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SIGN–CHANGING SOLUTIONS FOR AN ASYMPTOTICALLY LINEAR SCHRÖDINGER EQUATION

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Abstract. We study an asymptotically linear Schrödinger equation in the whole Euclidean space $\mathbb{R}^N$. By using a suitable linking theorem we prove that the problem admits not only the trivial solution but also either a sign–changing solution or one positive and one negative solution.

1. **Introduction.** Let us consider the following Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u) & x \in \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to +\infty. \end{cases} \tag{1.1}$$

In order to overcome the lack of compactness of the problem, we assume that the potential $V(x)$ has a "good" behaviour at infinity in such a way the Schrödinger operator $-\Delta + V(x)$ on $L^2(\mathbb{R}^N)$ has a discrete spectrum. More precisely, we suppose

(V_1) $V \in L^2_{loc}(\mathbb{R}^N)$, V bounded from below;

(V_2) there exist $r_0 > 0$ such that for any $h > 0$

$$meas\left(B_{r_0}(y) \cap V^h\right) \to 0 \quad \text{as } |y| \to +\infty,$$

where $meas(A)$ denotes the Lebesgue measure of A on $\mathbb{R}^N$, $B_{r_0}(y)$ is the ball of $\mathbb{R}^N$ centered at y with radius r_0 and $V^h = \left\{x \in \mathbb{R}^N : V(x) < h\right\}$.

Let $\{\lambda_j\}$ be the sequence of eigenvalues of the operator $-\Delta + V(x)$ (see proposition 2.1 in section 2). Concerning the nonlinearity f, setting $F(x,t) = \int_0^t f(x,s)ds$, we make the following assumptions:

(F_1) $f \in C(\mathbb{R}^N \times \mathbb{R})$; there exist $F_0 > 0$, $L \geq 0$ such that $|f(x,t)| \leq F_0\,|t|$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $f(x,t) + Lt$ is increasing in t;

(F_2) $\lambda_1 < \liminf_{t\to 0} \frac{f(x,t)}{t} \leq \limsup_{t\to 0} \frac{f(x,t)}{t} < \lambda_k$ uniformly for $x \in \mathbb{R}^N$;

(F_3) $2F(x,t) \leq \alpha t^2$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $\alpha < \lambda_{k+1}$;

(F_4) $\lim_{|t|\to\infty} \frac{2F(x,t)}{t^2} = \beta(x) \geq \lambda_k$ uniformly for $x \in \mathbb{R}^N$, where $\beta(x) \neq \lambda_k$;

(F_5) there exists $W(x) \in L^1(\mathbb{R}^N)$ such that the following alternative holds:

$$i) \qquad \begin{cases} f(x,t)t - 2F(x,t) \geq W(x) & \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R} \text{ and} \\ \lim_{|t|\to\infty} f(x,t)t - 2F(x,t) = +\infty & \text{for all } x \in \mathbb{R}^N \end{cases}$$

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or

$$ii) \qquad \begin{cases} f(x,t)t - 2F(x,t) \le W(x) & \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R} \text{ and} \\ \lim\limits_{|t|\to\infty} f(x,t)t - 2F(x,t) = -\infty & \text{for all } x \in \mathbb{R}^N. \end{cases}$$

We shall prove the following result.

Theorem 1.1. *Assume (V_1), (V_2), $(F_1) - (F_5)$. Then, (1.1) has either a sign-changing solution or one positive and one negative solution.*

Remark 1.2. The equation (1.1) with a potential V verifyng $(V_1) - (V_2)$ has been studied for a superlinear nonlinearity $f(x,t)$ in [9] and [10] while sign-changing solutions have been found in a different asymptotically linear case in [15] if V satisfies the following conditions

$(V_1)'$ $V \in L^\infty_{loc}(\mathbb{R}^N)$, $V \ge V_0 > 0$ on $\mathbb{R}^N$;

$(V_2)'$ for any $r > 0$ and for any $h > 0$ it is

$$meas\left(B_r(y) \cap V^h\right) \to 0 \quad \text{as } |y| \to +\infty.$$

Let us point out that, under our assumptions on $f(x,t)$, we can assume without loss of generality that V is strictly positive just replacing $V(x)$ with $V(x)+L$ and $f(x,u)$ with $f(x,u)+L$, L large enough. Moreover, (V_2) and $(V_2)'$ are equivalent since, fixed $r > 0$ we can always cover $B_r(y)$ by a finite number of balls $B_{r_0}(y_i)$ such that $|y_i| \to +\infty$ if $|y| \to +\infty$.

Remark 1.3. An existence result like Theorem 1.1 has been stated in [14] when the nonlinear term f satisfies the same assumptions (F_1)–(F_5) while the potential V verifies the following assumptions introduced by Bartsch and Wang in [4]:

$(V_1)''$ $V(x) = \lambda g(x) + 1$, $\lambda > 0$, $g \in C(\mathbb{R}^N, \mathbb{R})$, $g \gneqq 0$ and $\Omega = int(g^{-1}(0)) \ne \varnothing$;

$(V_2)''$ there exist $r_0 > 0$ and $h_0 > 0$ such that

$$meas\left(B_{r_0}(y) \cap V^{h_0}\right) \to 0 \quad \text{as } |y| \to +\infty;$$

$(V_3)''$ $\bar{\Omega} = g^{-1}(0)$ and $\partial\Omega$ is locally Lipschitz.

Clearly, $(V_1)''$ is very stronger than (V_1) while $(V_2)''$ is weaker than (V_2). Always if V verifies $(V_1)'' - (V_3)''$, the existence of one positive solution (respectively one changing sign-solution) to (1.1) with different asymptotically linear nonlinearities has been proved in [7] (resp. in [12]) while multiple solutions have been stated in [6] and in [8] if $f(x,t)$ is odd in t.

2. **The variational setting.**

From now on, following Remark 1.2 we assume that V verifies (V_1)–(V_2) with $V \ge V_0 > 0$ on $\mathbb{R}^N$. Hence, we can consider the Hilbert space

$$E = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2 \right) dx < \infty \right\}$$

endowed with the inner product $(u,v) = \int\limits_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\, dx$ for $u, v \in E$ and norm $\|u\| = (u,u)^{\frac{1}{2}}$. Clearly it is $E \hookrightarrow H^1(\mathbb{R}^N)$. In the sequel we will denote by $|u|_{L^2(\Omega)}$ the norm in $L^2(\Omega)$ and by $|u|_2$ the norm in $L^2(\mathbb{R}^N)$. In order to overcome the lack of compactness of the problem, the following proposition is crucial.

Proposition 2.1. *Let $V(x)$ verifying (V_1) and (V_2). Then the imbedding $E \hookrightarrow L^t(\mathbb{R}^N)$ is continuous if $t \in [2, 2^*]$ and compact if $t \in [2, 2^*[$.*
Hence, the spectrum of the self-adjoint realization of $-\Delta + V(x)$ in $L^2(\mathbb{R}^N)$ is

discrete, i.e. there exists an orthogonal basis of eigenfunctions $\{\psi_j\}$ in $L^2(\mathbb{R}^N)$, the corresponding eigenvalues λ_j tending to $+\infty$. Moreover the first eigenvalue is simple and any associated eigenfunction has constant sign.

Proof. The proof is essentially contained in [2] since E is a particular case of a class of weighted Sobolev spaces introduced in [1]. Really, Benci and Fortunato consider a strictly positive potential $V \in L^2_{loc}(\mathbb{R}^N)$ such that

$$\int_{B_1(x)} \frac{1}{V(y)}\, dy \to 0 \qquad \text{if} \quad |x| \to \infty,$$

but it is easy to prove that for V positive this assumption is equivalent to (V_2) (see [10, Proposition 3.1] and Remark 1.2). The last part of the proposition follows by [3, Theorem 3.4]. □

Remark 2.2. Let us recall that, if V verifies the assumptions $(V_1)'' - (V_3)''$, the operator $-\Delta + V(x)$ has a finite number of eigenvalues below the infimum of the essential spectrum (see [4]).

Let us consider the functional

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx, \qquad u \in E.$$

Then, standard arguments prove that $I \in C^1(E, \mathbb{R})$ and the weak solutions of (1.1) correspond to the critical points of I.

The proof of the our main result will be obtained by a suitable application of an abstract critical point theorem stated in [14]. For completeness, we recall here this theorem.

Let E be a Hilbert space with norm $\|u\|$ such that $E = N \oplus M$ and $\dim N < \infty$. Let G be a C^1 functional on E of the form $G(u) = \frac{1}{2}\|u\|^2 - J(u)$.

Let P denote a closed convex positive cone of E, $D_0^{(1)}$ and $D_0^{(2)}$ be open convex subsets of E, $S = E \backslash D_0^{(1)} \cup D_0^{(2)}$. In applications, $D_0^{(1)} \cup D_0^{(2)}$ contains all possible positive and negative critical points, with respect to the positive cone P, and S includes all possible sign-changing critical points. Hence, non trivial sign-changing solutions can be obtained by different chooses of $D_0^{(i)}$ and S.

Let us assume that

(H_1) $J'(D_0^{(i)}) \subset D_0^{(i)}$, $\quad i = 1, 2$;

(H_2) there exists $e_0 \in E$ such that $(1-t)D_0^{(i)} + te_0 \subset D_0^{(i)}$ for all $t \in [0, 1]$, $i = 1, 2$;

(H_3) there exist $\delta > 0$ and $z_0 \in N$ with $\|z_0\| = 1$ such that

$$B = \{u \in M : \|u\| \geq \delta\} \cup \{sz_0 + v : v \in M,\ s \geq 0,\ \|sz_0 + v\| = \delta\} \subset S.$$

As $D_0^{(i)}$ is convex, (H_2) is true if $D_0^{(1)} \cap D_0^{(2)} \neq \varnothing$.

In the sequel, we shall consider the following weak Palais-Smale condition, shortly (wPS), introduced in [13]:

(wPS) for any sequence $\{u_n\}$ such that $\{G(u_n)\}$ is bounded and $G'(u_n) \to 0$, either $\{u_n\}$ is bounded and has a convergent subsequence or $\|G'(u_n)\|\,\|u_n\| \to \infty$.

The following result holds (see [14, Theorem 2.1]).

Theorem 2.3. *Assume $(H_1) - (H_3)$. Let G be a C^1-functional on E verifying (wPS) which maps bounded sets to bounded sets and such that*

$$b_0 = \inf_M G \neq -\infty, \qquad a_0 = \sup_N G \neq \infty.$$

Then, G has a critical point in S with critical value $c \geq \inf_B G$.

Remark 2.4. The (wPS) assumption is a compactness condition stronger than the following weak Palais-Smale condition proposed by Cerami in [5]:

(C) any sequence $\{u_n\}$ such that $\{G(u_n)\}$ is bounded and $\|G'(u_n)\| \, \|u_n\| \to 0$, has a convergent subsequence.

3. **Proof of the main theorem.** From now on, we will denote by $\{\lambda_j\}$ the eigenvalues of $-\Delta + V(x)$ counting multiplicity, i.e. $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_n \leq ...$, and by M_{λ_n} the corresponding eigenspaces. Moreover, for any integer j let $E_j = \oplus_{n=1}^{j} M_{\lambda_n}$. In order to give the proof of Theorem 1.1, first we state some usefull lemmas. Really, those results have been already stated by Zou in [14] in a weaker form, since assumptions $(V_1)'' - (V_3)''$ only imply that the operator $-\Delta + V(x)$ has a finite number of eigenvalues below the infimum of the essential spectrum. Here, by combining the Zou's arguments and the discreteness of the spectrum of $-\Delta + V(x)$, we have the following lemmas.

Lemma 3.1. *Let F verifying (F_4). Then a constant $C_1 = C_1(k) > 0$ exists such that*

$$I(u) \leq C_1 \qquad \text{for all } u \in E_k.$$

Proof. As the dimension of E_k is finite, it suffices to prove that $I(u) \leq 0$ for $u \in E_k$, $\|u\|$ large enough. Arguing by contradiction, assume that a sequence $\{u_n\} \subset E_k$ exists such that $\|u_n\| \to \infty$ and $I(u_n) > 0$. Passing to a subsequence, we can assume that

$$\frac{u_n}{\|u_n\|} \to w_0 \quad \text{in } E_k \text{ with } \|w_0\| = 1.$$

By (F_4) it is

$$F(x,u) = \frac{\beta(x)}{2} |u|^2 + P(x,u),$$

where

$$P(x,u) = o(|u|^2) \qquad \text{uniformly for } x \in \mathbb{R}^N \text{ as } |u| \to \infty. \tag{3.1}$$

Clearly,

$$\|u\|^2 \leq \lambda_k |u|_2^2 \qquad \text{for any } u \in E_k.$$

By the standard elliptic theory and the Schechter-Simon Theorem on the unique continuation property for Schrödinger operators (see [11]), there exists $\varepsilon_o > 0$ such that

$$\|u\|^2 - \int_{\mathbb{R}^N} \beta(x) u^2 dx \leq -\varepsilon_o \|u\|^2 \qquad \text{for } u \in E_k.$$

Finally, by (3.1) and the Fatou Lemma we have

$$\begin{aligned} 0 \quad &\leq \quad \lim_n \frac{I(u_n)}{\|u_n\|^2} = \lim_n \left(\frac{1}{2} - \frac{1}{2} \int_{\mathbb{R}^N} \beta(x) \frac{u_n^2}{\|u_n\|^2} dx + \int_{\mathbb{R}^N} \frac{P(x,u_n)}{\|u_n\|^2} dx \right) \\ &\leq \quad \frac{1}{2} - \frac{1}{2} \int_{\mathbb{R}^N} \beta(x) w_0^2 dx + o(1) < -\frac{\varepsilon_o}{2} + o(1), \end{aligned}$$

which gives the contradiction. □

Lemma 3.2. *Let F verifying (F_1) and (F_2). Then, two strictly positive constants $\delta = \delta(k)$ and $C_2 = C_2(k)$ exist such that*

$$I(u) \geq C_2 \qquad \text{for any } u \in E_{k-1}^{\perp} \text{ with } \|u\| = \delta.$$

Proof. By the upper inequality in (F_2) there exist $t_0 > 0$ and $\lambda^* > 0$, $\lambda^* < \lambda_k$ such that $f(x,t)t \leq \lambda^* t^2$ for any $x \in \mathbb{R}^N$, $|t| \leq t_0$.
It follows that

$$2F(x,t) \leq \lambda^* t^2 \qquad \text{for any } x \in \mathbb{R}^N, \ |t| \leq t_0. \tag{3.2}$$

Moreover, by the inequality in (F_1) it follows

$$2F(x,t) \leq 2F_0 t^2 - F_0 t_0^2 \qquad \text{for any } x \in \mathbb{R}^N, \ |t| \geq t_0. \tag{3.3}$$

Since the sequence $\{\lambda_n\}$ goes to $+\infty$, we can choose m large enough such that

$$(2F_0 + \lambda_m - 2\lambda^*)(\lambda_k - \lambda^*) \geq 4(\lambda^*)^2 \tag{3.4}$$

$$(\lambda_m - 2F_0)(\lambda_k - \lambda^*) \geq 32(\lambda^*)^2 \tag{3.5}$$

$$\lambda_m - 2F_0 - 4|2F_0 - \lambda^*| \geq 0 \tag{3.6}$$

$$\lambda^* \lambda_m > 2F_0 \lambda_k. \tag{3.7}$$

Fixed $u \in E_{k-1}^{\perp}$, we write $u = v + w$ with $v \in M_{\lambda_k} \oplus ... \oplus M_{\lambda_m}$ and $w \in E_m^{\perp}$. Let

$$\xi_1 = \frac{2F_0 + \lambda_m}{4} w^2 + \frac{\lambda_k + \lambda^*}{4} v^2 - F(x,u).$$

First, we will prove that $\xi_1 \geq 0$, i.e. $\xi_1(x) \geq 0$ for all $x \in \mathbb{R}^N$.
We recall that

$$a^2 + b^2 \geq 2ab \qquad \text{for all } a, b \in \mathbb{R}. \tag{3.8}$$

If $|v + w| \leq t_0$, by (3.2), (3.8) and (3.4) we have

$$\begin{aligned} \xi_1 &\geq \frac{2F_0 + \lambda_m}{4} w^2 + \frac{\lambda_k + \lambda^*}{4} v^2 - \frac{1}{2}\lambda^*(v+w)^2 \\ &\geq \frac{2F_0 + \lambda_m - 2\lambda^*}{4} w^2 + \frac{\lambda_k - \lambda^*}{4} v^2 - \lambda^* |vw| \\ &\geq \left(\frac{(2F_0 + \lambda_m - 2\lambda^*)^{\frac{1}{2}} (\lambda_k - \lambda^*)^{\frac{1}{2}}}{2} - \lambda^* \right) |vw| \geq 0. \end{aligned}$$

If $|v + w| > t_0$, by (3.3) we obtain

$$\xi_1 \geq \frac{\lambda_m - 2F_0}{4} w^2 + \frac{\lambda_k + \lambda^* - 4F_0}{4} v^2 - 2F_0 vw + \frac{1}{2} F_0 t_0^2 = \xi_2 + \xi_3 \tag{3.9}$$

where

$$\xi_2 = \frac{\lambda_m - 2F_0}{8} w^2 + \frac{\lambda_k - \lambda^*}{4} v^2 - \lambda^* vw \tag{3.10}$$

and

$$\xi_3 = \frac{\lambda_m - 2F_0}{8} w^2 - \frac{2F_0 - \lambda^*}{2} v^2 - (2F_0 - \lambda^*) vw + \frac{1}{2} F_0 t_0^2. \tag{3.11}$$

Now, we want to estimate ξ_2 and ξ_3. If

$$\frac{\lambda_k - \lambda^*}{4} |v| - \lambda^* |w| \geq 0,$$

since (3.5) implies $\lambda_m - 2F_0 > 0$, by (3.10) it follows

$$\xi_2 \geq \frac{\lambda_m - 2F_0}{8} w^2 + \left(\frac{\lambda_k - \lambda^*}{4} |v| - \lambda^* |w| \right) |v| > 0.$$

On the other hand, if

$$\frac{\lambda_k - \lambda^*}{4} |v| - \lambda^* |w| \leq 0 \text{ or equivalently } |v| \leq \frac{4\lambda^* |w|}{\lambda_k - \lambda^*},$$

by (3.10) and (3.5) we obtain again

$$\xi_2 \geq \left(\frac{\lambda_m - 2F_0}{8} - \frac{4(\lambda^*)^2}{\lambda_k - \lambda^*} \right) w^2 + \frac{\lambda_k - \lambda^*}{4} v^2 \geq 0.$$

Moreover, by (3.11), (3.8) and (3.6)

$$\begin{aligned} \xi_3 &\geq \frac{\lambda_m - 2F_0 - 4|2F_0 - \lambda^*|}{8} w^2 - \frac{2F_0 - \lambda^* + |2F_0 - \lambda^*|}{2} v^2 + \frac{1}{2} F_0 t_0^2 \\ &\geq -(1 + |2F_0 - \lambda^*|) v^2 + \frac{1}{2} F_0 t_0^2. \end{aligned}$$

Denoting by C_m a constant such that

$$\|v\|_\infty \leq C_m \|v\| \qquad \text{for any } v \in M_{\lambda_k} \oplus ... \oplus M_{\lambda_m},$$

let

$$\delta = \left(\frac{F_0 t_0^2}{2(1 + |2F_0 - \lambda^*|) C_m^2} \right)^{\frac{1}{2}}.$$

Obviously, if $\|u\| = \delta$, then $\|v\|_\infty \leq C_m \|v\| \leq C_m \|u\| = C_m \delta$, hence $\xi_3 \geq 0$. By (3.9) we conclude that $\xi_1 \geq 0$. Therefore, by (3.7) for any $u \in E_{k-1}^\perp$ with $\|u\| = \delta$ we have

$$\begin{aligned} I(u) &\geq \frac{1}{4} \|v\|^2 + \frac{1}{4} \|w\|^2 + \frac{1}{4} \lambda_k \|v\|_2^2 + \frac{1}{4} \lambda_m \|w\|_2^2 - \int_{\mathbb{R}^N} F(x, u) dx \\ &\geq \frac{1}{4} \left(1 - \frac{\lambda^*}{\lambda_k} \right) \|v\|^2 + \frac{1}{4} \left(1 - \frac{2F_0}{\lambda_m} \right) \|w\|^2 + \int_{\mathbb{R}^N} \xi_1(x) dx \\ &\geq \frac{1}{4} \left(1 - \frac{\lambda^*}{\lambda_k} \right) \delta^2 > 0, \end{aligned}$$

hence the conclusion follows. □

From now on, assuming δ as in the previous lemma, let

$$M = E_k^\perp, \qquad z_0 \in E_{k-1}^\perp \setminus E_k^\perp = M_{\lambda_k} \text{ with } \|z_0\| = 1, \tag{3.12}$$

and

$$B = \{u \in M : \|u\| \geq \delta\} \cup \{u = sz_0 + v : s \geq 0,\ v \in M,\ \|u\| = \delta\}. \tag{3.13}$$

Lemma 3.3. *Let us assume that $(F_1) - (F_3)$ hold. Then a positive constant C_3 exists such that $\inf_B I(u) \geq C_3 > 0$. Moreover, $\inf_M I(u) \geq 0$.*

Proof. Fixing $u \in M$, by (F_3) we obtain

$$I(u) \geq \frac{1}{2} \|u\|^2 - \frac{\alpha}{2} \int_{\mathbb{R}^N} u^2 dx \geq \frac{1}{2} \left(1 - \frac{\alpha}{\lambda_{k+1}} \right) \|u\|^2 \geq 0, \tag{3.14}$$

hence $\inf_M I(u) \geq 0$. The conclusion of the proof follows by Lemma 3.2 and (3.14) taking $C_3 = \min \left\{ C_2,\ \frac{1}{2} \left(1 - \frac{\alpha}{\lambda_{k+1}} \right) \delta^2 \right\}$. □

Lemma 3.4. *Let us assume that* $(V_1)-(V_2)$ *and* $(F_3),(F_5)$ *hold. Then* I *satisfies the* (wPS) *condition, i.e. for any sequence* $\{u_n\}\subset E$ *such that* $\{I(u_n)\}$ *is bounded and* $I'(u_n)\to 0$ *either* $\{u_n\}$ *is bounded and has a convergent subsequence or* $\|I'(u_n)\|\,\|u_n\|\to\infty$.

Proof. Let $\{u_n\}$ be such that

$$\{I(u_n)\} \text{ is bounded,} \tag{3.15}$$
$$I'(u_n)\to 0, \tag{3.16}$$
$$\{\|I'(u_n)\|\,\|u_n\|\} \text{ is bounded.} \tag{3.17}$$

Our aim is to prove that $\{u_n\}$ is bounded and has a convergent subsequence. By (3.15) and (3.17) it follows that

$$\left|I(u_n)-\frac{1}{2}\langle I'(u_n),u_n\rangle\right| = \left|\int_{\mathbb{R}^N}\left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right)dx\right| \le C_4 \tag{3.18}$$

while by (3.15) and (F_3) it is

$$\frac{1}{2}\|u_n\|^2 \le C_5 + \int_{\mathbb{R}^N} F(x,u_n)dx \le C_5 + \frac{\alpha}{2}|u|_2^2. \tag{3.19}$$

Arguing by contradiction, assume that $\|u_n\|\to\infty$, then by (3.19) up to subsequence it is

$$1 \le C_3 \lim_n \frac{|u|_2^2}{\|u_n\|^2}. \tag{3.20}$$

Now, if we set for any $h>0$ and $R>0$

$$\begin{aligned} A_1(R,h) &= \left\{x\in\mathbb{R}^N : |x|>R,\ V(x)\ge h\right\},\\ A_2(R,h) &= \left\{x\in\mathbb{R}^N : |x|>R,\ V(x)< h\right\},\end{aligned}$$

by [7, Lemma 2.1] for any $\varepsilon>0$ there exists $R_{\varepsilon,h}>0$ such that

$$|u|^2_{L^2(A_2(R_{\varepsilon,b},h))} \le \frac{\varepsilon}{2}\|u\|^2 \qquad \text{for any } u\in E. \tag{3.21}$$

Moreover,

$$\int_{A_1(R,h)} u^2dx \le \frac{1}{h}\int_{A_1(R,h)} V(x)u^2dx \le \frac{1}{h}\int_{\mathbb{R}^N} V(x)u^2dx$$

then for any $\varepsilon>0$ choosing h_ε large enough it is

$$\int_{A_1(R,h_\varepsilon)} u^2dx \le \frac{\varepsilon}{2}\|u\|^2. \tag{3.22}$$

Hence, (3.21) and (3.22) imply that for any $\varepsilon>0$ there exist $h_\varepsilon>0$ and $R_\varepsilon = R_{\varepsilon,h_\varepsilon}>0$ such that

$$\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} u^2dx = \int_{A_1(R_\varepsilon,h_\varepsilon)} u^2dx + \int_{A_2(R_\varepsilon,h_\varepsilon)} u^2dx \le \varepsilon\|u\|^2 \qquad \text{for any } u\in E.$$

By the last inequality and (3.20) we obtain that there exists $R>0$ such that

$$\lim_n \int_{B_R(0)} \frac{u_n^2}{\|u_n\|^2}dx \ge C_6,$$

then, $\lim_n |u_n|^2 = +\infty$ on a subset of $\mathbb{R}^N$ with positive measure. By (F_5) we deduce that

$$\left| \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x, u_n) u_n - F(x, u_n) \right) dx \right| \to \infty$$

which contradicts (3.18), therefore $\{\|u_n\|\}$ is bounded. Since E is compactly imbedded in $L^2(\mathbb{R}^N)$, by (3.16) we conclude in a standard way that $\{u_n\}$ has a convergent subsequence. $\square$

In order to apply the abstract Theorem 2.3 we choose the following subsets. Let $P = \{u \in E : u \geq e_1\}$ where e_1 is a positive eigenfunction corresponding to λ_1. Clearly, P is closed and convex. Choosing e_1 small enough, in [8] it has been proved that all positive solutions of the equation (1.1) belong to P while all negative solutions belong to $-P$.
The following result states that suitable neighborhoods of these sets are invariant sets.

Lemma 3.5. *Assume that* $(V_1)-(V_2)$ *and* (F_1) *hold. Then, there exist* $\varepsilon_0 > 0$ *such that*

$$J'(\pm D_0(\varepsilon)) \subset \pm D_0(\varepsilon) \qquad \text{for all } \varepsilon \in (0, \varepsilon_0),$$

where $D_0(\varepsilon) = \{u \in E : dist(u, P) < \varepsilon\}$. *Moreover,*

$$dist\left((M_{\lambda_1})^\perp, \pm P\right) > 0.$$

Proof. The first part of the lemma has been proved in [8, Theorem 4.2]. For the second part, see [8, Lemma 5.3] or better [14, Remark 4.1]. $\square$

Proof of Theorem 1.1. Let M and B as in (3.12) and (3.13). Setting $N = E_k$, by Lemmas 3.1 and 3.3 it is

$$\inf_M I(u) \geq 0, \qquad \sup_N I(u) < \infty.$$

Fixing $0 < \varepsilon < dist\left((M_{\lambda_1})^\perp, P\right)$, let

$$D_0^1 = D_0(\varepsilon), \qquad D_0^2 = \varnothing, \qquad S = E \setminus W, \qquad W = D_0^1 \cup D_0^2.$$

Then, assumptions (H_1) and (H_3) follow by the previous Lemma while (H_2) holds choosing $e_0 = e_1$; moreover, I maps bounded sets to bounded sets by (F_3) and satisfies (wPS) condition by Lemma 3.4. Whence, by Theorem 2.3 I has a nontrivial critical point $u_1 \in S$ with $I(u_1) \geq \inf_B I(u) > 0$. As $u_1 \notin D_0(\varepsilon)$, u_1 is either negative or sign-changing.
On the other hand, choosing $0 < \varepsilon < dist\left((M_{\lambda_1})^\perp, -P\right)$,

$$D_0^1 = \varnothing, \qquad D_0^2 = -D_0(\varepsilon), \qquad S = E \setminus W, \qquad W = D_0^1 \cup D_0^2, \qquad e_0 = -e_1,$$

by applying again Theorem 2.3 we find a nontrivial critical point $u_2 \in S$ with $I(u_2) \geq \inf_B I(u) > 0$ which is either positive or sign-changing. In any case, the found solution is nontrivial since its critical value is strictly positive while by (F_1) it is $I(0) = I'(0) = 0$. $\square$

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