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ELLIPTIC QUASI-VARIATIONAL INEQUALITIES AND APPLICATIONS

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Abstract. It is the main objective of this paper to discuss applications of the abstract results recently evolved in [16, 17] to the solvability of variational inequalities with constraints depending on the unknown functions, which are called quasi-variational inequalities.

1. **Introduction.** In this paper we consider a system of an elliptic partial differential equation

$$-\kappa\Delta\theta = h(\theta, u) \text{ in } \Omega, \quad \theta = 0 \quad \text{on } \Gamma, \tag{1.1}$$

coupled with a variational inequality

$$\begin{cases} u \in H^1(\Omega)^2 \text{ with } u \in K_0(\theta) \text{ a.e. in } \Omega, \\ \displaystyle\int_\Omega \{\nabla a(u)\cdot\nabla(u-z) + g(u)\cdot(u-z)\}dx \le \int_\Omega f\cdot(u-z)dx, \\ \qquad\qquad \forall z \in H^1(\Omega)^2 \text{ with } z \in K_0(\theta) \text{ a.e. in } \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $1 \le N \le 3$, with smooth boundary $\Gamma := \partial\Omega$; $h(\cdot,\cdot) : \mathbf{R}\times\mathbf{R}^2 \to \mathbf{R}$ is a smooth function, $a(\cdot) : \mathbf{R}^2 \to \mathbf{R}^2$ and $g(\cdot) : \mathbf{R}^2 \to \mathbf{R}^2$ are smooth vector fields, f is a vector function given in $L^2(\Omega)^2$ and κ is a positive constant; $\theta \to K_0(\theta)$ is a smooth set-valued mapping from $\mathbf{R}$ into the space of all compact convex subsets of $\mathbf{R}^2$. Note in (1.2) that

$$\nabla a(u)\cdot\nabla(u-z) := \nabla a_1(u)\cdot\nabla(u_1-z_1) + \nabla a_2(u)\cdot\nabla(u_2-z_2)$$

for $u := (u_1, u_2)$ and $z = (z_1, z_2)$, where $a(u) := (a_1(u), a_2(u))$.

This type of systems arise in many reaction-diffusion phenomena in material or biological sciences as the mathematical descriptions of their equilibrium states.

Our approach to this system (P):={(1.1),(1.2)} is based on the recent development [16, 17] on abstract quasi-variational inequalities of the elliptic type. Under suitable assumptions on h, for each $u \in H^1(\Omega)^2$ problem (1.1) has a unique solution

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θ. Denoted this solution θ by Λu, we see that (P) is written as a quasi-variational inequality of the form:

$$\begin{cases} u \in K(u), \\ \displaystyle\int_\Omega \{\nabla a(u) \cdot \nabla(u-z) + g(u)\cdot(u-z)\}dx \le \int_\Omega f\cdot(u-z)dx, \ \forall z \in K(u), \end{cases} \tag{1.3}$$

where $K(u) := \{z \in H^1(\Omega)^2; z \in K_0(\Lambda u) \text{ a.e. in } \Omega\}$.

The concept of quasi-variational inequality was ealier introduced by A. Bensoussan and J. L. Lions [2] and has been evolved by many mathematicians, in connection with various free boundary problems (cf. [3, 6, 7, 9, 10]). For a monograph treating quasi-variational inequalities we refer to C. Baiocchi and A. Capelo [1]. There were two approaches to quasi-variational inequalities. They are called compactness method and monotonicity method. The monotonicity method which was proposed by L. Tartar [8] is powerful for a class of problems having the monotonicity structure, but such a nice structure is not able to be expected in so many free boundary problems. In this point it is worthwhile to improve the results due to J. L. Joly and U. Mosco [4, 5] based on the compactness method so as to apply to a wider class of problems. Quite recently, a new development of their results has been given by the authers [16, 17] in order to solve a class of phase transition or reaction-diffusion problems. In this paper we give an application of the abstract result in [16] to a quasi-variational inequality arising from phase transition phenomena.

Quasi-variational inequalities have been evolved from various points of views; for instance, in [13, 14] M. A. Noor showed some existence results by a monotonicity method with ordering property, and in a recent book [12] many variational and quasi-variational inequalities are treated from the mechanical point of view. But our problem (Theorem 2), which we want to solve in this paper, is not able to be treated in these frameworks.

We shall recall an abstract existence result for quasi-variational inequalities in section 2, and state our main result of this paper on the existence of a solution of (P), making assumptions on the data h, a, g, f and $K_0(\theta)$, in section 3. Moreover we shall prove some lemmas in section 4 and give a complete proof of our main result in section 5.

2. **An abstract existence result.**

In this section, we use the following notation. Let X be a real reflexive Banach space, and let X^* be the dual space of X. We assume that X and X^* are strictly convex. We denote the duality pairing between X^* and X by $< \cdot, \cdot >$. By an operator or a mapping A from X into X^* we mean that A is multivalued, namely for each $v \in X$ the value Av is a subset of X^*, and the domain $D(A)$ is the set $\{v \in X;\ Av \neq \emptyset\}$. Also, A is called *bounded* , if A maps bounded sets in X into bounded sets in X^*

Our abstract elliptic quasi-variational inequalities are formulated for the class of "*semi-monotone*" operators from $X \times X$ into X^*. The semi-monotonicity of nonlinear operators is defined as follows.

Definition. An operator $\widetilde{A}(\cdot,\cdot) : X \times X \to X^*$ is called *semi-monotone*, if $D(\widetilde{A}) = X \times X$ and the following conditions (SM1) and (SM2) are fulfilled:

(SM1): For any fixed $v \in X$, the mapping $u \to \tilde{A}(v, u)$ is maximal monotone from $D(\widetilde{A}(v,\cdot)) = X$ into X^*.

(SM2): Let u be any element of X and $\{v_n\}$ be any sequence in X such that $v_n \to v$ weakly in X. Then, for every $u^* \in \widetilde{A}(v,u)$ there exists a sequence $\{u_n^*\}$ in X such that $u_n^* \in \widetilde{A}(v_n,u)$ and $u_n^* \to u^*$ in X^* as $n \to +\infty$.

Next, we propose a class of closed and convex constraints $\{K(v)\}$, depending on $v \in X$, which satisfies the following properties $(K1)$ and $(K2)$:

(K1): If $v_n \to v$ weakly in X, then for each $w \in K(v)$ there is a sequence w_n in X such that $w_n \in K(v_n)$ and $w_n \to w$ in X.

(K2): If $v_n \to v$ weakly in X, $w_n \in K(v_n)$ and $w_n \to w$ weakly in X, then $w \in K(v)$.

We recall an abstract class of quasi-variational inequalities of the form: To find $(u^*, u) \in X^* \times X$ such that

$$P(g^*) \begin{cases} u \in K(u), \ u^* \in \tilde{A}(u,u), \\ < u^* - g^*, w - u > \leq 0, \ \forall w \in K(u), \end{cases} \tag{2.1}$$

where g^* is prescribed in X^* as a datum.

Theorem 1 (cf. [11]). *Let $\widetilde{A} : D(\tilde{A}) = X \times X \to X^*$ be a bounded semi-monotone operator; write simply Au for $\widetilde{A}(u,u)$, $u \in X$. Let $\{K(v)\}_{v \in X}$ be a family of non-empty, closed and convex sets in X such that $(K1)$ and $(K2)$ are satisfied. Suppose that there is a bounded, closed, and convex subset G_0 of X such that*

$$K(v) \cap G_0 \neq \emptyset \quad for \ \ \forall v \in X, \tag{2.2}$$

and

$$\inf_{v^* \in Av} \frac{< v^*, v - w >}{|v|_X} \to \infty \quad as \ \ |v|_X \to \infty \ uniformly \ in \ w \in G_0. \tag{2.3}$$

Then, for any $g^* \in X^*$, problem $P(g^*)$ has at least one solution u.

Quasi-variational inequality $P(g^*)$ is able to be treated in a more general setting. With a function $\varphi(\cdot,\cdot) : X \times X \to [0,\infty]$ such that for each $v \in X$, $w \to \varphi(v,w)$ is a proper, l.s.c. and convex function on X and the mapping $v \to \varphi(v,\cdot)$ is continuous in a certain weak sense from X into the space of all proper, l.s.c. and convex functions, problem $P(g^*)$ is described as a variational inequality

$$< u^* - g^*, w - u > + \varphi(u,u) \leq \varphi(u,w), \ \forall w \in X; \tag{2.4}$$

in fact, it is enough to choose the indicator function $I_{K(v)}(\cdot)$ of $K(v)$ as $\varphi(v,\cdot)$ in order to reformulate $P(g^*)$ in the form (2.4). For this type of generalization of the above theorem, see [12].

3. **Applications.** For the precise formulation of our quasi-variational problem (2.1), we introduce some notation and assumptions. Hereafter, we take $H^1(\Omega)^2$ as X, and make assumptions (H1)-(H3); especially assumption (H1) is essential for our problem setting:

(H1): To each $\theta \in \mathbf{R}$, a non-empty, compact and convex subset $K_0(\theta) \subset \mathbf{R^2}$ is assigned, and there exists a mapping $X_\theta : \mathbf{R^2} \to \mathbf{R^2}$ such that

- **(1):** $X_\theta \in \mathbf{D}^2(\mathbf{R}^2)$ for any $\theta \in \mathbf{R}$, where $\mathbf{D}^2(\mathbf{R}^2)$ is the metric space of all C^2−diffeomorphisms from $\mathbf{R}^2$ onto itself, equipped with the usual topology,
- **(2):** $X_\theta(K_0(0)) = K_0(\theta)$ for any $\theta \in \mathbf{R}$,
- **(3):** the mapping $\theta \to X_\theta$ is $\mathbf{C}^2$-class from $\mathbf{R}$ into $\mathbf{D}^2(\mathbf{R}^2)$, namely X_θ, $\frac{d}{d\theta}X_\theta$ and $\frac{d^2}{d\theta^2}X_\theta$ are continuous from $\mathbf{R}$ into $\mathbf{D}^2(\mathbf{R}^2)$.

By (H1), it is clear that for any two parameters $\theta_1,\ \theta_2 \in \mathbf{R}$, the composite mapping $X_{\theta_1,\theta_2} := X_{\theta_2} \circ X_{\theta_1}^{-1}$ belongs to $\mathbf{D}^2(\mathbf{R}^2)$ and satisfies $X_{\theta_1,\theta_2}(K_0(\theta_1)) = K_0(\theta_2)$ as well as it is of C^2-class with respect to θ_1 and θ_2 as a mapping from $\mathbf{R}^2$ into $\mathbf{D}^2(\mathbf{R}^2)$.

(H2): The function h on $\mathbf{R} \times \mathbf{R}^2$ satisfies the following conditions.

- **(1):** There exists $M_0 > 0$ such that $|h| \le M_0$ on $\mathbf{R} \times \mathbf{R}^2$.
- **(2):** $h(\theta, v)$ is Lipschitz continuous in $v \in \mathbf{R}^2$ for each $\theta \in \mathbf{R}$, and it is non-increasing in $\theta \in \mathbf{R}$ for each $v \in \mathbf{R}^2$.

From (H2) we observe that for each $v \in H^1(\Omega)^2$ problem

$$-\kappa\Delta\theta = h(\theta, v) \text{ in } \Omega,\ \theta = 0 \text{ on } \Gamma, \tag{3.1}$$

has one and only one solution θ in $H_0^1(\Omega) \cap W^{2,p}(\Omega)$ for all $1 \le p < \infty$ and moreover there exists a positive constant M_*, independent of $v \in H^1(\Omega)^2$, such that

$$|\theta| + |\nabla\theta| \le M_* \text{ on } \Omega. \tag{3.2}$$

By condition (H2), theorem 1 in chapter 1 and theorem 1 in chapter 2 in [11] show that the solution of problem (3.1) belongs to $W^{2,p}(\Omega) \cap H_0^1(\Omega)$ for every $2 \le p < \infty$. Using this property, we can define an operator $\Lambda : H^1(\Omega)^2 \to H_0^1(\Omega) \cap (\cap_{2 \le p < \infty} W^{2,p}(\Omega))$ $(\subset C^1(\overline{\Omega}))$ by putting $\Lambda v = \theta$, where $v \in H^1(\Omega)^2$ and θ is the solution of (3.1).

Next we make an assumption (H3) for the vector fields $a(u)$ and $g(u)$.

(H3): $a(\cdot),\ g(\cdot)$ are vector fields from $\mathbf{R}^2$ into itself which satisfy respectively the following conditions (1) and (2):

- **(1):** $a(v) := (a_1(v), a_2(v))$ for $v := (v_1, v_2) \in \mathbf{R}^2$ and all of the partial derivatives $a_{ik}(v) := \frac{\partial a_i(v)}{\partial v_k},\ i,k = 1,2$, are bounded on $\mathbf{R}^2$, namely

$$|a_{ik}(v)| \le M(a),\ \forall v \in \mathbf{R}^2,\ i,k = 1,2, \tag{3.3}$$

where $M(a)$ is a positive constant depending only on $a(\cdot)$, and moreover each component $a_i(v),\ i = 1,2$, of $a(v)$ is of C^2-class on $\mathbf{R}^2$ such that

$$\begin{aligned} &a_{11}(v)|\xi|^2 + (a_{12}(v) + a_{21}(v))\xi\cdot\zeta + a_{22}(v)|\zeta|^2 \ge c_0(|\xi|^2 + |\zeta|^2),\\ &\forall v \in \mathbf{R}^2,\ \forall \xi,\ \zeta \in \mathbf{R}^N, \end{aligned} \tag{3.4}$$

where c_0 is a positive constant and the notation $|\cdot|$ denotes the norm in $\mathbf{R}^N$ as well as in $\mathbf{R}$,

- **(2):** $g(v) := (g_1(v), g_2(v))$ and each component $g_i(v),\ i = 1,2$, are Lipschitz continuous in $v \in \mathbf{R}^2$ and for each bounded set B in $\mathbf{R}^2$ there is a positive constant $M(B)$ such that

$$g(v)\cdot(v - w) \ge c_1|v|^2 - M(B),\ \forall v \in \mathbf{R}^2,\ \forall w \in B, \tag{3.5}$$

where c_1 is a positive constant independent of any subset B of X.

Theorem 2. *Assume that (H1) - (H3) hold. Let $\kappa > 0$ and $f = (f_1, f_2) \in L^2(\Omega)^2$. Then, system*

$$(QVI)\ \begin{cases} -\kappa\Delta\theta = h(\theta, u) \ \text{in } \Omega,\\ \theta = 0 \ \text{on } \Gamma,\\ u \in H^1(\Omega)^2,\ u \in K_0(\theta) \ \text{a.e. in } \Omega,\\ \displaystyle\int_\Omega \{\nabla a(u)\cdot\nabla(u - z) + g(u)\cdot(u - z)\}\,dx \le \int_\Omega f\cdot(u - z)dx,\\ \forall z \in H^1(\Omega)^2 \ \text{with } z \in K_0(\theta) \ \text{a.e. in } \Omega, \end{cases}$$

has at least one solution $\{\theta, u\}$ such that $\theta \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ for any $1 \le p < \infty$ and $u \in H^1(\Omega)^2$.

This theorem is obtained as an application of Theorem 1. The proof will be given in Section 5.

4. **Lemmas.** In this section we suppose that (H1) - (H3) hold. We obtain some regularity properties of $K_0(\theta)$ and X_{θ_1,θ_2} from (H1).

Lemma 1. *For each positive number M, there exist positive constants $C_i(M)$, $i = 1,2$, such that*

(1): *if $|\theta_k| \le M$, $k=1,2,3,4$, $|v_i| \le M$, $i=1,2$, $v=(v_1,v_2)$, then*

$$\left|X^{(i)}_{\theta_1,\theta_2}(v) - X^{(i)}_{\theta_3,\theta_4}(v)\right| \le C_1(M)\left(|\theta_1-\theta_3| + |\theta_2-\theta_4|\right),$$

(2): *if $\theta_k \in H^1_0(\Omega)\cap W^{2,p}(\Omega)$ for all $p\in[1,\infty)$, $v := (v_1,v_2) \in H^1(\Omega)^2$, $|\theta_k| \le M$ and $|v_i| \le M$ a.e. on Ω for $i=1,2$, $k=1,2$, then*

$$\left|\frac{\partial X^{(i)}_{\theta_1,\theta_2}(v)}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right| \le C_2(M)(|\nabla\theta_1| + |\nabla\theta_2| + |\nabla v| + 1)(|\nabla(\theta_1-\theta_2)| + |\theta_1-\theta_2|)$$

$$a.e.\ on\ \Omega,\ i=1,2,\ j=1,2,\cdots,N,$$

where $X^{(i)}_{\theta_1,\theta_2}(v)$ is the i-component of $X_{\theta_1,\theta_2}(v)$.

Proof. By the mean value theorem, there exists $\xi \in \mathbf{R}^2$ on the segment joinning points (θ_1,θ_2) and (θ_3,θ_4) such that

$$\begin{aligned}\left|X^{(i)}_{\theta_1,\theta_2}(v) - X^{(i)}_{\theta_3,\theta_4}(v)\right| \le & \left|\left(\frac{\partial X^{(i)}_{\sigma,\tau}(v)}{\partial\sigma}\right)_{(\sigma,\tau)=\xi}\right||\theta_1-\theta_3| \\ & + \left|\left(\frac{\partial X^{(i)}_{\sigma,\tau}(v)}{\partial\tau}\right)_{(\sigma,\tau)=\xi}\right||\theta_2-\theta_4|\end{aligned}$$

Since the partial derivatives of $X^{(i)}_{\sigma,\tau}(v_1,v_2)$ are bounded on the set

$$S_M := \{(\sigma,\tau,v_1,v_2); |\sigma|\le M,\ |\tau|\le M,\ |v_1|\le M,\ |v_2|\le M\},$$

there is a positive constant $C_1(M)$, depending only on M, such that

$$\left|X^{(i)}_{\theta_1,\theta_2}(v) - X^{(i)}_{\theta_3,\theta_4}(v)\right| \le C_1(M)\left(|\theta_1-\theta_3| + |\theta_2-\theta_4|\right). \tag{4.1}$$

Thus (1) is obtained.

Next, we show (2). By the chain rule, we have for $i=1,2$, $j=1,2,\cdots,N$,

$$\begin{aligned}&\left|\frac{\partial X^{(i)}_{\theta_1,\theta_2}(v)}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right| = \left|\frac{\partial X^{(i)}_{\theta_1,\theta_2}(v)}{\partial x_j} - \frac{\partial X^{(i)}_{\theta_1,\theta_1}(v)}{\partial x_j}\right| \\ \le\ & \left|\left(\frac{\partial X^{(i)}_{\sigma,\theta_2}(v)}{\partial\sigma}\right)_{\sigma=\theta_1} - \left(\frac{\partial X^{(i)}_{\sigma,\theta_1}(v)}{\partial\sigma}\right)_{\sigma=\theta_1}\right|\left|\frac{\partial\theta_1}{\partial x_j}\right| \\ & + \left|\left(\frac{\partial X^{(i)}_{\theta_1,\tau}(v)}{\partial\tau}\right)_{\tau=\theta_2} - \left(\frac{\partial X^{(i)}_{\theta_1,\tau}(v)}{\partial\tau}\right)_{\tau=\theta_1}\right|\left|\frac{\partial\theta_2}{\partial x_j}\right| \\ & + \left|\left(\frac{\partial X^{(i)}_{\theta_1,\tau}(v)}{\partial\tau}\right)_{\tau=\theta_1}\right|\left|\frac{\partial\theta_2}{\partial x_j} - \frac{\partial\theta_1}{\partial x_j}\right| + \left|\frac{\partial X^{(i)}_{\theta_1,\theta_2}(v)}{\partial v_1} - \frac{\partial X^{(i)}_{\theta_1,\theta_1}(v)}{\partial v_1}\right|\left|\frac{\partial v_1}{\partial x_j}\right| \\ & + \left|\frac{\partial X^{(i)}_{\theta_1,\theta_2}(v)}{\partial v_2} - \frac{\partial X^{(i)}_{\theta_1,\theta_1}(v)}{\partial v_2}\right|\left|\frac{\partial v_2}{\partial x_j}\right|\end{aligned}$$

Since all of the first derivatives of $X^{(i)}_{\sigma,\tau}(v)$ are Lipschitz continuous on the set S_M, there exists a positive constant $C_2(M)$ such that

$$\left|\frac{\partial X^{(i)}_{\sigma,\tau}(v)}{\partial\sigma}\right| + \left|\frac{\partial X^{(i)}_{\sigma,\tau}(v)}{\partial\tau}\right| \le C_2(M),\ \forall(\sigma,\tau,v_1,v_2)\in S_M,$$

$$\left|\left(\frac{\partial X^{(i)}_{\sigma,\tau}(v)}{\partial\sigma}\right)_{\sigma=\sigma_1,\tau=\tau_1} - \left(\frac{\partial X^{(i)}_{\sigma,\tau}(v)}{\partial\sigma}\right)_{\sigma=\sigma_2,\tau=\tau_2}\right| \le C_2(M)(|\sigma_1-\sigma_2|+|\tau_1-\tau_2|),$$

$$\left|\left(\frac{\partial X^{(i)}_{\sigma,\tau}(v)}{\partial\tau}\right)_{\sigma=\sigma_1,\tau=\tau_1} - \left(\frac{\partial X^{(i)}_{\sigma,\tau}(v)}{\partial\tau}\right)_{\sigma=\sigma_2,\tau=\tau_2}\right| \le C_2(M)(|\sigma_1-\sigma_2|+|\tau_1-\tau_2|),$$

$$\left|\frac{\partial X^{(i)}_{\sigma_1,\tau_1}(v)}{\partial v_k} - \frac{\partial X^{(i)}_{\sigma_2,\tau_2}(v)}{\partial v_k}\right| \le C_2(M)(|\sigma_1-\sigma_2|+|\tau_1-\tau_2|),\ k=1,2,$$

$$\forall(\sigma_j,\tau_j,v_1,v_2)\in S_M,\ j=1,2.$$

Therefore it follows that

$$\begin{aligned} &\left|\frac{\partial X^{(i)}_{\theta_1,\theta_2}(v)}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\right| \\ \le\ & C_2(M)\left(|\theta_1-\theta_2| + \left|\frac{\partial(\theta_1-\theta_2)}{\partial x_j}\right|\right)\left(\left|\frac{\partial\theta_1}{\partial x_j}\right| + \left|\frac{\partial\theta_2}{\partial x_j}\right| + \left|\frac{\partial v_1}{\partial x_j}\right| + \left|\frac{\partial v_2}{\partial x_j}\right| + 1\right)\end{aligned}$$

for all $(\theta_1,\theta_2,v_1,v_2)\in S_M$. Accordingly we have (2).

(Q.E.D.)

Next, we define an operator $\tilde{A}: H^1(\Omega)^2\times H^1(\Omega)^2 \to (H^1(\Omega)^2)^*$ associated with vector filelds $a(\cdot)$ and $g(\cdot)$ by

$$\begin{aligned} &< \tilde{A}(v,u), w > \\ &= \int_\Omega \{(a_{11}(v)\nabla u_1 + a_{12}(v)\nabla u_2)\cdot\nabla w_1 + (a_{21}(v)\nabla u_1 + a_{22}(v)\nabla u_2)\cdot\nabla w_2\}dx \\ &\quad + \int_\Omega g(v)\cdot w dx,\ \forall v,\ u:=(u_1,u_2),\ w:=(w_1,w_2)\in H^1(\Omega)^2 \end{aligned} \tag{4.2}$$

Lemma 2. *The operator $\tilde{A}: H^1(\Omega)^2\times H^1(\Omega)^2 \to (H^1(\Omega)^2)^*$ given by (4.2) is bounded and semi-monotone, and the coerciveness condition (2.3) holds for any bounded set G_0 in $H^1(\Omega)^2$.*

Proof. The boundedness of $\tilde{A}$ follows immediately from (4.2) and assumption (H3). For any fixed $v\in H^1(\Omega)^2$, the monotonicity of $\tilde{A}(v,\cdot)$ is due to (3.4) in (H3), and $u\to\tilde{A}(v,u)$ is continuous from $H^1(\Omega)^2$ into $(H^1(\Omega)^2)^*$ by the definition (4.2). Therefore $\tilde{A}(v,\cdot)$ is maximal monotone from $D(\tilde{A}(v,\cdot)) = H^1(\Omega)^2$ into $(H^1(\Omega)^2)^*$. Thus (SM1) holds.

Next, let $u,\ v,\ w\in H^1(\Omega)^2$ and $v_n\in H^1(\Omega)^2, n=1,2,\cdots$, such that $v_n\to v$ weakly in $H^1(\Omega)^2$, and hence $v_n\to v$ in $L^2(\Omega)^2$. Then,

$$\begin{aligned} &| < \tilde{A}(v_n,u) - \tilde{A}(v,u), w > | \\ \le\ & \sum_{i,k}\int_\Omega |a_{ik}(v_n)-a_{ik}(v)|(|\nabla u_1| + |\nabla u_2|)(|\nabla w_1| + |\nabla w_2|)dx \\ &+ \int_\Omega |g(v_n)-g(v)||w|dx. \end{aligned}$$

This implies that $\tilde{A}(v_n, u)$ converges to $\tilde{A}(v, u)$ in $(H^1(\Omega)^2)^*$, since $|a_{ik}(v_n) - a_{ik}(v)|(|\nabla u_1| + |\nabla u_2|)$ converges in $L^2(\Omega)$ by the Lebesgue dominated convergence theorem. Thus (SM2) holds.

For any $v := (v_1, v_2) \in H^1(\Omega)^2$ and any $w \in G_0$ we have by (H3)

$$\begin{aligned} < \tilde{A}(v,v), v - w > \quad \geq \quad & \int_\Omega \{c_0(|\nabla v_1|^2 + |\nabla v_2|^2) + c_1|v|^2\}dx \\ & - \int_\Omega M(a)(|\nabla v_1| + |\nabla v_2|)(|\nabla w_1| + |\nabla w_2|)dx - M(G_0)|\Omega| \end{aligned}$$

where $M(G_0)$ is a positive constant depending only on G_0 and $|\Omega|$ denotes the volume of Ω. It follows from this inequality that (2.3) holds.

(Q.E.D.)

5. **Proof of Main Theorem.** For simplicity, let $X := H^1(\Omega)^2$. Also, let $\Lambda : X \to H_0^1(\Omega) \cap (\cap_{1 \leq p < \infty} W^{2,p}(\Omega))$ be the same operator as is defined by (3.1), and $\tilde{A} : X \times X \to X^*$ be the operator given by (4.2).

Now, for each $v \in X$ we define a subset $K(v)$ of X by

$$K(v) := \{w \in X; w(x) \in K_0(\theta(x)) \text{ for a.e. } x \in \Omega,\ \theta = \Lambda v\}. \tag{5.1}$$

Clearly $K(v)$ is closed and convex in X for every $v \in X$, and by (H1) and (3.2), $\cup_{v \in X} K(v)$ is bounded in $L^\infty(\Omega)^2$; namely there is a positive constant M_1 such that

$$|w| \leq M_1 \text{ a.e. on } \Omega \text{ for all } w \in \cup_{v \in X} K(v). \tag{5.2}$$

Furthermore $K(v)$ is non-empty. In fact, fixing $x_0 \in \Omega$ and $v_0 \in K_0(\theta(x_0))$, we consider a function

$$w(x) := X_{\theta(x_0), \theta(x)}(v_0),\ \forall x \in \Omega. \tag{5.3}$$

Since $\theta \in W^{2,p}(\Omega) \subset C^1(\overline{\Omega})$ for large p, we see from (H1) that $w(x) \in K_0(\theta(x))$ for all $x \in \Omega$ and $w \in C^1(\overline{\Omega})^2$. Hence $w \in K(v)$, so that $K(v) \neq \emptyset$. By the way, it follows from (5.3) that each component $w^{(i)},\ i = 1, 2$, of w safisfies

$$\left| \frac{\partial w^{(i)}(x)}{\partial x_j} \right| = \left| \left(\frac{X^{(i)}_{\theta(x_0),\tau}(v_0)}{\partial \tau} \right)_{\tau = \theta(x)} \frac{\partial \theta(x)}{\partial x_j} \right| \leq M_2,\ \forall x \in \Omega,\ j = 1, 2, \cdots, N, \tag{5.4}$$

where M_2 is a positive constant independent of $v \in X$; the existence of a such a constant M_2 is seen from (H1) and (3.2).

Next we are going to construct a bounded subset G_0 of X such that (2.2) and (2.3) hold. Using the same constants M_1 and M_2 as (5.2) and (5.4), we put

$$G_0 := \{w \in X; |w|_X \leq (M_1 + NM_2)|\Omega|^{\frac{1}{2}}\}. \tag{5.5}$$

As was seen above, $K(v) \cap G_0 \neq \emptyset$ for every $v \in X$. Thus (2.2) is obtained. By Lemma 2, (2.3) is obtained, too.

Finally, let us verify that $K(v)$ satisfies conditions (K1) and (K2). To do so, assuming that $v_n \to v$ weakly in X and $w \in K(v)$, we define

$$w_n(x) = X_{\theta(x), \theta_n(x)}(w(x)),\ \text{for a.e. } x \in \Omega,\ n = 1, 2, \cdots,$$

where $\Lambda v_n = \theta_n$. Then it is easy to see that $w(x) \in K_0(\theta(x))$ and $w_n(x) \in K_0(\theta_n(x))$, $n = 1, 2, \cdots$, for a.e. $x \in \Omega$. Note by (5.2) that

$$|w(x)| \leq M_1,\ |w_n(x)| \leq M_1,\ n = 1, 2, \cdots,\ \text{for a.e. } x \in \Omega.$$

Futhermore, for each component $w_n^{(i)}$ and $w^{(i)}$ of w_n and w, $i = 1, 2$, respectively, we see from Lemma 1 that

$$\begin{aligned} \left|w_n^{(i)} - w^{(i)}\right|^2_{H^1(\Omega)} &= \left|w_n^{(i)} - w^{(i)}\right|^2_{L^2(\Omega)} + \sum_{j=1}^{N} \int_\Omega \left| \frac{\partial X^{(i)}_{\theta,\theta_n}(w)}{\partial x_j} - \frac{\partial w^{(i)}}{\partial x_j} \right|^2 dx \\ \leq\ & C_1(M_3)^2 |\theta_n - \theta|^2_{L^2(\Omega)} \\ & + N \cdot C_2(M_3)^2 \int_\Omega (|\nabla\theta| + |\nabla\theta_n| + |\nabla w| + 1)^2 (|\nabla(\theta - \theta_n)| + |\theta - \theta_n|)^2 \, dx, \end{aligned}$$

where $M_3 := M_1 + M_*$ with the same constant M_* as in (3.2). Here we note that Λ is a compact operator from X into $C^1(\overline{\Omega})$ because Λ is solution operator of (3.1), whence θ_n converges to θ in $C^1(\overline{\Omega})$ by [11]. Therefore, the above inequality implies that w_n converges to w in X. Thus condition $(K1)$ was verified.

Next, in order to verify (K2), assume that $v_n \to v$ weakly in X, $w_n \in K(v_n)$ and $w_n \to w$ weakly in X. Then $\theta_n := \Lambda v_n \to \theta := \Lambda v$ in $C^1(\overline{\Omega})$. Also, we may assume that $v_n(x) \to v(x)$, $w_n(x) \to w(x)$ for a.e. $x \in \Omega$. Putting $\tilde{w}_n(x) := X_{\theta_n(x),\theta(x)}(w_n(x))$, we see that $\tilde{w}_n(x) \in K_0(\theta(x))$ for a.e. $x \in \Omega$. We have

$$\begin{aligned} |\tilde{w}_n(x) - w(x)| &\leq |\tilde{w}_n(x) - w_n(x)| + |w_n(x) - w(x)| \\ &= \left|X_{\theta_n(x),\theta(x)}(w_n(x)) - w_n(x)\right| + |w_n(x) - w(x)| \\ &\leq C_1(M_3)|\theta_n(x) - \theta(x)| + |w_n(x) - w(x)| \\ &\to 0, \end{aligned}$$

for a.e. $x \in \Omega$; the last convergence is due to Lemma 1, (1). Hence, it follows from the closedness of $K_0(\theta(x))$ in $\mathbf{R}^2$ that $w(x) \in K_0(\theta(x))$ for a.e. $x \in \Omega$. Hence $w \in K(v)$. Thus (K2) holds.

We have checked all the conditions of Theorem 1 for the operator $\tilde{A} : X \times X \to X^*$, the family $\{K(v)\}$ of convex sets in X and the bounded subset G_0 of X which are given by (4.2), (5.1) and (5.5), respectively. Also, it is easily seen that (QVI) is reformulated as the following problem:

$$\begin{cases} u \in K(u), \\ < Au - f, u - w > \leq 0, \ \forall w \in K(u). \end{cases}$$

Now, by Theorem 1, we see that (QVI) has at least one solution u. (Q.E.D.)

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