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ON THE LOCAL SOLVABILITY OF DARBOUX'S EQUATION

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Abstract. We reduce the question of local nonsolvability of the Darboux equation, and hence of the isometric embedding problem for surfaces, to the local nonsolvability of a simple linear equation whose type is explicitly determined by the Gaussian curvature.

Let (M^2, g) be a two-dimensional Riemannian manifold. A well-known problem is to ask, when can one realize this locally as a small piece of a surface in $\mathbb{R}^3$? That is, if the metric $g = g_{ij}dx^i dx^j$ is given in the neighborhood of a point, say $(x^1, x^2) = 0$, when do there exist functions $z_\alpha(x^1, x^2)$, $\alpha = 1, 2, 3$, defined in a possibly smaller domain such that $g = dz_1^2 + dz_2^2 + dz_3^2$? This equation may be written in local coordinates as the following determined system

$$\sum_{\alpha=1}^{3} \frac{\partial z_\alpha}{\partial x^i}\frac{\partial z_\alpha}{\partial x^j} = g_{ij}.$$

Due to its severe degeneracy, in the sense that every direction happens to be a characteristic direction, little information has been obtained by studying this system directly. However a more successful approach has been to reduce this system to the following single equation of Monge-Ampère type, known as the Darboux equation:

$$\det \nabla_{ij} z = K|g|(1 - |\nabla_g z|^2) \tag{1}$$

where ∇_{ij} are second covariant derivatives, K is the Gaussian curvature, ∇_g is the gradient with respect to g, and $|g| = \det g$. In fact, the local isometric embedding problem is equivalent to the local solvability of this equation (see the appendix).

Let us first recall the known results. Since equation (1) is elliptic if $K > 0$, hyperbolic if $K < 0$, and of mixed type if K changes sign, the manner in which K vanishes will play the primary role in the hypotheses of any result. The classical results state that a solution always exists in the case that g is analytic or $K(0) \neq 0$; these results may be found in [4]. C.-S. Lin provides an affirmative answer in [10] and [11] when g is sufficiently smooth and satisfies $K \geq 0$, or $K(0) = 0$ and $\nabla K(0) \neq 0$. When $K \leq 0$ and ∇K possesses a certain nondegeneracy, Han, Hong, and Lin [5] show that a smooth solution always exists if g is smooth. Lastly if the Gaussian curvature vanishes to finite order and the zero set $K^{-1}(0)$ consists of Lipschitz curves intersecting tranvsversely, then Han and the author [6] have proven

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the existence of smooth solutions if g is smooth. Related results may be found in [1], [2], [3], [7], [8].

A negative result has been obtained by Pogorelov [13] (see also [12]), who found a $C^{2,1}$ metric with no local C^2 isometric embedding in $\mathbb{R}^3$. More recently, the author [9] has constructed C^∞ examples of degenerate hyperbolic and mixed type Monge-Ampère equations of the form

$$\det(\partial_{ij} z + a_{ij}(p, z, \nabla z)) = k(p, z, \nabla z) \tag{2}$$

which do not admit a local solution, where $p = (x^1, x^2)$ and ∂_{ij} denote second partial derivatives. A fundamental part of the strategy in [9] is to reduce the local nonsolvability of (2), to the local nonsolvability of a quasilinear equation whose type is explicitly determined by the function k. It is the purpose of this article to show that the Darboux equation possesses a similar property for a large class of Gaussian curvatures.

We begin by partially constructing the Gaussian curvature. Here we will denote the coordinates x^1 and x^2 by x and y respectively. Define sequences of disjoint open squares $\{X^n\}_{n=1}^\infty$ and $\{X_1^n\}_{n=1}^\infty$ whose sides are aligned with the x and y-axes, and such that X^n, and X_1^n are centered at $q_n = (\frac{1}{n}, 0)$, $X^n \subset X_1^n$, and X^n, X_1^n have widths $\frac{1}{2n(n+1)}$, $\frac{1}{n(n+1)}$, respectively. Set $K \equiv 0$ in $\mathbb{R}^2 - \bigcup_{n=1}^\infty X_1^n$. Define

$$X = \{(x, y) \mid |x| < 1, |y| < 1\}$$

and let $\phi \in C^\infty(\overline{X})$ be such that ϕ vanishes to infinite order on ∂X, and either $\phi(q) > 0$ or $\phi(q) < 0$ for all $q \in X$ (here $\overline{X}$ denotes the closure of X). We now define K in X^n by

$$K(q) = \gamma_n \phi(4n(n+1)(q - q_n)), \qquad q \in \overline{X}^n,$$

where $\{\gamma_n\}_{n=1}^\infty$ is a sequence of positive numbers that are to be chosen with the property that $\lim_{n\to\infty} \gamma_n = 0$ in order to insure that $K \in C^\infty(\mathbb{R}^2)$. A description of how K should be prescribed in the remaining region $\bigcup_{n=1}^\infty (X_1^n - X^n)$ shall be given below.

Theorem 1. *Suppose that K adheres to the description given above, and that a local C^5 solution z of the Darboux equation exists in a domain containing the origin. Then in a neighborhood of a point on ∂X^n for some n sufficiently large, there exists a C^2 function u constructed from z which after an appropriate change of coordinates satisfies the equation*

$$\partial_{tt} u + K \partial_{ss} u = K f, \tag{3}$$

where $f \in C^0$ also depends on z and is strictly positive.

This theorem suggests a strategy for constructing smooth counterexamples to the local solvability of the Darboux equation, or equivalently the local isometric embedding problem. Namely, complete the construction of a smooth Gaussian curvature function in the region $\bigcup_{n=1}^\infty (X_1^n - X^n)$, in such a way that the linear equation (3) can have no local solution. Whether this is possible is still an open question, however as pointed out above, a similar strategy was successfully employed for the related Monge-Ampère equation (2). Note that in order for this strategy to be utilized for the Darboux equation, it must be shown that given a smooth function K there always exists a locally defined smooth metric g having Gaussian curvature K. This may be accomplished in the following way. Let Ω be a neighborhood of the origin, and let $G \in C^{\infty}(\Omega)$ be the unique solution of the equation

$$\partial_{xx}G + KG = 0, \quad G(0,y) = 1, \quad \partial_x G(0,y) = 0.$$

By choosing Ω sufficiently small we have that $G > 0$. Then

$$g = dx^2 + G^2 dy^2$$

is a smooth Riemannian metric and has Gaussian curvature K in the domain Ω.

The first step in verifying Theorem 1, will be to show that certain second covariant derivatives of any solution of (1) cannot vanish on ∂X^n for n sufficiently large. Suppose that a local solution $z \in C^2$ of (1) exists, so that upon rewriting the equation we have

$$b^{ij}\nabla_{ij}z = 2K(1 - |\nabla_g z|^2), \tag{4}$$

where the Einstein summation convention concerning raised and lowered indices has been used (this convention will also be utilized in what follows) and

$$(b^{ij}) = |g|^{-1}\begin{pmatrix} \nabla_{22}z & -\nabla_{12}z \\ -\nabla_{12}z & \nabla_{11}z \end{pmatrix}.$$

Then integrating by parts yields

$$\int_{X^n} b^{ij}\nabla_{ij}z d\omega_g = -\int_{X^n} \nabla_j z \nabla_i b^{ij} d\omega_g + \int_{\partial X^n} b^{ij} n_i \nabla_j z d\sigma_g, \tag{5}$$

where $d\omega_g$ and $d\sigma_g$ are the elements of area and length with respect to g, and (n_1, n_2) is the unit outer normal to ∂X^n also with respect to g. In order to calculate the interior term on the right-hand side we note that b^{ij} is a contravariant 2-tensor, so that

$$\nabla_i b^{ij} = \partial_i b^{ij} + \Gamma^i_{il} b^{lj} + \Gamma^j_{il} b^{il}$$

where Γ^l_{ij} are Christoffel symbols. Therefore

$$\begin{aligned}\nabla_i b^{i1} =& |g|^{-1}(\partial_1\nabla_{22}z - \partial_2\nabla_{12}z) + |g|^{-2}(-\partial_1|g|\nabla_{22}z + \partial_2|g|\nabla_{12}z) \\ &+ |g|^{-3/2}(\partial_1|g|^{1/2}\nabla_{22}z - \partial_2|g|^{1/2}\nabla_{12}z) + \Gamma^1_{il}b^{il} \\ =& |g|^{-1}(\partial_1\nabla_{22}z - \partial_2\nabla_{12}z + |g|\Gamma^1_{ij}b^{ij}) - \Gamma^i_{ij}b^{j1},\end{aligned}$$

after making use of the identity

$$\Gamma^i_{ij} = |g|^{-1/2}\partial_j|g|^{1/2}.$$

Moreover direct computation shows that

$$\begin{aligned}&\partial_1\nabla_{22}z - \partial_2\nabla_{12}z + |g|\Gamma^1_{ij}b^{ij} \\ =& -\Gamma^j_{j2}\partial_{12}z + \Gamma^j_{j1}\partial_{22}z \\ &+ (\partial_2\Gamma^i_{12} - \partial_1\Gamma^i_{22} - \Gamma^1_{11}\Gamma^i_{22} + 2\Gamma^1_{12}\Gamma^i_{12} - \Gamma^1_{22}\Gamma^i_{11})\partial_i z \\ =& |g|(\Gamma^j_{j2}b^{12} + \Gamma^j_{j1}b^{11}) \\ &+ (\partial_2\Gamma^i_{12} - \partial_1\Gamma^i_{22} - \Gamma^1_{11}\Gamma^i_{22} + 2\Gamma^1_{12}\Gamma^i_{12} - \Gamma^1_{22}\Gamma^i_{11} - \Gamma^j_{j2}\Gamma^i_{12} + \Gamma^j_{j1}\Gamma^i_{22})\partial_i z,\end{aligned}$$

and we observe that the coefficient of $\partial_i z$ is in fact a curvature term. More precisely, if it is denoted by χ^i then

$$\chi^i = \partial_2\Gamma^i_{12} - \partial_1\Gamma^i_{22} + \Gamma^j_{12}\Gamma^i_{j2} - \Gamma^j_{22}\Gamma^i_{j1} = -R^i_{212} = -g^{i1}|g|K$$

where R^i_{jkl} is the Riemann tensor. We now have

$$\partial_1\nabla_{22}z - \partial_2\nabla_{12}z + |g|\Gamma^1_{ij}b^{ij} = |g|(\Gamma^j_{j2}b^{12} + \Gamma^j_{j1}b^{22} - g^{i1}K\partial_i z)$$

so that

$$\nabla_i b^{i1} = -Kz^1. \tag{6}$$

Similarly

$$\nabla_i b^{i2} = -Kz^2. \tag{7}$$

With the help of (4), (6), and (7) it follows that (5) becomes

$$\begin{aligned}&\int_{X^n} K(2-3|\nabla_g z|^2)d\omega_g\\ &=\int_{\partial X^n}|g|^{-1/2}[(\nabla_1 z\nabla_{22}z-\nabla_2 z\nabla_{12}z)\overline{n}_1+(\nabla_2 z\nabla_{11}z-\nabla_1 z\nabla_{12}z)\overline{n}_2]d\sigma,\end{aligned} \tag{8}$$

where $(\overline{n}_1,\overline{n}_2)$ is the Euclidean unit outer normal to ∂X^n and $d\sigma$ is the Euclidean element of length.

The integral equality (8) will now be used to show that certain second covariant derivatives of any solution of the Darboux equation cannot vanish on ∂X^n for n sufficiently large. Let $-\mathrm{v}_n$, $+\mathrm{v}_n$ represent the left and right vertical portions of ∂X^n, respectively, and let $+\mathrm{h}_n$, $-\mathrm{h}_n$ represent the top and bottom horizontal portions of ∂X^n, respectively.

Lemma 2. *Suppose that K satisfies the hypotheses of Theorem 1. Then it is not possible for a C^2 solution z of (1) to satisfy the following property for any n sufficiently large:*

$$\nabla_{22}z|_{\pm\mathrm{v}_n}=0, \qquad \nabla_{11}z|_{\pm\mathrm{h}_n}=0. \tag{9}$$

Proof. We proceed by contradiction and assume that property (9) holds. Then since $K|_{\partial X^n}=0$, the Darboux equation implies that $\nabla_{12}z|_{\partial X^n}=0$. Therefore the right-hand side of (8) vanishes. However this yields a contradiction, as the left-hand side is nonzero for large n. To see this last fact observe that according to the appendix, any solution of the Darboux equation yields an isometric embedding $F=(z_1,z_2,z)$ of the metric g. So that by performing an appropriate rigid body motion of this embedding, to obtain $\overline{F}=AF$ where A is an orthogonal matrix, we can ensure that the new third component $\overline{z}$ of $\overline{F}$ satisfies $|\nabla\overline{z}|(0,0)=0$. Furthermore the appendix also shows that $\overline{z}$ must satisfy the Darboux equation, and so we have $2-3|\nabla_g\overline{z}|^2>1$ inside X^n if n is chosen sufficiently large. Therefore since K never vanishes on X^n, integral equality (8) yields a contradiction. □

In light of Lemma 2, there must exist a point $p\in\partial X^n$ at which one of the given second covariant derivatives is nonzero. As arguments similar to those presented below may be applied if $p\in-\mathrm{v}_n$ or $p\in\pm\mathrm{h}_n$, we assume without loss of generality that $p\in+\mathrm{v}_n$ so that $\nabla_{22}z(p)\neq 0$. It follows that after a change of coordinates near p, a solution u of equation (3) may be constructed. The following lemma will complete the proof of Theorem 1.

Lemma 3. *Suppose that there exists a C^5 solution z of the Darboux equation satisfying $\nabla_{22}z(p)\neq 0$. Then there exists a C^3 local change of coordinates near $p=(p^1,p^2)$ given by*

$$t=x-p^1, \qquad s=s(x,y),$$

and a C^2 solution u of the equation

$$\partial_{tt}u+K\partial_{ss}u=Kf,$$

where $f\in C^0$ and is strictly positive if n is sufficiently large.

Proof. The desired coordinates (t, s) will be chosen to eliminate the mixed second covariant derivative appearing in (4). Since b^{ij} is a contravariant 2-tensor, under a coordinate change $\overline{x}^i = \overline{x}^i(x^1, x^2)$ it transforms by

$$\overline{b}^{ij} = b^{lm} \frac{\partial \overline{x}^i}{\partial x^l} \frac{\partial \overline{x}^j}{\partial x^m}.$$

Therefore by setting $t = \overline{x}^1 = x - p^1$, we seek $s = \overline{x}^2$ such that

$$\overline{b}^{12} = b^{11}\partial_x s + b^{12}\partial_y s = 0, \qquad s(p^1, y) = cy, \tag{10}$$

where c is a nonzero constant to be determined. Observe that since $b^{11} = |g|^{-1}\nabla_{22}z \neq 0$ near p, the line $x = p^1$ is noncharacteristic for (10). Thus the theory of first order partial differential equations guarantees the existence of a unique local solution $s \in C^3$, in light of the fact that $b^{11}, b^{12} \in C^3$.

We now calculate each of the new coefficients for the Darboux equation. First note that $\overline{b}^{11} = b^{11}$, and with the help of (10)

$$\begin{aligned}\overline{b}^{22} =& b^{11}(\partial_x s)^2 + 2b^{12}\partial_x s \partial_y s + b^{22}(\partial_y s)^2 \\ =& (b^{11})^{-1}(\partial_y s)^2 \det b^{ij} \\ =& (|g|b^{11})^{-1}(\partial_y s)^2 K(1 - |\nabla_g z|^2).\end{aligned}$$

Therefore in the new coordinates Darboux's equation (4) is given by

$$b^{11}\overline{\nabla}_{11}z + K\overline{f}\,\overline{\nabla}_{22}z = 2K(1 - |\nabla_g z|^2), \tag{11}$$

where $\overline{\nabla}_{ij}$ denote covariant derivatives with respect to the new coordinates (t, s) and

$$\overline{f} = (|g|b^{11})^{-1}(\partial_y s)^2(1 - |\nabla_g z|^2).$$

Notice that if we choose

$$c = b^{11}|g|^{1/2}(1 - |\nabla_g z|^2)^{-1/2}(p),$$

then $(b^{11})^{-1}\overline{f}(p) = 1$. Moreover by setting

$$u(t, s) = z(t, s) - \int_0^t \left(\int_0^{t'} (\overline{\Gamma}_{11}^1 \partial_t z + \overline{\Gamma}_{11}^2 \partial_s z)(t'', s)dt'' \right) dt'$$

we have $\partial_{tt}u = \overline{\nabla}_{11}z$, so that (11) becomes

$$\partial_{tt}u + K\partial_{ss}u = Kf$$

with

$$f = (b^{11})^{-1}[2(1 - |\nabla_g z|^2) + (\overline{f}(p) - \overline{f})\overline{\nabla}_{22}z] + \overline{\Gamma}_{22}^1 \partial_t z + \overline{\Gamma}_{22}^2 \partial_s z + \partial_{ss}(u - z).$$

Lastly we observe that $f(t, s) > 0$ in a sufficiently small neighborhood of p if n is large, since as in the proof of Lemma 2 we may assume that $|\nabla z|(0, 0) = 0$. □

Appendix

Here we show that the local isometric embedding problem is equivalent to the local solvability of the Darboux equation (1). Assume that there exists a local C^2 embedding $F = (z_1, z_2, z_3)$ for a given metric g. Then according to the Gauss equations

$$\nabla_{ij}F = h_{ij}\nu,$$

where h_{ij} are the components of the second fundamental form with respect to a unit normal ν. Then by taking the Euclidean inner product of this equation with the vector $\vec{k} = (0, 0, 1)$, we obtain

$$\det \nabla_{ij} z = K|g|(\nu \cdot \vec{k})^2$$

where for convenience we denote z_3 by z. Furthermore, if $\times$ represents the cross product operation between two vectors in $\mathbb{R}^3$ then

$$(\nu \cdot \vec{k})^2 = 1 - \left| \frac{(\partial_1 F \times \partial_2 F) \times \vec{k}}{|\partial_1 F \times \partial_2 F|} \right|^2 = 1 - g^{ij} \partial_i z \partial_j z = 1 - |\nabla_g z|^2,$$

where g^{ij} are components of the inverse matrix $(g_{ij})^{-1}$. Clearly the remaining two components of F must also satisfy equation (1). Conversely, if a local solution of (1) exists for a given metric g and $|\nabla_g z| < 1$, then a calculation shows that $g - dz^2$ is a Riemannian metric and is flat. It follows that there exists a local change of coordinates $z_1 = z_1(x^1, x^2)$, $z_2 = z_1(x^1, x^2)$ such that $g - dz^2 = dz_1^2 + dz_2^2$.

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