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MULTIPLE SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS

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Abstract. This paper discusses the quasilinearization process, that is, reduction of a given boundary value problem to a quasi-linear one. Quasi-linear BVP has a solution of definite type which is induced by oscillatory properties of the linear part of a system. Conditions are given for a solution of a modified (quasi-linear) BVP to satisfy also the original problem. It is shown that multiple quasilinearizations with different linear parts if possible lead to multiple solutions.

1. **Introduction.** Consider the boundary value problem

$$x'' = f(t, x, x'), \quad t \in I := [0, 1], \tag{1}$$

$$x(0) = A, \quad x(1) = B. \tag{2}$$

Function $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and Lipschitzian with respect to x and x'.

The classical result states that the problem is solvable if f is bounded (the Picard theorem). Conditions for existence of a solution can be formulated also in terms of the lower and upper functions. Recall that functions α and β are the lower and upper functions for the problem (1), (2) if:

1. $\alpha(t) \le \beta(t)$;
2. $\alpha'' \ge f(t, \alpha, \alpha')$, $\beta'' \le f(t, \beta, \beta')$,
3. $\alpha(0) \le A \le \beta(0)$, $\quad \alpha(1) \le B \le \beta(1)$.

If additionally the Nagumo type condition holds then the problem (1), (2) has a solution $x(t)$ such that $\alpha(t) \le x(t) \le \beta(t)$ $\forall t \in I$.

L. Jackson and K. Schrader ([2], Th. 3.2) proved that in presence of α and β ($\alpha < \beta$ on $(0, 1)$) there exists a solution $\xi(t)$ with the property (B'), that is, there exists a sequence of solutions $\{x_n\}$ such that: (i) $x_n \to \xi$ and $x_n' \to \xi'$uniformly on $[0, 1]$; (ii) $\Delta_n = \xi - x_n \ne 0$ and has the same sign for all $n \ge 1$ and $0 \le t \le 1$, or for all $n \ge 1$ and $0 < t \le 1$; (iii) for each $0 < \delta < \frac{1}{2}$ there is a constant c depending on δ but not on n and t such that $|\Delta_n'(t)| \le c|\Delta_n(t)|$ for all $n \ge 1$ and $\delta \le t \le 1 - \delta$.

A solution $\xi(t)$ has then the following remarkable feature: equation of variations with respect to $\xi(t)$

$$y'' = f_x(t, \xi, \xi')y + f_{x'}(t, \xi, \xi')y' \tag{3}$$

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is disconjugate in $(0,1)$. The latter means that there exists no nontrivial solution of (3) with two zeros in $(0,1)$. By Sturm separation theorem this is equivalent to the statement that a solution of (3) with the initial conditions

$$y(0) = 0, \quad y'(0) = 1 \tag{4}$$

does not vanish in $(0,1)$.

By reasons which will be explained later we will call $\xi(t)$ **a zero type solution.** The existence of proper lower and upper functions is always accompanied by the existence of a 0-type solution. We will say that α and β are *the proper* lower and upper functions if in addition to the above three properties α and β are not identical. It is an easy matter to show that then (recall that f is Lipschitzian and solutions of the Cauchy problems are uniquely defined by the initial data) $\alpha(t) < \beta(t)$ as $t \in (0,1)$.

The problem

$$x'' = f(t,x), \quad x(0) = A, \quad x(1) = B \tag{5}$$

has a solution $x(t)$ if the lower and upper functions exist (a Nagumo type condition is not needed). This condition is necessary and sufficient for solvability. Evident but important consequence of the above observation is the following: if the problem has not a 0-type solution then α and β must be identical with $x(t)$. For example consider $x'' + k^2 x = h(t), \quad t \in [0,1]$; if $\alpha'' + k^2\alpha \geq h(t)$ and $\beta'' + k^2\beta \leq h(t)$ then $(\beta - \alpha)'' + k^2(\beta - \alpha) \leq 0$. The inequality $u'' + k^2 u \leq 0$ has not nonnegative (but not identical to zero) solutions if $k > \pi$. Then finding of α and β is not easier than finding a solution itself.

Let us discuss another drawback of the method of lower and upper functions. Usage of the method of lower and upper functions is often based on reduction the original equation (1) to a quasi-linear one of the form $x'' = F(t,x,x')$, where F is continuous and bounded. Recall the Conti's theorem [1] which states in adapted for our purposes form that the quasi-linear problem

$$x'' + p(t)x' + q(t)x = F(t,x,x'), \quad x(0) = A, \quad x(1) = B, \tag{6}$$

is solvable if the homogeneous problem

$$x'' + p(t)x' + q(t)x = 0, \quad x(0) = 0,\ x(1) = 0 \tag{7}$$

has only the trivial solution. It appears that the Conti's theorem is applied in a very restricted form.

In what follows we discuss the process of quasi-linearization for the second order equations and establish similar results for two-dimensional differential systems.

2. **Scalar quasi-linear problems.** Consider the quasi-linear problem (6).

Definition 2.1. We will say that the linear part $(L_2x)(t) := x'' + p(t)x' + q(t)x$ is i-nonresonant with respect to the boundary conditions (2), if a solution $x(t)$ of the Cauchy problem

$$(L_2x)(t) = 0, \quad x(0) = 0, \quad x'(0) = 1 \tag{8}$$

has exactly i zeros in the interval $(0,1)$ and $x(1) \neq 0$.

Definition 2.2. We will say that $\xi(t)$ is an i-type solution of the problem (1), (2) (resp.: (6)), if for small enough $r > 0$ the difference $u(t;r) = x(t;r) - \xi(t)$ has exactly i zeros in $(0,1)$ and $u(1;r) \neq 0$, where $x(t;r)$ is a solution of (1) (resp.: equation in (6)), which satisfies the initial conditions

$$x(0;r) = \xi(0), \quad x'(0;r) = \xi'(0) + r. \tag{9}$$

Remark 2.1. An i-type solution ξ of the problem (1), (2) (resp.: (6)) has the following characteristics in terms of the variational equation: a solution $y(t)$ of the variational equation

$$y'' = f_x(t, \xi(t), \xi'(t))y + f_{x'}(t, \xi(t), \xi'(t))y',$$
$$(\text{resp.:} \quad y'' + p(t)y' + q(t)y = F_x(t, \xi(t), \xi'(t))y + F_{x'}(t, \xi(t), \xi'(t))y')$$

subject to the initial conditions

$$y(0) = 0, \ y'(0) = 1, \tag{10}$$

either has exactly i zeros in the interval $(0, 1]$, or it has exactly i zeros in the interval $(0, 1)$ and $y(1) = 0$. The cases of the i-th zero of $y(t)$ being at $t = 1$ or $(i+1)$-th zero being at $t = 1$ are not excluded and the respective examples can be constructed.

Theorem 2.1 ([3], Th. 2.1)**.** *Quasi-linear problem (6) with an i-nonresonant linear part $(L_2x)(t)$ has an i-type solution.*

The proof can be found in [3], [4].

2.1. **Quasilinearization and multiple solutions.** Consider the problem (1), (2).

Definition 2.3. Let equations $x'' = f(t, x, x')$ and $x'' + p(t)x' + q(t)x = F(t, x, x')$, where the linear part is i-nonresonant in the interval I, be equivalent in the domain

$$D_1 = \{(t, x, x') : \quad 0 \le t \le 1, \quad |x| < N, \quad |x'| < N_1\} \tag{11}$$

in the sense that any solution $x : I \to \mathbb{R}$ of (1) with a graph in D_1 is also a solution of equation in (6) and vice versa. Suppose that any solution $x(t)$ of the quasi-linear problem (6) satisfies the estimates

$$|x(t)| < N, \quad |x'(t)| < N_1 \quad \forall t \in I. \tag{12}$$

We will say then that the problem (1), (2) *allows for quasilinearization* with respect to a domain D_1 and a linear part $(L_2x)(t)$.

Remark 2.2. Estimates of the form (12) for any solution $x(t)$ of the problem (6) can be obtained in terms of the respective Green's function and a constant M which bounds the modulus of $F(t, x, x')$ in (6) ([3]).

Remark 2.3. If the right sides in (1) and/or equation in (6) do not depend on x', then N_1 can be set to $+\infty$.

The following results in [3], [4] form a basis for application of the quasilinearization process for proving the existence of multiple solutions.

Theorem 2.2. *If the problem (1), (2) allows for quasilinearization with respect to some domain D_1 and some i-nonresonant linear part $(L_2x)(t)$, then it has an i-type solution.*

Theorem 2.3. *Suppose that the problem (1), (2) allows for quasilinearization with respect to D_1 and i-nonresonant linear part $(L_2x)(t)$, and, at the same time, it allows for quasilinearization with respect to a domain*

$$D_2 = \{(t, x, x') : \quad 0 \le t \le 1, \quad |x| < K, \quad |x'| < K_1\}$$

and j-nonresonant linear part $(l_2x)(t)$, where $i \ne j$.

Then the problem (1), (2) has at least 2 solutions of different types.

Corollary 2.1. *Suppose that the problem (1), (2) allows for quasilinearization with respect to n essentially different (in the sense of Definition 2.1) linear parts and n domains of the form (11). Then it has at least n different solutions.*

3. **Quasi-linear problems for systems.** In what follows we show that the same idea (of quasilinearization) can be applied also to other objects such as a two-dimensional differential system. Consider the quasi-linear system

$$\begin{cases} x' + a_{11}x + a_{12}y &= F_1(t,x,y), \\ y' + a_{21}x + a_{22}y &= F_2(t,x,y), \end{cases} \tag{13}$$

where functions F_1, F_2 are continuous, bounded and satisfy the Lipschitz conditions with respect to x and y. The boundary conditions are

$$x(0) = 0, \quad x(1) = 0. \tag{14}$$

Assume that

(A) the binary quadratic form $a_{21}u^2 + (a_{22} - a_{11})uv - a_{12}v^2$ is positive definite.

Introduce polar coordinates for solutions $(x,\, y)$ of the system (13) as

$$x(t) = r(t)\, \sin\phi(t), \quad y(t) = r(t)\, \cos\phi(t). \tag{15}$$

Equation for the angular function $\phi(t)$ is

$$\phi'(t) = a_{21}\sin^2\phi(t) + (a_{22} - a_{11})\sin\phi(t)\cos\phi(t) - a_{12}\cos^2\phi(t). \tag{16}$$

Since $\sin\phi(t)$ and $\cos\phi(t)$ cannot vanish simultaneously and in view of the condition **(A)** the angular function $\phi(t)$ is monotonically increasing.

Notice that the boundary conditions (14) in polar coordinates take the form

$$\phi(0) = 0, \quad \phi(1) = \pi. \tag{17}$$

Definition 3.1. A linear part $(LX)(t) := X' + A\,X$ (the vectorial form of the left side in (13)) is called *i-nonresonant with respect to the boundary conditions (14)* if the angular function $\phi(t)$, defined by the initial condition $\phi(0) = 0$, satisfies the inequalities

$$i\,\pi < \phi(1) < (i+1)\,\pi \qquad i \in \{0, 1, \ldots\}. \tag{18}$$

For instance, the linear part in $x' - y = 0,\ \ y' + k^2x = 0$ is 0-nonresonant (resp.: 1-nonresonant, 2-nonresonant) with respect to the boundary conditions (14) if $0 < k^2 < \pi^2$ (resp.: $\pi^2 < k^2 < 4\pi^2,\ 4\pi^2 < k^2 < 9\pi^2$).

3.1. **Main result for quasi-linear systems.** Consider the quasi-linear system (13).

Lemma 3.1. *Suppose that the linear part in (13) is i-nonresonant with respect to the boundary conditions (14) for some non-negative integer i. Then a set S of all solutions $X(t) = (x(t), y(t))$ to the problem (13), (14) is non-empty and compact in $C_2^1([0,1], R^2)$.*

Proof. The non-emptyness (the solvability) follows from classical results ([1], for instance) since the linear homogeneous BVP has the trivial solution only. This is a consequence of i-nonresonantness of the linear part in (13). Compactness follows from the integral representation of a solution of the problem (13), (14) via the Green's function and standard evaluations in order to show that the Arzela - Ascoli criterium is satisfied. □

Corollary 3.1. *A set of initial values $(0, y(0))$ of solutions to the problem (13), (14) is compact in R.*

Proof. A set $S_1 = \{(0, y(0)) \in R : X \in S\}$ is the image of a continuous map $M : C_2^1([0,1]) \to R$ defined by $M(x,y) = (0, y(0))$. Since S is compact S_1 is compact also. Notice that there exist maximal and minimal elements in S_1, respectively, $(0, y_{max})$ and $(0, y_{min})$. The respective solutions of the BVP (13), (14) are denoted (in vectorial form) ξ_{max} and ξ_{min}. So the initial values of ξ_{max} are $(0, y_{max})$. If the BVP (13), (14) has a unique solution then $\xi_{max} = \xi_{min}$. □

Let $\xi = (\xi_1, \xi_2)$ be certain solution of the BVP (13), (14). In order to classify solutions of the BVP introduce (local) polar coordinates as

$$x(t) - \xi_1(t) = \rho(t)\,\sin\Theta(t), \quad y(t) - \xi_2(t) = \rho(t)\,\cos\Theta(t), \tag{19}$$

where $(x,\, y)$ is a solution of (13), (14).

Definition 3.2. Suppose that $\rho_0 = \rho(0) > 0$ is small and $\Theta(0) = 0$. We say that ξ is *an i-type solution* of the BVP (13), (14) if Θ satisfies

$$i\,\pi < \Theta(1;\rho_0) < (i+1)\,\pi, \quad i \in \{0, 1, \ldots\}. \tag{20}$$

Lemma 3.2. *Suppose that the linear part $(LX)(t) := X' + A\,X$ in (13) is i-nonresonant with respect to the boundary conditions (14). Let $\xi = (\xi_1, \xi_2)$ be any element of S and $X = (x, y)$ be any solution of (13), (14), $|y(0)| = \rho_0$.*

Then $\Theta(t)$, introduced by (19), satisfies

$$i\,\pi < \Theta(1;\rho_0) < (i+1)\,\pi \tag{21}$$

for large enough ρ_0.

Proof. Notice that both $X(t;\rho_0)$ and $\xi(t)$ are solutions of the equation (13) and $\Theta(0) = 0$. The normalized functions

$$u_1 = \frac{1}{\rho_0}(x(t;\rho_0) - \xi_1(t)), \quad u_2 = \frac{1}{\rho_0}(y(t;\rho_0) - \xi_2(t))$$

satisfy the system

$$\begin{cases} u_1' + a_{11}u_1 + a_{12}u_2 = \dfrac{1}{\rho_0}[F_1(t,x,y) - F_1(t,\xi_1,\xi_2)], \\ u_2' + a_{21}u_1 + a_{22}u_2 = \dfrac{1}{\rho_0}[F_2(t,x,y) - F_2(t,\xi_1,\xi_2)]. \end{cases} \tag{22}$$

The right sides in (22) tend to zero uniformly in $t \in [0,1]$ as $\rho_0 \to +\infty$. The functions $u_1(t)$, $u_2(t)$ tend to solutions $v_1(t)$, $v_2(t)$ of the homogeneous equation

$$\begin{cases} v_1' + a_{11}v_1 + a_{12}v_2 = 0, \\ v_2' + a_{21}v_1 + a_{22}v_2 = 0, \end{cases}$$

which satisfy the initial conditions $\phi(0) = 0$, $\sqrt{v_1^2(0) + v_2^2(0)} = 1$, where $\phi(t)$ is the angular function for $(v_1(t), v_2(t))$. Therefore $\Theta(t) \to \phi(t)$ as $\rho_0 \to +\infty$, uniformly in $t \in [0,1]$. As a consequence, $\Theta(1)$ satisfies for given i the inequalities (18) together with $\phi(1)$. □

The main theorem follows.

Theorem 3.3. *Suppose that the linear part $(LX)(t) = X' + A\,X$ is i-nonresonant with respect to the boundary conditions (14) and the condition* **(A)** *is satisfied. Then a solution ξ_{max} (described in Corollary 3.1) is an i-type solution.*

Proof. Consider a solution $\xi_{max} = (\xi_1, \xi_2)$, mentioned in Corollary 3.1 and neighboring solutions $X(t) = (x, y)$ such that (19) hold for small enough $\rho_0 > 0$. Suppose that ξ_{max} is not an i-type solution. There are two possibilities.

Consider the case: for any $\varepsilon > 0$ there exists $0 < \rho_0 < \varepsilon$ such that either $\Theta(1; \rho_0) = i\pi$ or $\Theta(1; \rho_0) = (i+1)\,\pi$. Therefore $x(t)$ solves the BVP (13), (14) and ξ_{max} is not a maximal solution.

Consider the second possibility: either $\Theta(1; \rho_0) < i\,\pi$ or $\Theta(1; \rho_0) > (i+1)\,\pi$ for small ρ_0. To be definite, consider the first inequality. By Lemma 3.2 $\Theta(1; \rho_0)$ satisfies the inequality $\Theta(1; \rho_0) > i\,\pi$ for large enough $\rho_0 > 0$. Then, by continuity of $\Theta(1; \rho_0)$, there exists ρ_{0*} such that $\Theta(1; \rho_{0*}) = i\,\pi$. Therefore ξ_{max} is not a maximal solution in the sense of Corollary 3.1. □

3.2. **Application for systems.** Consider a two-dimensional differential system of the form

$$\begin{cases} x' = \lambda^2 y + h(t)x, \\ y' = -\mu^2 |x|^{p-1}\, x, \end{cases} \tag{23}$$

where $t \in [0, 1] := I$, $h \in C(I, R)$, $\lambda \neq 0$, $\mu \neq 0$, $p > 1$, with the boundary conditions of Dirichlet type

$$x(0) = x(1) = 0. \tag{24}$$

The boundary conditions (24) in polar coordinates take the form

$$\varphi(0) = 0, \qquad \varphi(1) = \pi + \pi n, \tag{25}$$

where $n = 0, 1, 2, \ldots$. The given system (23) is equivalent to a system

$$\begin{cases} x' - \lambda^2 y = h(t)x, \\ y' + k^2 x = k^2 x - \mu^2 |x|^{p-1}\, x, \end{cases} \tag{26}$$

where the coefficient k satisfies

$$\sin(\lambda k) \neq 0. \tag{27}$$

In order to truncate the function $f_k(x) := k^2 x - \mu^2 |x|^{p-1}\, x$ we calculate the value of this function at the point of local extremum x_0. Set

$$M_k := |f_k(x_0)| = \mu^{\frac{2}{1-p}} \cdot \left(\frac{k^2}{p}\right)^{\frac{p}{p-1}} \cdot |p-1|. \tag{28}$$

Choose N_k such that

$$|x| \leq N_k \Rightarrow |f_k(x)| \leq M_k.$$

Computation gives that

$$N_k = \left(\frac{k^2}{\mu^2}\right)^{\frac{1}{p-1}} \cdot \gamma, \tag{29}$$

where γ is a positive root of the equation $\gamma^p = \gamma + (p-1) \cdot p^{\frac{p}{1-p}}$.

Instead of the function $f_k(x)$ consider

$$F_k(x) := f_k\big(\delta(-N_k, x, N_k)\big)$$

and instead of the function $h(t)x$ consider

$$H_k(t,\, x) := h(t) \cdot \delta(-N_k, x, N_k),$$

where

$$\delta(u,v,z)=\begin{cases} z, & v>z\\ v, & u\le v\le z\\ u, & v<u.\end{cases}$$

Then the multiple modified quasi-linear systems for different values of k

$$\begin{cases} x'-\lambda^2 y=H_k(t,\,x),\\ y'+k^2x=F_k(x)\end{cases} \tag{30}$$

are equivalent to the given nonlinear one (23) in the respective domains

$$\Omega_k=\{(t,x,y):\ 0\le t\le 1,\ |x|<N_k,\ y\in R\}. \tag{31}$$

The obtained quasi-linear problems (30), (24) are solvable. The respective solutions $X_k(t):=\big(x_k(t),\,y_k(t)\big)$ can be written in the integral form

$$\begin{cases} x_k(t)=\displaystyle\int_0^1\Big(G_k^{11}(t,s)H_k\big(s,\,x(s)\big)+G_k^{12}(t,s)F_k\big(x(s)\big)\Big)ds,\\ y_k(t)=\displaystyle\int_0^1\Big(G_k^{21}(t,s)H_k\big(s,\,x(s)\big)+G_k^{22}(t,s)F_k\big(x(s)\big)\Big)ds,\end{cases} \tag{32}$$

where $G_k^{ij}(t,\,s)$ $(i,\,j=1,\,2)$ are the elements of the Green's matrix to the respective homogeneous problem

$$\begin{cases} x'-\lambda^2 y=0,\\ y'+k^2x=0,\end{cases}\qquad x(0)=x(1)=0. \tag{33}$$

(The first index shows a number of a row, the second shows a number of a column).

Then

$$\begin{cases} |x_k(t)|\le\Gamma_k^{11}\cdot A_h\cdot N_k+\Gamma_k^{12}\cdot M_k,\\ |y_k(t)|\le\Gamma_k^{21}\cdot A_h\cdot N_k+\Gamma_k^{22}\cdot M_k,\end{cases} \tag{34}$$

where $A_h:=\max\limits_{t\in I}\big|h(t)\big|$ and Γ_k^{ij} are the estimates of the elements $G_k^{ij}(t,\,s)$ to the Green's matrix. If moreover the inequality

$$\Gamma_k^{11}\cdot A_h\cdot N_k+\Gamma_k^{12}\cdot M_k\ <\ N_k \tag{35}$$

holds, then the solution $X_k(t)$ is located in the domain of equivalence Ω_k (31). It means that $X_k(t)$ also solves the original nonlinear problem (23), (24).

Thus the inequality (35) is a sufficient condition for the solvability of the given nonlinear problem (23), (24).

Since the Green's matrix of the homogeneous linear problem (33) is given by

$$G_k(t,\,s)=\begin{cases}\begin{pmatrix}\dfrac{-\cos(\lambda ks)\sin(\lambda k(t-1))}{\sin(\lambda k)} & \dfrac{\lambda\sin(\lambda ks)\sin(\lambda k(t-1))}{k\sin(\lambda k)}\\ -\dfrac{k\cos(\lambda ks)\cos(\lambda k(t-1))}{\lambda\sin(\lambda k)} & \dfrac{\sin(\lambda ks)\cos(\lambda k(t-1))}{\sin(\lambda k)}\end{pmatrix}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if}\qquad 0\le s\le t\le 1,\\[2ex] \begin{pmatrix}-\dfrac{\sin(\lambda kt)\cos(\lambda k(s-1))}{\sin(\lambda k)} & \dfrac{\lambda\sin(\lambda kt)\sin(\lambda k(s-1))}{k\sin(\lambda k)}\\ -\dfrac{k\cos(\lambda kt)\cos(\lambda k(s-1))}{\lambda\sin(\lambda k)} & \dfrac{\cos(\lambda kt)\sin(\lambda k(s-1))}{\sin(\lambda k)}\end{pmatrix}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if}\qquad 0\le t< s\le 1,\end{cases} \tag{36}$$

then

$$|G_k^{11}(t,\,s)|\le\frac{1}{|\sin(\lambda k)|}=:\Gamma_k^{11}\quad\text{and}\quad |G_k^{12}(t,\,s)|\le\left|\frac{\lambda}{k\sin(\lambda k)}\right|=:\Gamma_k^{12}. \tag{37}$$

Taking into consideration the expressions for M_k (28), N_k (29) and the estimates Γ_k^{11}, Γ_k^{12} (37), the inequality (35) reduces to

$$\frac{A_h\cdot\gamma}{|\sin(\lambda k)|}+\frac{|\lambda k|}{|\sin(\lambda k)|}\cdot p^{\frac{p}{1-p}}\cdot(p-1)<\gamma. \tag{38}$$

In other words, if the inequality (38) holds then the original nonlinear problem (23), (24) has a solution of definite type. It is an i-type solution, if the linear part

$$(LX)(t):=X'+\begin{pmatrix}0 & -\lambda^2\\ k^2 & 0\end{pmatrix}X \tag{39}$$

is i-nonresonant with respect to the given boundary conditions (24). The following theorem is valid.

Theorem 3.4. *If the inequality*

$$\frac{A_h\cdot\gamma}{|\sin(\lambda k)|}+\frac{|\lambda k|}{|\sin(\lambda k)|}\cdot p^{\frac{p}{1-p}}\cdot(p-1)<\gamma,$$

where γ is the positive root of the equation $\gamma^p=\gamma+(p-1)\cdot p^{\frac{p}{1-p}}$, holds for some value k such that $\lambda k\in(i\pi,\,(i+1)\pi)$, $\quad(i=0,\,1,\,2,\ldots)$, then there exists an i-type solution of the problem (23), (24).

Corollary 3.2. *If there exist numbers k_j, $(j=0,\,1,\,2,\ldots,n)$ such that $\lambda k_j\in(j\pi,\,(j+1)\pi)$ and satisfy the inequality (38), then there exist at least $n+1$ solutions of different types to the nonlinear problem (23), (24).*

We have obtained the results (see Table 1), which show that certain values of λk in the form $\lambda k=\dfrac{\pi}{2}+\pi n,\ n=0,\,1,\,2,\ldots$ for some values of p and A_h are good for the inequality (38) to be satisfied.

Table 1. Results of calculations for the BVP (23), (24).

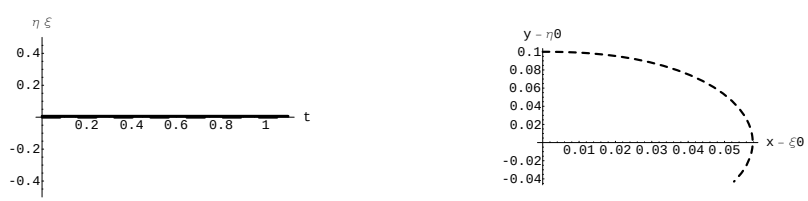
p	γ	A_h	$\lambda k = \frac{\pi}{2} + \pi n$
$\frac{3}{2}$	1.25089	$A_h \le 0.81396$	$\frac{\pi}{2}$
		$A_h \le 0.44189$	$\frac{\pi}{2}; \frac{3\pi}{2}$
		$A_h \le 0.06982$	$\frac{\pi}{2}; \frac{3\pi}{2}; \frac{5\pi}{2}$
$\frac{5}{4}$	1.28132	$A_h \le 0.89957$	$\frac{\pi}{2}$
		$A_h \le 0.69872$	$\frac{\pi}{2}; \frac{3\pi}{2}$
		$A_h \le 0.49786$	$\frac{\pi}{2}; \frac{3\pi}{2}; \frac{5\pi}{2}$
		$A_h \le 0.29701$	$\frac{\pi}{2}; \frac{3\pi}{2}; \frac{5\pi}{2}; \frac{7\pi}{2}$
		$A_h \le 0.09616$	$\frac{\pi}{2}; \frac{3\pi}{2}; \frac{5\pi}{2}; \frac{7\pi}{2}; \frac{9\pi}{2}$

This table may be interpreted as a set of multiplicity results for two-point nonlinear boundary value problem (23), (24).

3.3. **Example.** Consider the boundary value problem

$$\begin{cases} x' = y + 0.3\cos 2t \cdot x, \\ y' = -10|x|^{\frac{1}{3}}x, \end{cases} \qquad x(0) = x(1) = 0. \tag{40}$$

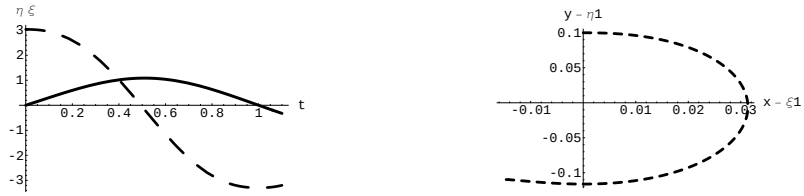
In accordance with Corollary 3.2 and calculations (see Table 1) there exist at least 3 solutions of different types to the given nonlinear problem (40). We have computed them. The zero type solution is the trivial one ($(\xi_0, \eta_0)(t) \equiv (0, 0)$).



a) b)

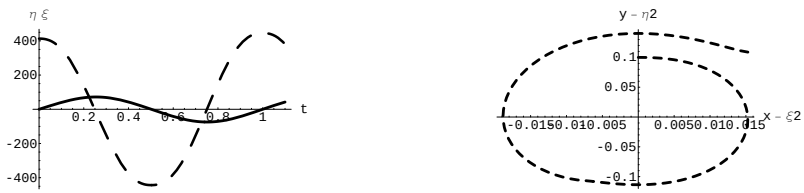
Fig. 1. 0-type solution of the BVP (40):

a) $\xi_0(t) \equiv 0,\ \eta_0(t) \equiv 0$;
b) phase portrait of the difference between neighboring solution $\big(x(t;\delta),\ y(t;\delta)\big)$ and $\big(\xi_0(t),\ \eta_0(t)\big),\ t \in [0,\ 1]$, if $\delta = 0.1$.



a) b)

Fig. 2. 1-type solution $(\xi_1, \eta_1)(t)$ of the BVP (40):
a) $\xi_1(0) = 0,\ \eta_1(0) = 3.033$;
b) phase portrait of the difference between neighboring solution $\big(x(t;\delta),\ y(t;\delta)\big)$ and $\big(\xi_1(t),\ \eta_1(t)\big),\ t \in [0,\ 1]$, if $\delta = 0.1$.



a) b)

Fig. 3. 2-type solution $(\xi_2, \eta_2)(t)$ of the BVP (40):
a) $\xi_2(0) = 0,\ \eta_2(0) = 413.139$;
b) phase portrait of the difference between neighboring solution $\big(x(t;\delta),\ y(t;\delta)\big)$ and $\big(\xi_2(t),\ \eta_2(t)\big),\ t \in [0,\ 1]$, if $\delta = 0.1$.

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