \Gamma(\alpha+\beta)t^{\alpha-1}\left(1-\frac{s}{\tau}\right)^{\alpha+\beta-1} - (t-s)^{\alpha-1} \\ &\geq t^{\alpha-1}\left[\left(1-\frac{s}{\tau}\right)^{\alpha+\beta-1} - \left(1-\frac{s}{t}\right)^{\alpha-1}\right] \\ &\geq t^{\alpha-1}\left(1-\frac{s}{\tau}\right)^{\alpha-1}\left[\left(1-\frac{s}{\tau}\right)^{\beta} - 1\right] > 0. \end{aligned}$$

The proof is complete. ☐

We will seek a fixed point of an operator associated with (8) using the Leray-Schauder Nonlinear Alternative and the Krasnosel'skiĭ-Zabeiko fixed point theorem [9]. For completeness we state these theorems below.

Theorem 2.3 (Leray-Schauder Nonlinear Alternative)**.** *Let X be a normed linear space. Let $\mathcal{C} \subset X$ be a convex set and let U be open in $\mathcal{C}$ such that $0 \in U$. Let $T : \overline{U} \to \mathcal{C}$ be a continuous and compact mapping. Then either*

1. *the mapping T has a fixed point in $\overline{U}$, or*
2. *there exists a $u \in \partial U$ and a $\lambda \in (0,1)$ such that $u = \lambda T u$.*

Theorem 2.4. *Let $\mathcal{B}$ be a Banach Space. Let $T : \mathcal{B} \to \mathcal{B}$ be a completely continuous mapping and let $L : \mathcal{B} \to \mathcal{B}$ be a bounded linear mapping such that 1 is not an eigenvalue of L. Suppose that*

$$\lim_{\|u\|\to\infty} \frac{\|Tu - Lu\|}{\|u\|} = 0. \tag{9}$$

Then T has a fixed point in $\mathcal{B}$.

3. **Existence and Nonexistence of Solutions.** In this section we establish our existence and nonexistence theorems. For a given $\tau \in (0,T]$, define the Banach space $X = (C[0,\tau], \|\cdot\|)$ where $\|u\| = \max_{t\in[0,\tau]} |u(t)|$. Define the operator $T : X \to X$ by

$$\begin{aligned} Tu(t) =& \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{t^{\alpha-1}}{\tau^{\alpha+\beta-1}}\int_0^\tau (\tau-s)^{\alpha+\beta-1} f(s,u(s))\,ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))\,ds. \end{aligned} \tag{10}$$

By Lemma 2.1, if T has a fixed point in $C[0,\tau]$, then this fixed point is a solution of (1), (2). Furthermore, in view of Lemma 2.2 and the properties of f, this fixed point satisfies $u(t) > 0$ for all $t \in [0,\tau]$.

For our first theorem we will assume the following condition holds.

(H2) There exist non-negative functions $a_1, a_2 \in C[0,T]$ such that $f(t,z) \leq a_1(t) + a_2(t)z$, for all $t \in [0,T]$.

Theorem 3.1. *Assume conditions (H1) and (H2) hold. Suppose that $\tau \in (0,T)$ is such that*

$$0 < \frac{1}{\Gamma(\alpha)}\int_0^\tau [\Gamma(\alpha+\beta)\tau^{-\beta}(\tau-s)^{\alpha+\beta-1} + (\tau-s)^{\alpha-1}]a_1(s)\,ds < \infty$$

and

$$0 < \frac{1}{\Gamma(\alpha)}\int_0^\tau [\Gamma(\alpha+\beta)\tau^{-\beta}(\tau-s)^{\alpha+\beta-1} + (\tau-s)^{\alpha-1}]a_2(s)\,ds < 1.$$

Then there exists a positive solution of (1), (2).

Proof. From condition (H1) and the Dominated Convergence Theorem, the operator T is continuous and compact.

Define

$$A \equiv \frac{1}{\Gamma(\alpha)} \int_0^\tau [\Gamma(\alpha+\beta)\tau^{-\beta}(\tau-s)^{\alpha+\beta-1} + (\tau-s)^{\alpha-1}] a_1(s)\, ds$$

and

$$B \equiv \frac{1}{\Gamma(\alpha)} \int_0^\tau [\Gamma(\alpha+\beta)\tau^{-\beta}(\tau-s)^{\alpha+\beta-1} + (\tau-s)^{\alpha-1}] a_2(s)\, ds.$$

Let $U = \{u \in X : \|u\| < R\}$ where $R = \frac{A}{1-B}$. Then,

$$\begin{aligned} |Tu(t)| &\leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{\tau^{\alpha-1}}{\tau^{\alpha+\beta-1}} \int_0^\tau (\tau-s)^{\alpha+\beta-1} f(s,u(s))\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1} f(s,u(s))\, ds \\ &\leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \int_0^\tau \tau^{-\beta}(\tau-s)^{\alpha+\beta-1} a_1(s)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1} a_1(s))\, ds \\ &\quad + \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \int_0^\tau \tau^{-\beta}(\tau-s)^{\alpha+\beta-1} a_2(s)\, \|u\|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1} a_2(s)\, \|u\|\, ds \\ &= A + B\|u\|. \end{aligned}$$

Suppose there exists a $u \in \partial U$ and a $\lambda \in (0,1)$ such that $u = \lambda Tu$. Then for this u and λ, we have

$$R = \|u\| = \lambda \|Tu\| < A + B\|u\| = R,$$

which is a contradiction. By Theorem 2.3, there exists a fixed point $u \in \overline{U}$ of T. This fixed point is a positive solution of (1), (2) and the proof is complete. □

Theorem 3.2. *Suppose that (H1) holds and that*

(H3) $\lim_{|u|\to\infty} \frac{f(t,u)}{|u|} = \phi(t)$ *uniformly in* $[0,T]$.

Suppose that $\tau \in (0,T]$ *is such that*

$$0 < \frac{1}{\Gamma(\alpha)} \int_0^\tau [\Gamma(\alpha+\beta)\tau^{-\beta}(\tau-s)^{\alpha+\beta-1} + (\tau-s)^{\alpha-1}]\, |\phi(s)|\, ds < 1. \tag{11}$$

Then there exists a positive solution of (1), (2).

Proof. Standard arguments can be used to show that T is compact.

Let τ be such that (11) is satisfied. Define an operator $L : X \to X$ by

$$\begin{aligned} Lu(t) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{t^{\alpha-1}}{\tau^{\alpha+\beta-1}} \int_0^\tau (\tau-s)^{\alpha+\beta-1} \varphi(s) u(s)\, ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) u(s)\, ds. \end{aligned}$$

Then L is a bounded linear mapping. Furthermore, if u is such that $u = Lu$ and $u \not\equiv 0$, then

$$\begin{aligned} \|u\| &= \|Lu(t)\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^\tau \big(\Gamma(\alpha+\beta)\tau^{-\beta}(\tau-s)^{\alpha+\beta-1} + (\tau-s)^{\alpha-1}\varphi(s)\big)\, ds\, \|u\| \\ &< \|u\|, \end{aligned}$$

which is a contradiction. Consequently, $\lambda = 1$ is not an eigenvalue of L.

Fix $\varepsilon > 0$. By condition (H3), there exists an $N > 0$ such that if $|z| > N$ then

$$\left| \frac{f(s,z)}{|z|} - \varphi(s) \right| < \varepsilon \tag{12}$$

for all $s \in [0,\tau]$.

Set

$$B = \max\{f(s,z) :\ s \in [0,\tau], |z| \in [0,N]\}.$$

Note that if $|u(s)| \leq N$, then $\big|f(s,u(s)) - \varphi(s)|u(s)|\big| \leq B + \|\varphi\|N$. Pick $M > N$ such that $B + \|\varphi\|N < \varepsilon M$. Let $u \in X$ be such that $\|u\| \geq M$. If $s \in [0,\tau]$ is such that $|u(s)| \leq N$, then

$$\big|f(s,u(s)) - \varphi(s)|u(s)|\big| \leq B + \|\varphi\|N \leq \|u\|\varepsilon.$$

If $s \in [0,\tau]$ is such that $|u(s) > N$, then by (12),

$$\big|f(s,u(s)) - \varphi(s)|u(s)|\big| \leq \|u\|\varepsilon.$$

Hence, for all $s \in [0,\tau]$,

$$\big|f(s,u(s)) - \varphi(s)|u(s)|\big| \leq \|u\|\varepsilon.$$

For $\|u\| > M$, we have

$$\begin{aligned} |Tu(t) &- Lu(t)| \\ &\leq \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{t^{\alpha-1}}{\tau^{\alpha+\beta-1}} \int_0^\tau (\tau-s)^{\alpha+\beta-1} \big|f(s,u(s)) - \varphi(s)|u(s)|\big|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big|f(s,u(s)) - \varphi(s)|u(s)|\big|\, ds \\ &\leq \left(\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} \right) \tau^\alpha \varepsilon \|u\|. \end{aligned}$$

That is, for some constant C, $\|Tu - Lu\| \leq C\varepsilon\|u\|$. Hence, condition (9) of Theorem 2.4 holds. Since all the conditions of Theorem 2.4 hold, there exists a fixed point u of T. This solution u is a positive solution of the boundary value problem (1), (2) and the proof is complete. □

Our last theorem in this paper is a nonexistence theorem.

Theorem 3.3. *Suppose that*

$$\lim_{|z|\to 0^+} \frac{f(t,z)}{|z|} = 0, \qquad \lim_{|z|\to +\infty} \frac{f(t,z)}{|z|} = 0 \tag{13}$$

uniformly on $[0,T]$. *Then there exists an* $R > 0$ *such that if*

$$\left(\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} \right) \tau^\alpha < R \tag{14}$$

then the boundary value problem (1), (2) *has no positive solution.*

Proof. Fix $\varepsilon_1 > 0$. By (13) there exists $N_1 > 0$ and $N_2 > 0$ such that

$$f(t,z) \le |z|\varepsilon_1$$

for all z with $|z| < N_1$ and

$$f(t,z) \le |z|\varepsilon_1$$

for all z with $|z| > N_2$. Without loss of generality we take $N_1 \le N_2$.

Define

$$\varepsilon = \max\left\{\varepsilon_1, \max\left\{\frac{f(t,z)}{|z|} : N_1 \le |z| \le N_2, t \in [0,T]\right\}\right\}.$$

Then, for all $z \in \mathbb{R}$ and all $t \in [0,T]$,

$$f(t,z) < \varepsilon|z|.$$

Let $R = \frac{1}{\varepsilon}$ and suppose that $\tau \in (0,T]$ is such that (14) holds. Assume that u is a positive fixed point of T. Then,

$$\begin{aligned}
\|u\| &= \|Tu\| \\
&\le \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{t^{\alpha-1}}{\tau^{\alpha+\beta-1}}\int_0^\tau (\tau-s)^{\alpha+\beta-1} f(s,u(s))\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))\,ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^\tau \Gamma(\alpha+\beta)\tau^{-\beta}(\tau-s)^{\alpha+\beta-1} + (\tau-s)^{\alpha-1}\,ds\,\varepsilon\|u\| \\
&\le \left(\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\right)\tau^\alpha\varepsilon\|u\| \\
&< R\varepsilon\|u\| = \|u\|,
\end{aligned}$$

which is a contradiction. The proof is complete. □

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