DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **466–475**

STATE ESTIMATION FOR LINEAR IMPULSIVE DIFFERENTIAL SYSTEMS THROUGH POLYHEDRAL TECHNIQUES

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Abstract. The paper is devoted to the state estimation problem in control theory under uncertainty. The approach for estimating the reachable sets of the linear impulsive differential systems is presented. The reachable sets are approximated by the ones for special discrete time systems. The degree of convergence is established. The families of external and internal polyhedral (parallelepiped-valued and parallelotope-valued) estimates of the reachable sets of the auxiliary systems are introduced. Evolution of estimates is determined by systems of recurrence relations. The families which ensure the exact representations of the reachable sets of the auxiliary systems as well as the families of the touching and tight estimates are found. This technique gives the possibility to construct the guaranteed estimates (including ε-touching and ε-tight ones) for the reachable sets of the primary systems. The results of numerical simulations are presented.

1. **Introduction .** The solution of many problems of the theory of control under uncertainty in a "guaranteed formulation" is based on the investigation of trajectory tubes [8]. Models with set-membership description of uncertainties arise in a wide variety of applications [8, 12]. There are many approaches to solving these important problems. Different numerical methods were developed for approximating reachable sets and trajectory tubes of systems with classical (Lebesgue measurable) controls. Some of them are based on approximations of sets by polytopes with a large number of vertices and faces or by unions of finite number of points. Such methods may require much calculations for large dimension systems. Another approach is to approximate convex sets by domains of some fixed shape such as ellipsoids, parallelepipeds, zonotopes (see, for example, [2, 3, 5, 6, 7, 8, 9, 12] and references given there). The approach proposed by A.B.Kurzhanski is to approximate the trajectory tube by whole families of external (internal) tubes formed by domains of some fixed shape [8, 9, 7, 5]. The families are introduced in such a way that on the one hand they would ensure exact representations of solutions (through intersections of external estimates and unions of internal ones) and on the other hand each particular approximate tube would be determined by some system of evolutionary equations independently of the others (this further allows to parallelize computations). There are few works devoted to approximating reachable sets of differential systems with impulsive controls. There reachable sets are estimated

2000 *Mathematics Subject Classification.* Primary: 93B03, 93C05, 93C15; Secondary: 52B12, 93C55.

Key words and phrases. Dynamical systems, impulsive differential systems, reachable sets, set-valued state estimation, approximation, polyhedral estimates, parallelepipeds, parallelotopes.

The work is partially supported by the Russian Foundation for Basic Research (Grant 06-01-00483) and by the State Program for Support of Leading Scientific Schools of Russian Federation (Grant 4576.2008.1).

by polytopes with a large number of vertices [4], by unions of finite number of points [11], by single external ellipsoids [12]. Our aim is to investigate the capabilities of external and internal approximations (estimates) of reachable sets of linear impulsive systems by families of parallelepipeds (parallelotopes).

We approximate the reachable sets by the ones for special discrete time systems with integral bounds on inputs; the theorem about convergence is proved; error of apptoximation (in the sense of "Hausdorff" distance) is presented in the explicit form. In [6] the families of external and internal polyhedral estimates for reachable sets of this sort of discrete time systems were introduced and investigated. These results are reminded here for the completeness of the exposition. Unlike [5], in view of special system structure we have to construct polyhedral estimates for a convex hull of the union of parallelotopes. The novel family of touching external estimates with constant orientation matrices is introduced for time-invariant systems using the idea similar to that from [3] to apply the other order of set operations. Using the mentioned techniques we construct the families of guaranteed estimates (including ε-touching and ε-tight ones) for the reachable sets of the primary systems. The results of numerical simulations are presented.

The following notations are used below: $\stackrel{\triangle}{=}$ — sign of equality by definition; $\mathbb{R}^n$ — the n-dimensional vector space; $\top$ — the transposition symbol; $\|x\| = (x^\top x)^{1/2}$, $\|x\|_\infty = \max_{1\le i\le n}|x_i|$, $\|x\|_1 = \sum_{i=1}^n |x_i|$ — different vector norms for $x = (x_1,\dots,x_n)^\top \in \mathbb{R}^n$; $\mathrm{e}^i = (0,\dots,0,1,0,\dots,0)^\top$ — unit vector oriented along the axis $0x_i$ (the unit stands at i-position); $\mathrm{e} = (1,1,\dots,1)^\top$; $\mathbb{R}^{n\times m}$ — the linear space of real $n\times m$-matrices; I — the unit matrix; $\|A\| = \|A\|_\infty^1 = \max_{1\le i\le n}\sum_{j=1}^m |a_i^j|$ — the matrix norm for $A = \{a_i^j\} \in \mathbb{R}^{n\times m}$; $\operatorname{diag}\pi$ — diagonal matrix A with $a_i^i = \pi_i$ (π_i — components of the vector π); $\operatorname{Abs} A = \{|a_i^j|\}$ for $A = \{a_i^j\}$; $\operatorname{conv}\mathbb{R}^n$ — the set of all compact convex subsets $\mathcal{X} \subset \mathbb{R}^n$; $\operatorname{co}\mathcal{X}$ — convex hull of a set $\mathcal{X} \subset \mathbb{R}^n$; $\rho(l|\mathcal{X}) = \sup\{(x,l)|\, x \in \mathcal{X}\}$ — the support function of $\mathcal{X} \subseteq \mathbb{R}^n$ at $l \in \mathbb{R}^n$; $h(\mathcal{X},\mathcal{Y})$ — "Hausdorff" distance corresponding to $\|\cdot\|_\infty$: $h(\mathcal{X},\mathcal{Y}) = \max\{h_+(\mathcal{X},\mathcal{Y}), h_+(\mathcal{Y},\mathcal{X})\}$, where $h_+(\mathcal{X},\mathcal{Y}) = \min\{\lambda|\mathcal{X} \subseteq \mathcal{Y}+\lambda\mathcal{C}\}$ — semidistance, $\mathcal{C} = \{x|\|x\|_\infty \le 1\}$; $\mathbb{C}^r[0,\theta]$ — the space of continuous functions f: $[0,\theta] \to \mathbb{R}^r$; $\mathbb{BV}_\infty^r[0,\theta]$ — the space of functions U: $[0,\theta] \to \mathbb{R}^r$ continuous from the right for all $t \in (0,\theta]$, $U(0) = 0$, with bounded variation $\operatorname{Var}_\infty[U]_0^\theta = \sup_{\{t_j\}}\sum_{j=1}^m \|U(t_j) - U(t_{j-1})\|_\infty < \infty$.

2. **Problem formulation .** Consider a dynamic system with impulsive control (measure) $U(t)$ and with unknown but bounded initial condition

$$dx = A(t)\,x\,dt + B(t)dU, \qquad t \in T = [0,\theta], \tag{1}$$

$$U(\cdot) \in \mathbb{U} = \{U \in \mathbb{BV}_\infty^r[0,\theta] \mid \operatorname{Var}_\infty[U]_0^\theta \le \mu_0,\ U \in \mathbb{K}\}, \tag{2}$$

$$x(0) = x_0 \in \mathcal{X}_0. \tag{3}$$

Here $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times r}$ are given continuous and Lipschitz's matrix functions respectively, $\mu_0 \ge 0$ is a given number, $\mathbb{K}$ is a given cone in $\mathbb{BV}_\infty^r[0,\theta]$, $\mathcal{X}_0 \in \operatorname{conv}\mathbb{R}^n$ is a given set.

We are interested in *reachable sets* — sets of all possible states of system (1)–(3):

$$\mathcal{X}(t) = \bigcup_{x_0\in\mathcal{X}_0}\bigcup_{U\in\mathbb{U}} x(t;x_0,U),$$

$$x(t;x_0,U) = \Phi(t,0)\,x_0 + \int_0^t H(t,\tau)dU(\tau), \quad \text{where } H(t,\tau) = \Phi(t,\tau)B(\tau).$$

Here the integral is taken as the Riemann-Stieltjes one, $\Phi(t,\tau) = \Phi(t)\Phi(\tau)^{-1}$, where $\Phi(t)$ is the fundamental matrix solution to $\dot\Phi = A(t)\Phi$, $\Phi(0) = I$.

Set-valued map $\mathcal{X}(t)$, $t \in T$, defines a so-called *trajectory tube* $\mathcal{X}(\cdot)$.

Generally speaking, this map is generated by discontinuous functions of bounded variation. Nevertheless, it is known (see, for example, [11] and reference therein) that the map $\mathcal{X}(t)$ turns out to be discontinuous only at $t = 0$.

Let us introduce several definitions which are used below.

By the *parallelepiped* $\mathcal{P}(p, P, \pi) \subset \mathbb{R}^n$ we mean a set such that $\mathcal{P} = \mathcal{P}(p, P, \pi) = \{x \mid x = p + \sum_{i=1}^n p^i \pi_i \xi_i,\ \ |\xi_i| \le 1,\ i = 1, 2, \dots, n\}$. Here $p \in \mathbb{R}^n$; P is a nonsingular matrix with columns p^i of unit length[1]: $P = \{p^i\} \in \mathcal{M}_*^{n\times n}$, $\mathcal{M}_*^{n\times n} = \{P | \det P \ne 0, \|p^i\| = 1\}$; $\pi \in \mathbb{R}^n$, $\pi \ge 0$ (vector inequalities are understood componentwise). It may be said that p determines the center of the parallelepiped, P — orientation matrix, p^i — "directions" and π_i — values of its "semi-axes".

By the *parallelotope* $\mathcal{P}[p, \bar{P}] \subset \mathbb{R}^n$ (where $\bar{P} = \{\bar{p}^i\} \in \mathbb{R}^{n\times r}$, $r \le n$) we mean a set such that $\mathcal{P} = \mathcal{P}[p, \bar{P}] = \{x \,|\, x = p + \sum_{i=1}^r \bar{p}^i \xi_i,\ \ |\xi_i| \le 1,\ i = 1, 2, \dots, r\}$.

Each parallelepiped is a parallelotope and vice versa if the matrix $\bar{P} \in \mathbb{R}^{n\times n}$ and $\det \bar{P} \ne 0$. We will consider external and internal parallelepiped-valued and parallelotope-valued (shorter *polyhedral*) estimates for sets.

We call $\mathcal{P}$ *external (internal) estimate for* $\mathcal{Q} \subset \mathbb{R}^n$ if $\mathcal{P} \supseteq \mathcal{Q}$ ($\mathcal{P} \subseteq \mathcal{Q}$).

We call an external (internal) estimate of $\mathcal{Q} \in \operatorname{conv} \mathbb{R}^n$ *ε-tight (in direction l)* if $|\rho(\pm l|\mathcal{P}) - \rho(\pm l|\mathcal{Q})| \le \varepsilon \|l\|_1$.

We call a parallelepiped $\mathcal{P}(p, P, \pi)$ *ε-touching external estimate for* $\mathcal{Q}$ if it is ε-tight estimate in n specified directions $l^i = {P^{-1}}^\top \mathrm{e}^i$, $i = 1, 2, \dots, n$.

We come to known definitions of *tight* [9] and *touching* [5] estimates if $\varepsilon = 0$.

Consider the following *problems*:

- Find some external $\mathcal{P}^+(t) = \mathcal{P}(p^+(t), P^+(t), \pi^+(t))$ and internal $\mathcal{P}^-(t) = \mathcal{P}[p^-(t), \bar{P}^-(t)]$ polyhedral estimates for reachable sets $\mathcal{X}(t)$: $\mathcal{P}^-(t) \subseteq \mathcal{X}(t) \subseteq \mathcal{P}^+(t)$. Moreover introduce some families of estimates $\mathcal{P}^\pm(t)$.
- Introduce some families of ε-touching (ε-tight) estimates $\mathcal{P}^+(t)$ ($\mathcal{P}^-(t)$).
- Introduce some families of estimates $\mathcal{P}^+(t)$ which provide ε-approximations of the reachable sets in the following sense: $h(\mathcal{X}(t), \bigcap \mathcal{P}^+(t)) \le \varepsilon$.

The following assumption is assumed throughout the paper.

Assumption 1. $\mathcal{X}_0$ is a parallelepiped $\mathcal{X}_0 = \mathcal{P}_0$ and $\mathbb{K}$ is a cone of the form:

$$\mathbb{K} = \{U \in \mathbb{BV}_\infty^r[0, \theta] |\ \int_0^\theta h^\top(\tau) dU(\tau) \ge 0\ \forall h \in \mathbb{C}^r[0, \theta] \text{ such that } h(\tau) \in \mathcal{K}\ \forall \tau \in T\},$$
$$\mathcal{K} = \{v \in \mathbb{R}^r\ |v_i \ge 0, i \in I_\mathcal{K}\}, \quad \text{where } I_\mathcal{K} \subseteq \{1, 2, \dots, r\} \text{ or } I_\mathcal{K} = \emptyset.$$

Remark 1. Such $\mathbb{K}$ restricts jump vectors of $U(\cdot)$ to have specified nonnegative components (namely, components with numbers $i \in I_\mathcal{K}$; we have $\mathcal{K} = \mathbb{R}^r$ if $I_\mathcal{K} = \emptyset$). At that the set $\mathcal{R} = \mathcal{C} \cap \mathcal{K}$ is a parallelepiped (here $\mathcal{C} = \mathcal{P}(0, I, \mathrm{e})$ is the unit cube).

3. **Auxiliary discrete time systems .** Fix a natural number N and introduce the grid $\mathcal{T}_N$ of times $t_j = jh_N$, $j = 0, 1, \dots, N$, with the step $h_N = \theta N^{-1}$. Consider the following auxiliary discrete time system:

$$x[j] = A[j]x[j-1] + B[j]u[j], \quad j = 1, 2, \dots, N, \tag{4}$$

$$\sum_{j=1}^N \|u[j]\|_\infty \le \mu_0, \qquad u[j] \in \mathcal{K}, \quad j = 1, 2, \dots, N, \tag{5}$$

$$x[0] = x_0 \in \mathcal{X}_0 = \mathcal{P}_0, \tag{6}$$

where $A[j] = I + \int_{t_{j-1}}^{t_j} A(\tau) d\tau$, $B[j] = B(t_j)$, μ_0, $\mathcal{K}$ and $\mathcal{P}_0$ are the same as above.

[1] The normality condition $\|p^i\| = 1$ may be omitted to simplify formulas (particulary, it ensures the uniqueness of representation of a parallelepiped with nonzero values of semi-axes).

The *reachable set* $\mathcal{X}[k]$ of system (4)–(6) at time $k \in \{1, 2, \dots, N\}$ is a set of those points $x \in \mathbb{R}^n$, for each of which there exists a pair $\{x[0], u[\cdot]\}$ that satisfies (5), (6) and generates a solution $x[\cdot]$ of (4) that satisfies $x[k] = x$.

The following theorem displays the closeness of reachable sets of the primary and auxiliary systems as $h_N \to 0$.

Theorem 3.1. *Under the all above assumptions,*

$$\begin{gathered} h(\mathcal{X}(t_k), \mathcal{X}[k]) \le \varphi(h_N) = C \cdot h_N, \quad k = 1, 2, \dots, N, \quad \textit{where} \\ C = (\mu_0(C_A C_B + L_B + C_B C_3 \theta) + C_{\mathcal{X}_0} C_3 \theta) e^{C_A \theta}, \ C_3 = \tfrac{1}{2}(C_A)^2, \\ C_A = \max_{0 \le t \le \theta} \|A(t)\|, \ C_B = \max_{0 \le t \le \theta} \|B(t)\|, \ C_{\mathcal{X}_0} = \max_{x_0 \in \mathcal{X}_0} \|x_0\|_\infty, \end{gathered} \tag{7}$$

L_B is the Lipschitz constant for B: $\|B(t_1) - B(t_2)\| \le L_B |t_1 - t_2|$.

Before proving let us introduce following operators. The operator $\mathbf{R}^N$ maps a function $u^N[k]$ of discrete argument $k = 1, 2, \dots, N$ into $U^N(\cdot) \in \mathbb{BV}^r_\infty[0, \theta]$ by the formulas $U^N(t) = \sum_{k=1}^N u^N[k] \chi(t - t_k)$, where $\chi(t)$ is the Heaviside function: $\chi(t) = 1$ if $t \ge 0$, $\chi(t) = 0$ if $t < 0$. The operator $\mathbf{L}^N$, conversely, maps $U \in \mathbb{BV}^r_\infty[0, \theta]$ into $u^N[\cdot]$ by the formulas $u^N[k] = U(t_k) - U(t_{k-1})$, $k = 1, 2, \dots, N$.

It is not difficult to see that these operators satisfy the following properties. Let $u^N = \mathbf{L}^N U$, $U^N = \mathbf{R}^N \mathbf{L}^N U$. Then $\sum_{k=1}^N \|u^N[k]\|_\infty = \mathrm{Var}_\infty[U^N]_0^\theta \le \mathrm{Var}_\infty[U]_0^\theta$ and if $U \in \mathbb{K}$, then $u^N[k] \in \mathcal{K}$, $k = 1, 2, \dots, N$, $U^N \in \mathbb{K}$. Let $U^N = \mathbf{R}^N u^N$. Then $\mathrm{Var}_\infty[U^N]_0^\theta = \sum_{k=1}^N \|u^N[k]\|_\infty$ and if $u^N[k] \in \mathcal{K}$, $k = 1, 2, \dots, N$, then $U^N \in \mathbb{K}$.

Lemma 3.2. *Let $x(\cdot)$, $x^N(\cdot)$ and $x^N[\cdot]$ be solutions to (1)–(3), (1)–(3) and (4)–(6) respectively corresponding to the same initial condition x_0 and to controls $U(\cdot)$, $U^N(\cdot)$ and $u^N[\cdot]$ which are connected by relations $U^N = \mathbf{R}^N \mathbf{L}^N U$, $u^N = \mathbf{L}^N U$. Then, under the all above assumptions,*

$$\|x(t_k) - x^N(t_k)\|_\infty \le \varphi_1(t_k, h_N) \mathrm{Var}_\infty[U]_0^{t_k}, \quad k = 1, 2, \dots, N, \tag{8}$$

$$\|x^N(t_k) - x^N[k]\|_\infty \le \varphi_2(t_k, h_N)(\|x_0\|_\infty + C_B \mathrm{Var}_\infty[U]_0^{t_k}), \quad k = 1, 2, \dots, N, \tag{9}$$

where $\varphi_1(t, h_N) = h_N(C_A C_B + L_B) e^{C_A t}$, $\varphi_2(t, h_N) = h_N C_3 t e^{C_A t}$.

Proof. Inasmuch as $U^N = \mathbf{R}^N \mathbf{L}^N U$ we have

$$\begin{aligned} x^N(t_k) - \Phi(t_k, 0) x_0 &= \int_0^{t_k} H(t_k, \tau) dU^N(\tau) = H(t_k, 0)(U^N(0{+}0) - U^N(0)) \\ &+ \sum_{j=1}^{k-1} H(t_k, t_j)(U^N(t_j{+}0) - U^N(t_j{-}0)) + H(t_k, t_k)(U^N(t_k) - U^N(t_k{-}0)) \\ &= \sum_{j=1}^{k} H(t_k, t_j)(U(t_j) - U(t_{j-1})) = \sum_{j=1}^{k} H(t_k, t_j) \int_{t_{j-1}}^{t_j} dU(\tau). \end{aligned}$$

Introducing the notation $\Delta^1(t, \tau, \xi) = H(t, \tau) - H(t, \xi)$ and using the inequality of the type $\int_0^\theta h^\top(\tau) dU(\tau) \le \max_{0 \le t \le \theta} \|h(t)\|_1 \mathrm{Var}_\infty[U]_0^\theta$ valid for each $h \in \mathbb{C}^r[0, \theta]$ we have $\|x(t_k) - x^N(t_k)\|_\infty = \|\sum_{j=1}^k \int_{t_{j-1}}^{t_j} \Delta^1(t_k, \tau, t_j) dU(\tau)\|_\infty \le \max_{1 \le j \le k} \max_{\tau \in [t_{j-1}, t_j]} \|\Delta^1(t_k, \tau, t_j)\|_\infty^1 \mathrm{Var}_\infty[U]_0^{t_k}$. Adding and subtracting cross terms $\Phi(t_k, t_j) B(\tau)$ and using the inequalities of the type $\|\Phi(t, \tau)\|_\infty \le e^{C_A(\theta - \tau)}$ (it follows from the Gronwall-Bellman inequality) and $\|\Phi(t, \tau) - I\|_\infty = \|\int_t^\tau A(\xi) \Phi(\xi, \tau) d\xi\|_\infty \le C_A e^{C_A(t - \tau)}(t - \tau)$ we can easily obtain (8).

Similarly, if we denote $\Delta^2(t_k, j) = \Phi(t_k, t_j) - \Phi[k, j]$, where $\Phi[k, j] = A[k] A[k - 1] \cdots A[j + 1]$ if $k > j$ and $\Phi[k, j] = I$ if $k = j$, then $x^N(t_k) - x^N[k] = \Delta^2(t_k, 0) x_0 + \sum_{j=1}^k \Delta^2(t_k, j) B[j] u^N[j]$. Following arguments of [1, Ch. VIII, §4] for estimation of

errors of one-step methods for solving ordinary differential equations we can obtain the estimates $\|\Delta^2(t_k,j)\|^1_\infty \le \varphi_2(t_k,h_N)$ which lead to (9). □

Analogously to (9) we have

Lemma 3.3. *Let $x^N[\cdot]$ and $x^N(\cdot)$ be solutions to (4)–(6) and (1)–(3) corresponding to controls $u^N[\cdot]$ and $U^N = \mathbf{R}^N u^N$ respectively. Then*

$$\|x^N(t_k) - x^N[k]\|_\infty \le \varphi_2(t_k,h_N)(\|x_0\|_\infty + C_B \sum_{k=1}^{N} \|u^N[k]\|_\infty), \quad k = 1,2,\dots,N.$$

Proof of Theorem 3.1. If $x^* \in \mathcal{X}(t_k)$, then $x^* = x(t_k;x_0,U)$ is generated by some pair $\{x_0,U\}$ (may be not unique) satisfying (5),(6). If we construct $x^N(\cdot) = x(\cdot;x_0,U^N)$ and $x^N[\cdot] = x[\cdot;x_0,u^N]$ as in Lemma 3.2, then the point $x^{*N} \stackrel{\triangle}{=} x[k;x_0,u^N] \in \mathcal{X}[k]$. Estimating $\|x^* - x^{*N}\|_\infty$ using the triangle inequality and Lemma 3.2 we obtain $h_+(\mathcal{X}(t_k),\mathcal{X}[k]) \le \varphi(h_N)$, where $\varphi(h_N)$ is determined in (7). The converse inequality $h_+(\mathcal{X}[k],\mathcal{X}(t_k)) \le \varphi_2(t_k,h_N)(C_{\mathcal{X}_0}+C_B\mu_0)$ follows from Lemma 3.3. □

It is known [6] that the reachable sets $\mathcal{X}[k]$ satisfy some system of recurrence relations which involves operations with sets such as union and Minkowski's addition.

Theorem 3.4. [6] *Reachable sets $\mathcal{X}[k]$ of system (4)–(6) satisfy the relations*

$$\begin{aligned}
&\mathcal{X}[k] = \mathcal{X}^0[k] + \mathcal{X}^1[k], \quad k = 1,2,\dots,N,\\
&\mathcal{X}^0[k] = A[k]\mathcal{X}^0[k-1], \quad k = 1,2,\dots,N, \qquad \mathcal{X}^0[0] = \mathcal{X}_0,\\
&\mathcal{X}^1[k] = \mathrm{co}\,\{A[k]\mathcal{X}^1[k-1] \cup \mu_0 B[k]\mathcal{R}\}, \qquad \mathcal{X}^1[0] = \{0\},
\end{aligned}$$

where $\mathcal{R} = \mathcal{C} \cap \mathcal{K}$, $\mathcal{C} = \mathcal{P}(0,I,\mathrm{e})$.

Theorem 3.5. *If the system is time-invariant ($A[k] \equiv A$, $B[k] \equiv B$), then reachable sets $\mathcal{X}[k]$ of system (4)–(6) also satisfy the relations*

$$\begin{aligned}
&\mathcal{X}[k] = \mathcal{X}^0[k] + \mathrm{co}\,\mathcal{X}^3[k], \quad k = 1,2,\dots,N,\\
&\mathcal{X}^0[k] = A\mathcal{X}^0[k-1], \quad k = 1,2,\dots,N, \qquad \mathcal{X}^0[0] = \mathcal{X}_0,\\
&\mathcal{X}^3[k] = \mathcal{X}^3[k-1] \cup \mathcal{X}^2[k], \quad k = 1,2,\dots,N, \qquad \mathcal{X}^3[0] = \emptyset,\\
&\mathcal{X}^2[k] = A\mathcal{X}^2[k-1], \quad k = 2,3,\dots,N, \qquad \mathcal{X}^2[1] = \mu_0 B\mathcal{R}.
\end{aligned} \tag{10}$$

If $0 \in \mathcal{R}$ (it is so under Assumption 1), then we may replace $\mathcal{X}^3[0]{=}\emptyset$ by $\mathcal{X}^3[0]{=}\{0\}$.

Proof. It follows from [6] that $\mathcal{X}[k] = \mathcal{X}^0[k] + \mathrm{co}\,\mathcal{X}^3[k]$ where $\mathcal{X}^0[k] \stackrel{\triangle}{=} A^k\mathcal{X}_0$, $\mathcal{X}^3[k] \stackrel{\triangle}{=} \mu_0 \bigcup_{\alpha=1}^{k} A^{\alpha-1}B\mathcal{R}$, $k = 1,2,\dots,N$. Let us use the other (in comparison with Theorem 3.4) order of operations with sets for calculating $\mathcal{X}[k]$. Denote $\mathcal{X}^2[k] \stackrel{\triangle}{=} \mu_0 A^{k-1}B\mathcal{R}$, $k{=}1,2,\dots,N$. Then (10) is obtained by induction. □

Therefore we can calculate polyhedral estimates for $\mathcal{X}[k]$ if we are able to construct primary polyhedral estimates for results of the mentioned operations with parallelepipeds and parallelotopes.

4. **Primary polyhedral estimates .** Remind [5] that the touching external parallelepiped-valued estimate $\boldsymbol{P}^+_{P^+}(\mathcal{Q}) = \mathcal{P}(p^+,P^+,\pi^+)$ of $Q \subset \mathbb{R}^n$ with a given orientation matrix P^+ can be calculated using values of the support function of Q: $p^+ = P^+(\gamma^{(-)}+\gamma^{(+)})/2$, $\pi^+ = (\gamma^{(+)}-\gamma^{(-)})/2$, where $\gamma_i^{(\pm)} = \pm\rho(\pm(P^+)^{-1\top}\mathrm{e}^i|\mathcal{Q})$. The external estimates $\boldsymbol{P}^+_{P^+}(\mathcal{P}^1{+}\mathcal{P}^2)$, $\boldsymbol{P}^+_{P^+}(\mathcal{P}^1{\cup}\mathcal{P}^2)$ for the sum and union of parallelepipeds (parallelotopes) are determined by explicit formulas [5, 6]. Namely, $\boldsymbol{P}^+_{P^+}(\sum_{k=1}^2 \mathcal{P}(p^k,P^k,\pi^k)) = \mathcal{P}(\sum_{k=1}^2 p^k, P^+, \sum_{k=1}^2(\mathrm{Abs}\,((P^+)^{-1}P^k)))\pi^k)$ and $\boldsymbol{P}^+_{P^+}(\mathcal{P}^1{\cup}\mathcal{P}^2)$ is calculated using the formula $\rho(l|\mathcal{P}^1{\cup}\mathcal{P}^2) = \max_{1\le k\le 2}\rho(l|\mathcal{P}^k)$.

Internal parallelotope-valued estimates for $\mathcal{Q} = \mathcal{P}^1 + \mathcal{P}^2$, where $\mathcal{P}^k = \mathcal{P}[p^k, \bar{P}^k]$, $\bar{P}^1 \in \mathbb{R}^{n\times n}$, $\bar{P}^2 \in \mathbb{R}^{n\times r}$, can be constructed [6] in the form $\boldsymbol{P}^-_\Gamma(\mathcal{Q}) = \mathcal{P}[p^1 + p^2, \bar{P}^1 + \bar{P}^2\Gamma]$, where $\Gamma \in \mathcal{G} = \{\Gamma = \{\gamma^\beta_\alpha\} \in \mathbb{R}^{r\times n} | \max_{1\le\alpha\le r} \sum_{\beta=1}^n |\gamma^\beta_\alpha| \le 1\}$.

Parameters P^+, Γ specify the whole families of estimates.

5. **External and internal estimates for sets $\mathcal{X}[k]$.** View system (4)-(6) when $\det A[j] \ne 0$, $j = 1, 2, \dots, N$[2]. Consider the family $\mathfrak{P}^+$ of tubes $\mathcal{P}^+[\cdot]$ satisfying

$$\begin{aligned}
&\mathcal{P}^+[k] = \boldsymbol{P}^+_{P^+[k]}(\mathcal{P}^0[k] + \mathcal{P}^1[k]), \quad k = 1, 2, \dots, N,\\
&\mathcal{P}^0[k] = A[k]\mathcal{P}^0[k-1], \quad k = 1, 2, \dots, N, \qquad \mathcal{P}^0[0] = \mathcal{P}_0,\\
&\mathcal{P}^1[k] = \boldsymbol{P}^+_{P^+[k]}(A[k]\mathcal{P}^1[k-1] \cup \mu_0 B[k]\mathcal{R}), \quad k = 1, 2, \dots, N, \quad \mathcal{P}^1[0] = \mathcal{P}(0, I, 0).
\end{aligned}$$

Here orientation matrices $P^+[\cdot]$ serve as a parameter of the family. Let us also introduce the subfamily $\mathfrak{P}^{1+} \in \mathfrak{P}^+$ for which the orientation martices satisfy

$$P^+[k] = A[k]P^+[k-1], \; k = 1, \dots, N, \quad P^+[0] = P \qquad (P \in \mathcal{M}^{n\times n}_*).$$

For time-invariant systems let us also introduce the family $\mathfrak{P}^{2+} \notin \mathfrak{P}^+$ of tubes $\mathcal{P}^+[\cdot]$:

$$\begin{aligned}
&\mathcal{P}^+[k] = \boldsymbol{P}^+_P(\mathcal{P}^0[k] + \mathcal{P}^3[k]), \quad k = 1, 2, \dots, N,\\
&\mathcal{P}^0[k] = A\mathcal{P}^0[k-1], \quad k = 1, 2, \dots, N, \qquad \mathcal{P}^0[0] = \mathcal{P}_0,\\
&\mathcal{P}^3[k] = \boldsymbol{P}^+_P(\mathcal{P}^3[k-1] \cup \mathcal{P}^2[k-1]), \quad k = 1, 2, \dots, N, \quad \mathcal{P}^3[0] = \mathcal{P}(0, I, 0),\\
&\mathcal{P}^2[k] = A\mathcal{P}^2[k-1], \quad k = 1, 2, \dots, N, \qquad \mathcal{P}^2[0] = \mu_0 B\mathcal{R},
\end{aligned}$$

where the constant orientation matrix P, $\det P \ne 0$, is a parameter of the family.

The following theorems arise from Theorems 3.4 and 3.5 and facts of Section 4.

Theorem 5.1. [6] *Let $\mathcal{X}[k]$ be reachable sets for system (4)–(6). If $\mathcal{P}^+[\cdot] \in \mathfrak{P}^+$, then parallelepipeds $\mathcal{P}^+[k]$ are the external estimates for $\mathcal{X}[k]$: $\mathcal{X}[k] \subseteq \mathcal{P}^+[k]$, k=1, 2, …, N, whatever are $P^+[k] \in \mathcal{M}^{n\times n}_*$, k=1, 2, …, N. If $P \in \mathcal{M}^{n\times n}_*$ is an arbitrary matrix and $\mathcal{P}^+[\cdot]$ is the corresponding tube from $\mathfrak{P}^{1+}$, then $\mathcal{P}^+[k]$ are touching estimates for $\mathcal{X}[k]$ and $\mathcal{X}[k] = \bigcap\{\mathcal{P}^+[k] | \;\; P \in \mathcal{V}^0\}$. Here $\mathcal{V}^0$ denotes an arbitrary set of matrices $P \in \mathcal{M}^{n\times n}_*$ with the property that, for each nonzero vector $l \in \mathbb{R}^n$, there is a matrix P in this set such that l is collinear to some column of $(P^{-1})^\top$.*

Theorem 5.2. *Let $\mathcal{X}[k]$ be reachable sets for system (4)–(6) which is time-invariant. If $\mathcal{P}^+[\cdot] \in \mathfrak{P}^{2+}$, then $\mathcal{P}^+[k]$ are touching estimates for $\mathcal{X}[k]$ and $\mathcal{X}[k] = \bigcap\{\mathcal{P}^+[k] | \;\; P \in \mathcal{V}^0\}$, $k = 1, 2, \dots, N$, where $\mathcal{V}^0$ is the same as above.*

Proof. Using Theorem 3.5, facts of Section 4 and induction it is not difficult to obtain the relations $\mathcal{P}^0[k] = \mathcal{X}^0[k]$, $\mathcal{P}^2[k] = A\mathcal{X}^2[k]$, $\mathcal{P}^3[k] = \boldsymbol{P}^+_P(\mathcal{X}^3[k])$, $\mathcal{P}^+[k] = \boldsymbol{P}^+_P(\mathcal{X}[k])$, $k = 1, 2, \dots, N$. In this process the following properties of touching estimates are used: $\boldsymbol{P}^+_P(\boldsymbol{P}^+_P(\mathcal{X})) = \boldsymbol{P}^+_P(\mathcal{X})$, $\boldsymbol{P}^+_P(\sum_{k=1}^2 \mathcal{X}^k) = \sum_{k=1}^2 \boldsymbol{P}^+_P(\mathcal{X}^k)$, $\boldsymbol{P}^+_P(\boldsymbol{P}^+_P(\mathcal{X}^1) \cup \mathcal{X}^2) = \boldsymbol{P}^+_P(\mathcal{X}^1 \cup \mathcal{X}^2)$ which are valid for all bounded sets $\mathcal{X}, \mathcal{X}^1, \mathcal{X}^2 \subset \mathbb{R}^n$ and nonsingular matrices P (this properties may be verified by comparing values of support functions at $\pm(P^{-1})^\top \mathrm{e}^i$). □

Now consider the family $\mathfrak{P}^-$ of parallelotope-valued tubes $\mathcal{P}^-[\cdot]$ satisfying

$$\mathcal{P}^-[k] = \boldsymbol{P}^-_{\Gamma[k]}(A[k]\mathcal{P}^-[k-1] + h[k]B[k]\mathcal{R}), \quad k = 1, 2, \dots, N, \qquad \mathcal{P}^-[0] = \mathcal{P}_0,$$

where $h[k]$, $\Gamma[k]$ are scalar and matrix parameters such that

$$h[k] \ge 0, \quad \sum_{k=1}^N h[k] \le \mu_0, \qquad \Gamma[k] \in \mathcal{G} \qquad (k = 1, 2, \dots, N). \tag{11}$$

[2]This condition is satisfied for our case if $A[j]$ are constructed as above and h_N is sufficiently small. In general case, it is necessary for introducing and using subfamilies $\mathfrak{P}^{1+}$, $\mathfrak{P}^{1-}$ of estimates. Other results are true without it (but $\mathcal{X}[k]$, $\mathcal{P}^+[k]$, $\mathcal{P}^-[k]$ may appear to be degenerate).

Theorem 5.3. [6] *Let $\mathcal{X}[k]$ be reachable sets for system (4)–(6). If $\mathcal{P}^-[\cdot] \in \mathfrak{P}^-$, then $\mathcal{P}^-[k] \subseteq \mathcal{X}[k]$ and $\mathcal{X}[k] = \bigcup\{\mathcal{P}^-[k] \mid h[\cdot], \Gamma[\cdot]$ satisfy (11)$\}$, $k = 1, 2, \ldots, N$.*

Remark 2. For the case $\mathcal{K} = \mathbb{R}^n$, the subfamily $\mathfrak{P}^{1-} \in \mathfrak{P}^-$ of tight estimates is described in [6]. At that $\Gamma[\cdot]$, $h[\cdot]$ are calculated by some explicit formulas on the base of $l[\cdot]$ satisfying $l[j] = A[j]^{-1\top} l[j-1]$, $j = 1, 2, \ldots, N$, $l[0] = l$ ($l \in \mathbb{R}^n$, $l \neq 0$) and the corresponding $\mathcal{P}^-[k]$ are tight (in direction $l[k]$) estimates for $\mathcal{X}[k]$.

Below we consider systems (4)–(6) obtained by the described discretization of the primary system (1)–(3) with the step h_N and denote the introduced families of estimates by symbols $\mathfrak{P}^+_N$, $\mathfrak{P}^{1+}_N$, $\mathfrak{P}^{2+}_N$, $\mathfrak{P}^-_N$, $\mathfrak{P}^{1-}_N$.

Example 1. Consider an oscillatory system with $A(t) \equiv \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$, $B(t) \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mu_0 = 2$, $\mathcal{K} = \mathbb{R}^1$, $\theta = 2$, $\mathcal{X}_0 = \mathcal{P}((-0.5, 0)^\top, I, (0.5, 0.5)^\top)$ and system (4)–(6) for $N = 400$.

Figure 1(a) and 1(c) show some external tubes $\mathcal{P}^+[\cdot] \in \mathfrak{P}^{1+}_N$ and $\mathcal{P}^+[\cdot] \in \mathfrak{P}^{2+}_N$ respectively (drawing is carried out one time per each 4 stages). Figure 1(b) and 1(d) show $\mathcal{X}_0$ (in dash line) and several touching estimates $\mathcal{P}^+[N]$ for the reachable set $\mathcal{X}[N]$ for cases $\mathfrak{P}^{1+}_N$ and $\mathfrak{P}^{2+}_N$ respectively. In aggregate, these estimates "outline" $\mathcal{X}[N]$. Figure 1(e) and 1(f) present one of the tubes from $\mathfrak{P}^{1-}_N$ and several tight internal estimates for $\mathcal{X}[N]$ in comparing with touching estimates from $\mathfrak{P}^{2+}_N$.

Example 2. Consider the system with $A(t) \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B(t) \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mu_0 = 2$, $\theta = 2$, $\mathcal{X}_0 = \mathcal{P}((-0.5, 0)^\top, I, (0.5, 0.5)^\top)$ for two different cases when there is no constraints on the sign of the jumps of the control ($\mathcal{K} = \mathbb{R}^1$) and the jumps must be nonnegative ($\mathcal{K} = [0, \infty)$). Take $N = 400$.

Figure 2(a) and 2(b) correspond to the case 1 ($\mathcal{K} = \mathbb{R}^1$) and are similar to Figure 1(a) and 1(f) respectively but for the family $\mathfrak{P}^{1+}_N$. The rest four figures show the touching external estimates for $\mathcal{X}[N]$ from $\mathfrak{P}^{1+}_N$ (left) and $\mathfrak{P}^{2+}_N$ (right). Figure 2(c), 2(d) and Figure 2(e), 2(f) correspond to cases 1 and 2 respectively.

6. External and internal estimates for reachable sets $\mathcal{X}(t_k)$.

Corollary 1. *The reachable sets $\mathcal{X}(t_k)$ of the primary system (1)–(3) can be approximated, with any degree of accuracy, by intersections and unions of units of the introduced families $\mathfrak{P}^{1+}_N$, $\mathfrak{P}^{2+}_N$, $\mathfrak{P}^-_N$:*

$$h(\mathcal{X}(t_k), \bigcap_{P \in \mathcal{V}^0} \mathcal{P}^+[k]) \leq \varphi(h_N), \quad k = 1, 2, \ldots, N, \quad (\mathcal{P}^+[\cdot] \in \mathfrak{P}^{1+}_N \text{ or } \mathcal{P}^+[\cdot] \in \mathfrak{P}^{2+}_N),$$
$$h(\mathcal{X}(t_k), \bigcup_{\Gamma[\cdot], h[\cdot]} \mathcal{P}^-[k]) \leq \varphi(h_N), \quad k = 1, 2, \ldots, N, \quad (\mathcal{P}^-[\cdot] \in \mathfrak{P}^-_N).$$

In general, these units $\mathcal{P}^\pm[k]$ are not guaranteed to be external/internal estimates for the primary reachable sets $\mathcal{X}(t_k)$.

But we can construct such estimates by appropriate modifying (namely, "bloating" or "miniaturizing") mentioned units $\mathcal{P}^\pm[k]$. Consider

$$\mathcal{P}^\pm(t_k) = \mathcal{P}(p^+(t_k), P^+(t_k), \pi^+(t_k)) \quad \text{where } p^\pm(t_k) = p^\pm[k], \quad P^\pm(t_k) = P^\pm[k],$$
$$\pi^\pm(t_k) = \pi^\pm[k] \pm \varphi(h_N) \cdot \mathrm{Abs}\,(P^\pm[k]^{-1})\,\mathrm{e}, \quad \mathrm{e} = \{1, 1, \ldots, 1\}^\top.$$

Corollary 2. *If $\mathcal{P}^+[\cdot] \in \mathfrak{P}^+_N$ or $\mathcal{P}^+[\cdot] \in \mathfrak{P}^{2+}_N$, then $\mathcal{X}(t_k) \subseteq \mathcal{P}^+(t_k)$, $k = 1, 2, \ldots, N$, and if $\mathcal{P}^-[\cdot] \in \mathfrak{P}^-_N$, then $\mathcal{P}^-(t_k) \subseteq \mathcal{X}(t_k)$, $k = 1, 2, \ldots, N$.*

Proof. Using Theorems 3.1, 5.1–5.3 and facts from [5], [10, p.134] we have $\mathcal{P}^-(t_k) = \mathcal{P}^-[k] \dot{-} \varepsilon\mathcal{C} \subseteq (\mathcal{X}(t_k) + \varepsilon\mathcal{C}) \dot{-} \varepsilon\mathcal{C} = \mathcal{X}(t_k) \subseteq \mathcal{X}[k] + \varepsilon\mathcal{C} \subseteq \mathcal{P}^+[k] + \varepsilon\mathcal{C} \subseteq \mathcal{P}^+(t_k)$, where

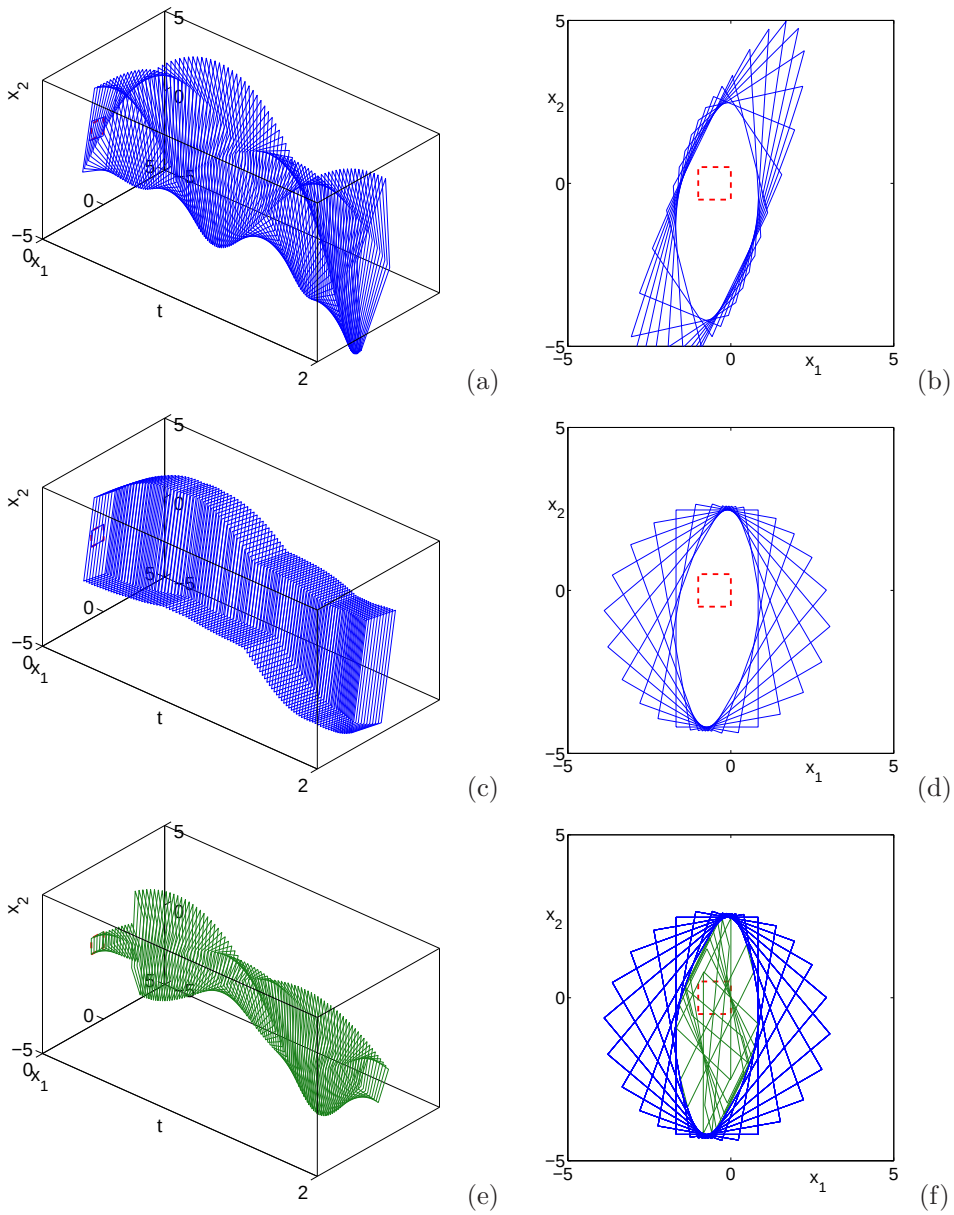

FIGURE 1. External and internal estimates for $\mathcal{X}[\cdot]$ and $\mathcal{X}[N]$ in Example 1

$\dot{-}$ denotes the sign of geometrical (Minkovski) difference of sets ($\mathcal{X}^1 \dot{-} \mathcal{X}^2 = \{x \in \mathbb{R}^n |\ x + \mathcal{X}^2 \subseteq \mathcal{X}^1\}$), $\varepsilon = \varphi(h_N)$. □

Using means of support functions, relations $\rho(l|\,\mathcal{C}) = \|l\|_1$ and $|\rho(l|\,\mathcal{X}) - \rho(l|\,\mathcal{Y})| \leq h(\mathcal{X}, \mathcal{Y}) \cdot \|l\|_1$, similar to that in [10, p.53], it is easy to verify the following assertions.

Corollary 3. *If $\mathcal{P}^+[\cdot] \in \mathfrak{P}_N^{1+}$ or $\mathfrak{P}_N^{2+}$, then $\mathcal{P}^+(t_k)$ are ε-touching estimates for $\mathcal{X}(t_k)$ and $h(\mathcal{X}(t_k), \bigcap_{P \in \mathcal{V}^0} \mathcal{P}^+(t_k)) \leq \varepsilon$ with $\varepsilon = 2\varphi(h_N) \to 0$ as $h_N \to 0$.*

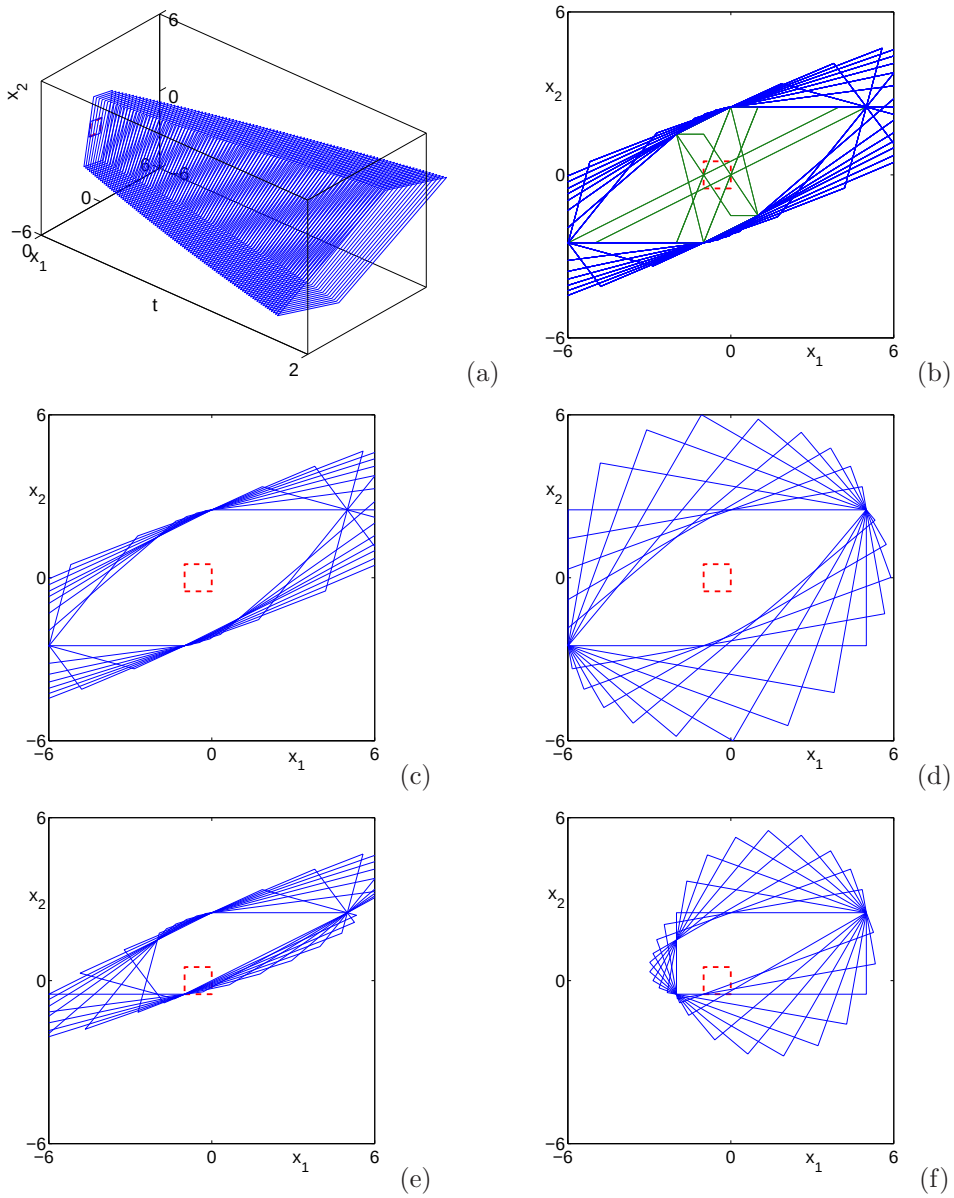

FIGURE 2. Estimates for $\mathcal{X}[\cdot]$ and $\mathcal{X}[N]$ in Example 2

Corollary 4. *If $\mathcal{P}^-[\cdot] \in \mathfrak{P}_N^-$ is such that $\mathcal{P}^-[k]$ are tight estimates for $\mathcal{X}[k]$ (in particular, $\mathcal{P}^-[\cdot] \in \mathfrak{P}_N^{1-}$) and $\mathcal{P}^-(t_k) \neq \emptyset$, $k = 1, 2, \ldots, N$, then $\mathcal{P}^-(t_k)$ are ε-tight estimates for $\mathcal{X}(t_k)$ with $\varepsilon = (1 + \|P^-[k]\|_\infty^1 \cdot \|P^-[k]^{-1}\|_\infty^1)\varphi(h_N)$.*

Thus we obtain the desired guaranteed estimates and, in particular, ε-touching external and ε-tight internal estimates for the primary reachable sets. Here ε is the same for all external estimates corresponding to two mentioned families $\mathfrak{P}_N^{1+}$ and $\mathfrak{P}_N^{2+}$. Unfortunately, it is not so for internal estimates.

Note that the obtained theoretical expression $\varphi(h_N)$ for approximation error may produce rather conservative modified estimates $\mathcal{P}^+(t_k)$. But in practice we

may be satisfied by estimates $\mathcal{P}^{\pm}[k]$ for auxiliary systems if they "practically" are not changed after bisection of the time step h_N. For example, in the two above examples, we have the same visual results for $N = 400$ and $N = 800$.

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Received July 2008; revised April 2009.

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