

STABILITY OF TRAVELING WAVEFRONTS FOR TIME-DELAYED REACTION-DIFFUSION EQUATIONS

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ABSTRACT. This paper is concerned with time-delayed reaction-diffusion equations. For all traveling wavefronts, they are proved to be stable time-asymptotically by the technical weighted energy method with the comparison principle together, which extends the wave stability results obtained in [7, 8]. Some numerical simulations are also carried out, which confirm our theoretical results.

1. Introduction. In this paper we consider the time-delayed reaction-diffusion equations for the population dynamics of a single species. When the birth rate function is considered to be isolated in location, namely, the local case, the equation is expressed as

$$\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} + du = pb(u(t-r, x)) \quad (1)$$

where $u(t, x)$ denotes the distribution of population of the single species in time t and at location x , $D > 0$ is the coefficient of diffusion, d is the coefficient of death for the species, $p > 0$ is the impact constant related to the birth rate, $r > 0$ is the mature age of the species, which is usually called the delay-time, and $b(u(t-r, x))$ is the birth rate function. However, more practically, we should consider the mature species' activities involving the whole space and that they move and marry in all region but not isolate in one spot. So the birth rate function should be nonlocal, and the equation is described as an integral-differential equation

$$\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} + du = p \int_{-\infty}^{\infty} b(u(t-r, x-y)) f_{\alpha}(y) dy, \quad (2)$$

where $f_{\alpha}(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{y^2}{4\alpha}}$ is the heat kernel, and α is a positive constant satisfying $0 < \alpha \leq rD$. $b(u)$ is the birth function, and satisfies

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(H₁) There exist $u_- = 0$ and $u_+ > 0$ such that $b(0) = 0$, $b'(0) = 1$, $pb(u_+) = du_+$ and $pb'(u_+) < d$;

(H₂) $0 \leq b'(u) \leq 1$ and $b''(u) \leq 0$ for $0 \leq u \leq u_+$.

The important prototypes are

$$b_1(u) = ue^{-au^q}, \quad b_2(u) = \frac{u}{1+au^q}, \quad a > 0, \quad q > 0,$$

which are studied in [7, 8]. When $q = 1$, $b_1(u)$ is just the Nicholson's birth rate function.

In this paper, we consider the Cauchy problem for the equations (1) and (2), respectively, where the initial data are given as

$$u(s, x) = u_0(s, x), \quad s \in [-r, 0], \quad x \in (-\infty, \infty). \tag{3}$$

Notice that both equations (1) and (2) have the same equilibria: $u_- := 0$ and u_+ . A traveling wavefront of the equation (1) is defined as a monotone solution of (1) in the form $\phi(x + ct)$ (c is the wave speed) connecting with the constant states u_{\pm} , which satisfies the following ordinary differential equation

$$c\phi' - D\phi'' + d\phi = pb(\phi(\xi - cr)), \quad \phi(\pm\infty) = u_{\pm}, \tag{4}$$

where $' = \frac{d}{d\xi}$, $\xi = x + ct$. Corresponding to equation (2), the traveling wavefront $\phi(x + ct)$ satisfies

$$c\phi' - D\phi'' + d\phi = p \int_{-\infty}^{\infty} b(\phi(\xi - cr + y))f_{\alpha}(y)dy, \quad \phi(\pm\infty) = u_{\pm}. \tag{5}$$

The population dynamics of a single species with age-structure has been intensively studied, for example, see [1]-[24] and the references therein. When $q = 1$ with $1 < \frac{p}{d} \leq e$, the existence of such traveling wavefronts for the equations (1) and (2) had been proved by So-Zou [20] and So-Wu-Zou [18], respectively, and the asymptotic behaviors of the critical wave speed to the delay-time r has been analyzed by Wu-Wei-Mei in [23, 22]. Furthermore, by the technical weighted energy method, the stability of the traveling wavefronts has been obtained by Mei-So-Li-Shen [10] and Mei-So [9], respectively, where the wave speed needs to be suitably large (i.e., the faster wave), and the initial perturbation around the wavefront is restricted to be sufficiently small. For the equation (1) with the local birth rate, when $r \ll 1$, the stability of the slower waves was proved by Lin-Mei [6]. For the birth function $b(u) = b_1(u)$ or $b_2(u)$, the existence of traveling wavefronts and their numerical simulations had been showed by Liang-Wu in [5]. Then, the stability for all wavefronts (no matter the fast waves or the slow waves) had been completely proved by Mei *et al* [7, 8]. In this paper, we consider the birth function to be more general, and try to extend the stability results obtained in [7, 8] to this general case. Namely, for any given traveling wavefront of (1) or (2) (no matter its speed is large or small), when the initial data decay to the wave exponentially in space as $x \rightarrow -\infty$, but the initial perturbation around the wavefront can be large in any other locations, then we will prove that the solution for (1) or (2) converges to the traveling wavefront time-exponentially. The main difficulty is to establish the first L^2 -energy estimate by selecting a suitable weight function.

The paper is organized as follows. In Section 2, we state our stability results, then prove them by the weighted energy method together with the comparison principle in Section 3. Finally, we carry out some numerical simulations, which confirm our theoretical results, and demonstrate also an interesting phenomenon

that the solution behaves as a traveling wavefront, which in the case of small delay-time is faster than in the case of large delay-time. Due to the page limit, the proofs are outlined, but the key steps in detail are provided.

2. Main Results. The existence of travelling wavefront for (1) or (2) can be similarly proved by the method of upper-lower solutions (c.f. [20, 18, 5]). That is, there exist a minimum speed $c^* > 0$ and a corresponding number $\lambda_* = \lambda(c^*) > 0$ satisfying

$$\Delta(\lambda_*, c^*) = 0, \quad \frac{\partial}{\partial \lambda} \Delta(\lambda_*, c^*) = 0, \quad (6)$$

where

$$\begin{aligned} \Delta(\lambda, c) &= \varepsilon p e^{-\lambda c r} - [c\lambda - D\lambda^2 + d], & \text{for Eq. (1), or,} \\ \Delta(\lambda, c) &= \varepsilon p e^{\alpha \lambda^2 - \lambda c r} - [c\lambda - D\lambda^2 + d], & \text{for Eq. (2),} \end{aligned}$$

such that for all $c > c^*$, the traveling wavefront $\phi(x + ct)$ of Eq. (1) (or (2)) connecting u_{\pm} exists uniquely (up to shift).

It is also noticed that

$$\Delta(\lambda_*, c) < 0, \quad c > c^*. \quad (7)$$

Now we are going to state our main results according to different equations. We define the weight function as

$$w(x) = e^{-2\lambda_* x}, \quad (8)$$

where $\lambda_* = \lambda(c^*)$. Obviously, $w(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $w(x) \rightarrow 0$ as $x \rightarrow +\infty$.

Theorem 2.1 (Local Eq. (1)). *For any given wavefront $\phi(x + ct)$ with the speed $c > c^*$, if the initial data satisfy*

$$u_- = 0 \leq u_0(s, x) \leq u_+, \quad \text{for } (s, x) \in [-r, 0] \times R, \quad (9)$$

and the initial perturbation is $u_0(s, x) - \phi(x + cs) \in C([-r, 0]; H_w^1(R))$, then the solution of (1) and (3) satisfies

$$u_- = 0 \leq u(t, x) \leq u_+, \quad \text{for } (t, x) \in R_+ \times R, \quad (10)$$

and

$$u(t, x) - \phi(x + ct) \in C([0, \infty); H_w^1(R)). \quad (11)$$

In particular, the solution $u(t, x)$ converges to the wavefront $\phi(x + ct)$ exponentially in time

$$\sup_{x \in R} |u(t, x) - \phi(x + ct)| \leq C e^{-\mu_1 t}, \quad t \geq 0 \quad (12)$$

for some positive number μ_1 .

Theorem 2.2 (Nonlocal Eq. (2)). *For any given wavefront $\phi(x + ct)$ with the speed $c > c^*$, if α satisfies*

$$e^{\lambda_*^2 \alpha} < \frac{c\lambda_* - D\lambda_*^2 + d}{c^*\lambda_* - D\lambda_*^2 + d}, \quad (13)$$

and the initial data satisfy

$$u_- = 0 \leq u_0(s, x) \leq u_+, \quad \text{for } (s, x) \in [-r, 0] \times R, \quad (14)$$

and the initial perturbation is $u_0(s, x) - \phi(x + cs) \in C([-r, 0]; H_{w_2}^1(R))$, then the solution of (2) and (3) satisfies

$$u_- = 0 \leq u(t, x) \leq u_+, \quad \text{for } (t, x) \in R_+ \times R, \quad (15)$$

and

$$u(t, x) - \phi(x + ct) \in C([0, \infty); H_w^1(R)). \quad (16)$$

In particular, the solution $u(t, x)$ converges to the wavefront $\phi(x + ct)$ exponentially in time

$$\sup_{x \in R} |u(t, x) - \phi(x + ct)| \leq Ce^{-\mu_2 t}, \quad t \geq 0 \tag{17}$$

for some positive number μ_2 .

3. Proof of Main Results. In this section, we prove only Theorem 2.1. The proof of Theorem 2.2 will be omitted due to the page limit. As shown in [6, 9, 10], we can similarly establish the following comparison principle. The detail of proof is omitted.

Lemma 3.1 (Comparison Principle). *Let $\bar{u}(t, x)$ and $\underline{u}(t, x)$ be the solutions of (1) and (3) with the initial data $\bar{u}_0(s, x)$ and $\underline{u}_0(s, x)$, respectively. If*

$$u_+ \geq \bar{u}_0(s, x) \geq \underline{u}_0(s, x) \geq u_-, \quad \text{for } (s, x) \in [-r, 0] \times R, \tag{18}$$

then

$$u_+ \geq \bar{u}(t, x) \geq \underline{u}(t, x) \geq u_-, \quad \text{for } (t, x) \in R_+ \times R. \tag{19}$$

For given initial data $u_0(s, x)$ satisfying $u_- \leq u_0(s, x) \leq u_+$ for $(s, x) \in [-r, 0] \times R$, define

$$\begin{cases} v_0^+(s, x) = \max\{u_0(s, x), \phi(x + cs)\}, \\ v_0^-(s, x) = \min\{u_0(s, x), \phi(x + cs)\}, \end{cases} \quad \text{for } (s, x) \in [-r, 0] \times R, \tag{20}$$

then

$$u_- \leq v_0^-(s, x) \leq u_0(s, x) \leq v_0^+(s, x) \leq u_+ \quad \text{for } (s, x) \in [-r, 0] \times R \tag{21}$$

$$u_- \leq v_0^-(s, x) \leq \phi(x + cs) \leq v_0^+(s, x) \leq u_+ \quad \text{for } (s, x) \in [-r, 0] \times R. \tag{22}$$

Denote $v^+(t, x)$ and $v^-(t, x)$ as the corresponding solutions of Eqs. (1) and (3) with respect to the above mentioned initial data $v_0^+(s, x)$ and $v_0^-(s, x)$, then by the Comparison Principle we have

$$u_- \leq v^-(t, x) \leq u(t, x) \leq v^+(t, x) \leq u_+ \quad \text{for } (t, x) \in R_+ \times R, \tag{23}$$

$$u_- \leq v^-(t, x) \leq \phi(x + ct) \leq v^+(t, x) \leq u_+ \quad \text{for } (t, x) \in R_+ \times R. \tag{24}$$

We are going to take three steps to prove the theorem.

Step 1: The convergence of $v^+(t, x)$ to $\phi(x + ct)$. Let $\xi = x + ct$, $v(t, \xi) = v^+(t, x) - \phi(x + ct)$, and $v_0(s, \xi) = v_0^+(t, x) - \phi(x + cs)$. Then $v(t, x) \geq 0$ satisfies

$$\begin{cases} v_t + cv_\xi - Dv_{\xi\xi} + dv - pb'(\phi(\xi - cr))v(t - r, \xi - cr) = pQ(t - r, \xi - cr), \\ v(s, \xi) = v_0(s, \xi), \quad (s, x) \in [-r, 0] \times R, \end{cases} \tag{25}$$

where

$$Q(t - r, \xi - cr) = b(\phi + v) - b(\phi) - b'(\phi)v \tag{26}$$

with $\phi = \phi(\xi - cr)$ and $v = v(t - r, \xi - cr)$.

Since $b''(\phi) \leq 0$ for $\phi \in [u_-, u_+]$, namely $b'(\phi)$ is decreasing in ϕ , and noticing also the increasing monotonicity of $\phi(\xi)$ for $\xi \in (-\infty, \infty)$, it can be verified that $0 \leq b'(u_+) = b'(\phi(+\infty)) < b'(\phi(\xi)) < b'(\phi(-\infty)) = b'(0) \leq 1$, and (by the Taylor's formula) $Q(t - r, \xi - cr) = b(\phi + v) - b(\phi) - b'(\phi)v = \frac{b''(\tilde{v})}{2}v^2 < 0$ for some \tilde{v} between v and $\phi + v$. As showed in [6], multiplying (25) by $e^{2\mu_1 t}w(\xi)v$ (μ_1 will be determined later) and integrating it over $R \times [0, t]$ with respect to ξ and t , then using the Young's

inequality $|De^{2\mu_1 t} w' v v_\xi| = De^{2\mu_1 t} w \left| v_\xi \cdot \frac{w'}{w} v \right| \leq De^{2\mu_1 t} w v_\xi^2 + \frac{D}{4} e^{2\mu_1 t} \left(\frac{w'}{w} \right)^2 w v^2$, and the fact $b'(\phi(\xi)) > 0$ for $\xi \in (-\infty, \infty)$ which implies $|pe^{2\mu_1 t} w(\xi) b'(\phi(\xi - cr)) v(t, \xi) v(t - r, \xi - cr)| \leq pe^{2\mu_1 t} w(\xi) b'(\phi(\xi - cr)) \left[\frac{\eta}{2} v^2(t, \xi) + \frac{1}{2\eta} v^2(t - r, \xi - cr) \right]$, where η is selected as $\eta = e^{-c^2 r / (2D)}$, and noting also (by the change of variables $\xi - cr \rightarrow \xi$, $\tau - r \rightarrow \tau$)

$$\begin{aligned} & \int_0^t \int_R \frac{p}{2\eta} e^{2\mu_1 \tau} w(\xi) b'(\phi(\xi - cr)) v^2(\tau - r, \xi - cr) d\xi d\tau \\ &= \int_{-r}^{t-r} \int_R \frac{p}{2\eta} e^{2\mu_1(\tau+r)} w(\xi + cr) b'(\phi(\xi)) v^2(\tau, \xi) d\xi d\tau \\ &\leq \frac{p}{2\eta} e^{2\mu_1 r} \int_0^t \int_R e^{2\mu_1 \tau} w(\xi + cr) b'(\phi(\xi)) v^2(\tau, \xi) d\xi d\tau \\ &\quad + \frac{p}{2\eta} e^{2\mu_1 r} \int_{-r}^0 \int_R e^{2\mu_1 \tau} w(\xi + cr) b'(\phi(\xi)) v_0^2(\tau, \xi) d\xi d\tau, \end{aligned}$$

as well as noting $Q(t - r, \xi - cr) < 0$ as shown before, finally we can obtain

$$\begin{aligned} & e^{2\mu_1 t} \|v(t)\|_{L_w^2}^2 + \int_0^t \int_R e^{2\mu_1 \tau} B(\xi) w(\xi) v^2(\tau, \xi) d\xi d\tau \\ &\leq \|v_0(0)\|_{L_w^2}^2 + \frac{p}{\eta} e^{2\mu_1 r} \int_{-r}^0 \int_R e^{2\mu_1 \tau} w(\xi + cr) b'(\phi(\xi)) v_0^2(s, \xi) d\xi d\tau \\ &\quad + \int_0^t \int_R e^{2\mu_1 \tau} w(\xi) v(\tau, \xi) Q(\tau - r, \xi - cr) d\xi d\tau \\ &\leq C \left(\|v_0(0)\|_{L_w^2}^2 + \int_{-r}^0 \|v_0(\tau)\|_{L_w^2}^2 d\tau \right), \end{aligned} \tag{27}$$

where

$$B(\xi) := A(\xi) - 2\mu_1 - \frac{p}{\eta} (e^{2\mu_1 r} - 1) b'(\phi(\xi)) \frac{w(\xi + cr)}{w(\xi)}, \tag{28}$$

$$\begin{aligned} A(\xi) := & -c \frac{w'(\xi)}{w(\xi)} + 2d - \frac{D}{2} \left(\frac{w'(\xi)}{w(\xi)} \right)^2 - p\eta b'(\phi(\xi - cr)) \\ & - \frac{p}{\eta} b'(\phi(\xi)) \frac{w(\xi + cr)}{w(\xi)}. \end{aligned} \tag{29}$$

Lemma 3.2. *Let $\eta = e^{-\lambda_* cr}$. Then*

$$A(\xi) \geq C_0 \tag{30}$$

for all $\xi \in R$, where $C_0 := |\Delta(\lambda_*, c)| > 0$.

Proof. Notice that (H₂). A straightforward calculation gives

$$\begin{aligned} A(\xi) &= 2\lambda_* c + 2d - 2D\lambda_*^2 - p\eta b'(\phi(\xi - cr)) - \frac{p}{\eta} b'(\phi(\xi)) e^{-2\lambda_* cr} \\ &\geq 2\lambda_* c + 2d - 2D\lambda_*^2 - p\eta - \frac{p}{\eta} e^{-2\lambda_* cr} \\ &= 2 \left(c\lambda_* + d - D\lambda_*^2 - p e^{-\lambda_* cr} \right) \\ &= |\Delta(\lambda_*, c)| =: C_0 > 0. \end{aligned} \tag{31}$$

□

Lemma 3.3. *Let $\mu'_1 > 0$ be the solution to*

$$C_0 - 2\mu'_1 - \frac{p}{\eta}(e^{2\mu'_1 r} - 1) = 0. \tag{32}$$

When $0 < \mu_1 < \mu'_1$, then

$$B(\xi) > 0 \text{ for all } \xi \in R. \tag{33}$$

Proof. Notice that $\frac{w(\xi+cr)}{w(\xi)} \leq 1$ and $b'(\phi(\xi)) < 1$ for all $\xi \in R$, from (30), then we obtain

$$\begin{aligned} B(\xi) &= A(\xi) - 2\mu_1 - \frac{p}{\eta}(e^{2\mu_1 r} - 1)b'(\phi(\xi)) \frac{w(\xi + cr)}{w(\xi)} \\ &\geq C_0 - 2\mu_1 - \frac{p}{\eta}(e^{2\mu_1 r} - 1) > 0 \text{ for all } \xi \in R \end{aligned} \tag{34}$$

by selecting $0 < \mu_1 < \mu'_1$ to be small. The proof is complete. □

Applying (34) to (27), dropping positive term $\int_0^t \int_R e^{2\mu_1 \tau} B(\xi)w(\xi) v^2(\tau, \xi) d\xi d\tau$, we then have the following estimate.

Lemma 3.4 (Basic Energy Estimate). *It holds*

$$e^{2\mu_1 t} \|v(t)\|_{L_w^2}^2 \leq C \left(\|v_0(0)\|_{L_w^2}^2 + \int_{-r}^0 \|v_0(\tau)\|_{L_w^2}^2 d\tau \right). \tag{35}$$

Furthermore, differentiating (25) with respect to ξ , and multiplying it by $e^{2\mu_1 t} w(\xi) v_\xi(t, \xi)$, then integrating the resultant equation with respect to (t, x) over $[0, t] \times R$, and using the basic energy estimates (35), we can prove

Lemma 3.5. *It holds*

$$e^{2\mu_1 t} \|v_\xi(t)\|_{L_w^2}^2 \leq C \left(\|v_0(0)\|_{H_w^1}^2 + \int_{-r}^0 \|v_0(\tau)\|_{H_w^1}^2 d\tau \right). \tag{36}$$

Combining (35) and (36), we finally prove

Lemma 3.6. *It holds*

$$\|(v^+ - \phi)(t)\|_{H_w^1}^2 = \|v(t)\|_{H_w^1}^2 \leq C e^{-2\mu_1 t} \left(\|v_0(0)\|_{H_w^1}^2 + \int_{-r}^0 \|v_0(\tau)\|_{H_w^1}^2 d\tau \right). \tag{37}$$

Notice that, $w(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$, we cannot expect $H_w^1(R) \hookrightarrow C(R)$. However, for any compact interval $I = (-\infty, \bar{\xi}]$ for some large $\bar{\xi} \gg 1$, we may have the Sobolev's embedding result $H_w^1(I) \hookrightarrow C(I)$, which combining with (37) gives the following L^∞ -convergence.

Lemma 3.7. *It holds*

$$\sup_{x \in I} |v^+(t, x) - \phi(x + ct)| = \sup_{\xi \in I} |v(\xi, t)| \leq O(1)e^{-\mu_1 t}. \tag{38}$$

Furthermore, we can prove the following convergence for $\xi > \bar{\xi} \gg 1$.

Lemma 3.8. *It holds*

$$\lim_{\xi \rightarrow +\infty} |v(\xi, t)| \leq O(1)e^{-\mu''_1 t}, \tag{39}$$

where $\mu''_1 := d - pb'(u_+) > 0$.

Proof. Since $Q(t-r, x-y) \leq 0$, from (25) we then have

$$\frac{\partial v}{\partial t} + c \frac{\partial v}{\partial \xi} - D \frac{\partial^2 v}{\partial \xi^2} + dv - pb'(\phi(\xi - cr))v(t-r, x) \leq 0. \quad (40)$$

Taking limits to (40) as $\xi \rightarrow \infty$, and noting the boundedness of $v(t, x)$ which implies $\partial_\xi v(t, \infty) = 0$ and $\partial_{\xi\xi} v(t, \infty) = 0$, we have

$$\frac{d}{dt}v(t, \xi) + dv(t, \infty) - pb'(u_+)v(t-r, \infty) \leq 0.$$

Integrating the above inequality with respect to t gives

$$v(t, \infty) + d \int_0^t v(s, \infty) ds - pb'(u_+) \int_0^t v(s-r, \infty) ds \leq v_0(0, \infty). \quad (41)$$

By the change of variable $s-r \rightarrow s$, we have

$$\begin{aligned} pb'(u_+) \int_0^t v(s-r, \infty) ds &= pb'(u_+) \int_{-r}^{t-r} v(s, \infty) ds \\ &\leq pb'(u_+) \int_0^t v(s, \infty) ds + pb'(u_+) \int_{-r}^0 v_0(s, \infty) ds. \end{aligned} \quad (42)$$

Substituting (42) to (41), we get

$$v(t, \infty) + [d - pb'(u_+)] \int_0^t v(s, \infty) ds \leq C.$$

Applying the Gronwall's inequality, and noting (H₁), i.e., $d - pb'(u_+) > 0$, we prove

$$v(t, \infty) \leq Ce^{-\mu_1'' t}, \quad \mu_1'' := d - pb'(u_+) > 0.$$

□

Combining Lemmas 3.6 and 3.7, and taking $\mu_1 < \min\{\mu_1', \mu_1''\}$, we finally prove the following convergence.

Lemma 3.9. *It holds*

$$\sup_{x \in R} |(v^+(t, x) - \phi(x + ct))| \leq Ce^{-\mu_1 t}. \quad (43)$$

Step 2: The convergence of $v^-(t, x)$ to $\phi(x + ct)$. Let $\xi = x + ct$, $v(t, \xi) = \phi(x + ct) - v^-(t, x)$, and $v_0(s, \xi) = \phi(x + cs) - v_0^-(s, x)$. As shown in the above, we can similarly prove that $v^-(t, x)$ converges to $\phi(x + ct)$, i.e.,

Lemma 3.10. *It holds*

$$\sup_{x \in R} |(v^-(t, x) - \phi(x + ct))| \leq Ce^{-\mu_1 t}. \quad (44)$$

Step 3: The convergence of $u(t, x)$ to $\phi(x + ct)$.

Lemma 3.11. *It holds*

$$\sup_{x \in R} |u(t, x) - \phi(x + ct)| \leq Ce^{-\mu_1 t}, \quad t \geq 0. \quad (45)$$

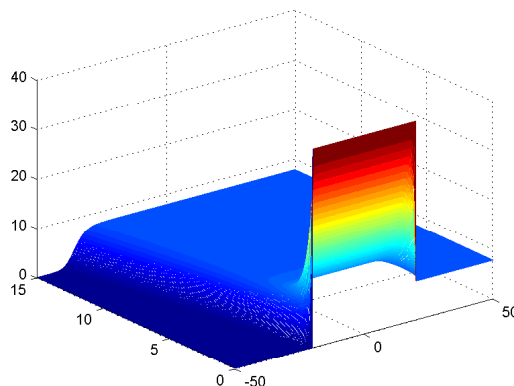


FIGURE 1. 3-D graph for the solution $u(t, x)$

Proof. Since the initial data are $v_0^-(x, s) \leq u_0(x, s) \leq v_0^+(x, s)$, from Lemma 3.1, the corresponding solutions of (1) and (3) satisfy

$$v^-(t, x) \leq u(t, x) \leq v^+(t, x), \quad (t, x) \in R_+ \times R.$$

Thanks to Lemmas 3.9 and 3.10, namely,

$$\sup_{x \in R} |v^-(t, x) - \phi(x + ct)| \leq Ce^{-\mu_1 t}, \quad \sup_{x \in R} |v^+(t, x) - \phi(x + ct)| \leq Ce^{-\mu_1 t}.$$

then applying the squeeze theorem, we finally prove

$$\sup_{x \in R} |u(t, x) - \phi(x + ct)| \leq Ce^{-\mu_1 t}, \quad t > 0.$$

This completes the proof. □

4. Numerical Simulations. We carry out some numerical simulations. Set $a = 0.1, p = 2.17, d = 1, q = 1, r = 0.5$ and the initial data as

$$u_0(s, x) = \begin{cases} u_- = 0, & x < -20 \\ 40, & -20 \leq x \leq 20 \\ u_+ = \frac{1}{a} \ln \frac{p}{d}, & x > 20, \end{cases}$$

which implies that the initial perturbation around any wavefront for $-20 \leq x \leq 20$ is really large, and the initial data even exceeds the bound $[u_-, u_+] = [0, 7.7472717]$.

In computation, the sizes of the time step and space step are chosen as $t = 0.04$ and $x = 0.08$. Although the original model assumes the spatial domain is the whole domain, a finite computational domain $[-L, L]$ is imposed. Here, we let $L = 800$, then the computational domain is sufficiently large so that numerical boundary effect is ignorable. The final computed time is 120.

By using the Crank-Nicholson scheme, we numerically study the stability of the traveling waves. We present the solution in two figures, where Figure 1 is the 3-D graph for the solution $u(t, x)$, and Figure 2 is the 2-D graphs for $u(t, x)$ at different time $t = 0, 5, 10, \dots, 120$. As showed in these figures, after a short time, the solution $u(s, x)$ behaves like a traveling wavefront. This demonstrates the stability of traveling wavefronts.

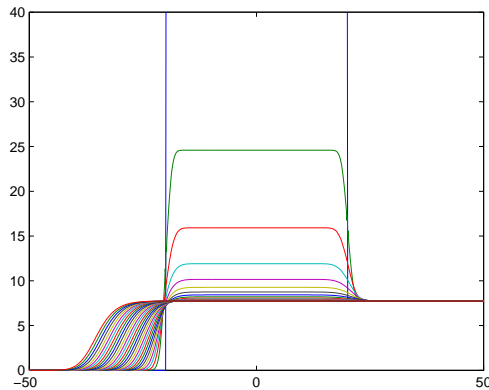


FIGURE 2. 2-D graphs of the solution $u(t, x)$ at different time $t = 0, 5, 10, \dots, 120$

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