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STABILITY ANALYSIS FOR TWO DIMENSIONAL ALLEN-CAHN EQUATIONS ASSOCIATED WITH CRYSTALLINE TYPE ENERGIES

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Abstract. This paper is devoted to the stability analysis for two dimensional interfaces in solid-liquid phase transitions, represented by some types of Allen-Cahn equations. Each Allen-Cahn equation is derived from a free energy, associated with a two dimensional Finsler norm, under the so-called crystalline type setting, and then the Wulff shape of the Finsler norm is supposed to correspond to the basic structural unit of masses of pure phases (crystals). Consequently, special piecewise smooth Jordan curves, based on Wulff shapes, will be exemplified in the main theorems, as the geometric representations of the stability condition.

1. **Introduction.** Let Ω be a two dimensional bounded domain with a Lipschitz boundary $\Gamma := \partial\Omega$. Let $\kappa > 0$ be a given (small) constant. Let $f \in C(\mathbb{R}^2)$ be a norm in $\mathbb{R}^2$, and let f° be the dual norm of f.

In this paper, the following type of evolution equation:

$$(E;\theta)_f \qquad u'(t) + \kappa\partial V_f(u(t)) + \partial I_{[-1,1]}(u(t)) \ni u(t) + \theta(t) \ \text{ in } L^2(\Omega), \quad t > 0;$$

with a forcing term $\theta \in L^2_{\rm loc}([0,+\infty); L^2(\Omega))$, is considered, where "$\,'\,$" denotes the time-derivative of functions, $\partial I_{[-1,1]}$ denotes the subdifferential of the indicator function $I_{[-1,1]}$ on the closed interval $[-1,1]$, and ∂V_f denotes the subdifferential of the L^2-restriction $V_f := \mathrm{Var}_f|_{L^2(\Omega)}$ of the following convex function Var_f on $L^1(\Omega)$:

$$z \in L^1(\Omega) \mapsto \mathrm{Var}_f(z) := \inf\left\{ \varliminf_{i\to+\infty} \int_\Omega f^\circ(\nabla\varphi_i)\,dx \;\middle|\; \begin{array}{l} \{\varphi_i\} \subset C^1(\Omega), \\ \varphi_i \to z \text{ in } L^1(\Omega) \\ \text{as } i \to +\infty \end{array} \right\}; \qquad (1)$$

known as "anisotropic total variation".

Equation $(E;\theta)_f$ is a type of "Allen-Cahn equation". As is well known, Allen-Cahn equation is a generic term of kinetic equations that are motivated to describe the dynamics of solid-liquid phase transitions, and each of them is usually derived as a gradient flow of an energy functional, called "free energy". In the case of equation $(E;\theta)_f$, the corresponding free energy is given by:

$$u \in L^2(\Omega) \mapsto \kappa V_f(u) + \int_\Omega \left\{ I_{[-1,1]}(u) - \frac{1}{2}\,u^2 - \theta u \right\} dx; \qquad (2)$$

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and in the context, θ is the (relative) temperature assuming the critical temperature to be 0, and u is the (nonconserved) order parameter which indicates the physical situation of material taking values into the closed interval $[-1,1]$. Indeed, the indicator function $I_{[-1,1]}$ as in (2) is to constrain the range of the order parameter u onto the desired closed interval $[-1,1]$, and due to this term, it is further seen that the density function:

$$u \in \mathbb{R} \mapsto I_{[-1,1]}(u) - \frac{1}{2}u^2 - \theta u;$$

of the integral part in (2) has a graph of double-well type, for each $\theta \in \mathbb{R}$. This property is quite important to characterize the bi-stability of solid and liquid phases, which are respectively indicated by values -1 and 1 of the order parameter u. In view of this, the integral part is often called as "bulk energy".

On the other hand, the convex function V_f as in (2) is called as "interfacial energy", and it is inserted to characterize the geometric patterns of the interfaces. Here, the unit ball $\mathcal{W}_f := \{\xi \in \mathbb{R} \,|\, f(\xi) < 1\}$, generated by the norm f, is called "Wulff shape", and it is supposed to be closely related to the anisotropic effects, appearing in the interface formations.

One of the simplest admissible choices for f is just the Euclidean norm:

$$f(\xi) := \sqrt{\xi_1^2 + \xi_2^2}, \quad \text{for all } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2; \tag{3}$$

and then the Wulff shape draws the usual unit open disk in $\mathbb{R}^2$. Additionally, since the interfacial energy V_f, in this case, forms the so-called total variation functional, the corresponding expression (2) of the free-energy just coincides with that proposed by Visintin [19, Chapter VI].

Under the setting by (3), the stability analysis for interfaces in steady-states has been addressed in several papers (e.g. [16, 18]), and special smooth Jordan curves, based on sufficiently large circles, have been exemplified as the geometric representations of the stability condition. But, at the same time, these results imply that the instability will be observed at nonsmooth parts of the interfaces, and then the setting by (3) may not be made for the representation of some angulate shapes, as in crystalline structures.

Hence, in this paper, we adopt the following mathematical formula, as one of possible setting of the norm f:

$$f(\xi) := \max_{k \in \mathbb{Z}} \nu_k \cdot \xi = \max_{0 \le k < 2m} \nu_k \cdot \xi, \quad \text{for all } \xi \in \mathbb{R}^2; \tag{4}$$

where $m \ge 2$ is a fixed integer, and $\nu_k := (\cos\frac{k\pi}{m}, \sin\frac{k\pi}{m})$, $k \in \mathbb{Z}$. The setting, kindred to (4), has been known as "crystalline type setting", and has been utilized by several authors, as a possible expression of anisotropy, from various viewpoints, such as "anisotropic mean curvature flow" (e.g. Bellettini-Caselles-Chambolle-Novaga [5], Giga-Rybka [9], Ishiwata [11] and Novaga-Paolini [15]), and "anisotropic gradient flow" (e.g. Giga-Ohtsuka-Schätzle [8] and Moll [14]). Actually, since the Wulff shape, under (4), forms a regular even-gon centered at the origin, we can figure out that the Wulff shape and the $2m$-unit vectors ν_k $(0 \le k < 2m)$ correspond to the basic structural unit of crystals and the crystal orientations, respectively. In this light, some special cases, such as the case of $m = 3$, can be expected to capture the crystalline structures in concrete phenomena, such as in ice crystals.

Now, the main focus of this paper will be on the stability analysis for the interfaces in steady-states, under the crystalline type setting by (4), and the main novelties found in this paper will be:

i) exemplification of a class of steady-state solutions of $(E;\theta)_f$, under (4);
ii) geometric characterizations for the stability of the interfaces, represented by the steady-state solutions as in i).

Consequently, it will be shown that special piecewise smooth Jordan curves, based on Wulff shapes (even-gons), will be built in the stability condition, and they will make the situation more variable than that as in the setting by (3).

Notation. For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X. For an abstract Hilbert space H, we denote by $(\cdot,\cdot)_H$ the inner product in H, and for any proper l.s.c. and convex function Φ on H, we denote by $D(\Phi)$, $\partial\Phi$ and $D(\partial\Phi)$ the effective domain of Φ, the subdifferential of Φ and the domain of $\partial\Phi$, respectively.

2. **Preliminaries.** Throughout this paper, $\Omega \subset \mathbb{R}^2$ is fixed as a bounded domain with a Lipschitz boundary Γ, and the product space $(0,+\infty)\times\Omega$ of the space-time coordinate system is denoted by Q. For any open subset $D \subset \Omega$, the external part $\Omega \setminus \overline{D}$ in Ω is denoted by D^{ex}. The class of all Borel subsets in Ω is denoted by $\mathcal{B}(\Omega)$. For each dimension $d \in \mathbb{N}$, we denote by $\mathcal{L}^d$ the d-dimensional Lebesgue measure, and we use this measure unless otherwise specified. Also, we denote by $\mathcal{H}^d$ the Hausdorff measure in each observing dimension $d \in \mathbb{N}$. Additionally, for any $r > 0$, any $x \in \mathbb{R}^2$ and any norm $f \in C(\mathbb{R}^2)$, we denote by $W_f(x;r)$ the open set $(x + r\mathcal{W}_f) \cap \Omega$.

In this section, we briefly check the key-properties of our Allen-Cahn equation $(E;\theta)_f$. To this end, let us start with recalling fundamentals relative to the space $BV(\Omega)$, that closely links to the effective domain of the interfacial energy.

For any $z \in L^1(\Omega)$, we call z a function of bounded variation (or simply a BV-function), iff the distributional gradient Dz of z forms a (vectorial) Radon measure on $\mathcal{B}(\Omega)$. Then, the total variation measure $|Dz|$ of Dz forms a (scalar valued) Radon measure on $\mathcal{B}(\Omega)$, such that:

$$|Dz|(\Omega) = \sup\left\{ \int_\Omega z \operatorname{div}\varphi \, dx \;\middle|\; \begin{array}{l} \varphi \in C^1_{\mathrm{c}}(\Omega;\mathbb{R}^2) \text{ and} \\ |\varphi| \le 1 \text{ on } \Omega \end{array} \right\} < +\infty.$$

In usual, the class of all BV-functions is denoted by $BV(\Omega)$, and then the functional space $BV(\Omega)$ forms a Banach space endowed with the norm:

$$|z|_{BV(\Omega)} := |z|_{L^1(\Omega)} + |Dz|(\Omega) \text{ for any } z \in BV(\Omega).$$

Also, since $\Omega \subset \mathbb{R}^2$, the space $BV(\Omega)$ is continuously embedded into $L^2(\Omega)$ and compactly embedded into $L^1(\Omega)$ (cf. [1, Chapter 3], [7, Chapter 5] or [10, Chapter 1]). Additionally, for any $z \in BV(\Omega)$, it is well-known (cf. [1, Chapters 1 and 3]) that the variation measure Dz can be decomposed into the absolutely continuous part Dz^a for $\mathcal{L}^2$ and the singular part Dz^s for $\mathcal{L}^2$, in the following way:

$$Dz = Dz^a + Dz^s, \quad Dz = \tfrac{Dz}{|Dz|}\,|Dz|, \quad Dz^a = \nabla z \mathcal{L}^2 \text{ and } Dz^s = \tfrac{Dz^s}{|Dz^s|}\,|Dz^s|;$$

where $\frac{Dz}{|Dz|}$, ∇z and $\frac{Dz^s}{|Dz^s|}$ are the Radon-Nikodým densities of Dz for $|Dz|$, Dz^a for $\mathcal{L}^2$ and Dz^s for its total variation $|Dz^s|$, respectively. In particular, the density ∇z coincides with the "approximate differential" of z (a.e. in Ω), proposed and studied in [1, Definition 3.70 and Theorem 3.83].

On the basis of the theory of BV-functions, including the above, the anisotropic total variation Var_f, given in (1), is characterized as follows (cf. [1, Section 5.5]):

(V1) the functional $z \in L^1(\Omega) \mapsto \mathrm{Var}_f(z)$ defines a proper l.s.c. and convex function on $L^1(\Omega)$, and its effective domain coincides with $BV(\Omega)$. Therefore, the L^2-restriction $V_f := \mathrm{Var}_f|_{L^2(\Omega)}$ is proper l.s.c. and convex on $L^2(\Omega)$, and the effective domain $D(V_f)$ coincides with $BV(\Omega) \cap L^2(\Omega)$;

(V2) for any $z \in BV(\Omega)$, there exists a Radon measure $f^\circ(Dz)$ on $\mathcal{B}(\Omega)$ such that

(i) $f^\circ(Dz)$ is absolutely continuous for $|Dz|$, and $\mathrm{Var}_f(z) = \int_\Omega f^\circ(Dz)$,

(ii) $\int_B f^\circ(Dz) = \int_B f^\circ(\nabla z)\,dx + \int_B f^\circ(\frac{Dz^s}{|Dz^s|})\,|Dz^s|$, for any $B \in \mathcal{B}(\Omega)$.

In addition to the above, taking account of the theory of T-monotonicity [13, Section 2], we have the following lemma.

Lemma 2.1. *Let us define a proper l.s.c. and convex function Φ_f on $L^2(\Omega)$, by putting:*

$$\Phi_f(z) := V_f(z) + \int_\Omega I_{[-1,1]}(z)\,dx, \quad \textit{for any } z \in L^2(\Omega).$$

Then, it follows that:

(i) $D(\Phi_f) = \left\{\ \zeta \in BV(\Omega) \ \middle|\ |\zeta| \le 1, \ \text{a.e. in } \Omega\ \right\}$, *and for any $r > 0$, the level-set* $\left\{\ \zeta \in L^2(\Omega)\ \middle|\ \Phi_f(\zeta) \le r\ \right\}$ *is compact in $L^2(\Omega)$;*

(ii) $(z_1^* - z_2^*, (z_1 - z_2)^+)_{L^2(\Omega)} \ge 0$, *for all* $[z_k^*, z_k] \in \partial\Phi_f$ $(k = 1, 2)$*;*

(iii) $D(\partial\Phi_f) = D(\partial V_f) \cap D(\Phi_f)$, *and* $\partial\Phi_f(z) = \partial V_f(z) + \partial I_{[-1,1]}(z)$ *in* $L^2(\Omega)$, *for all* $z \in D(\partial\Phi_f)$.

Proof. We omit to present here the detailed proof, which follows from an easy adaptation of the argument in [18, Lemmas 2.1-2.2 and Theorem 3.1] related to the case of (3). □

Remark 1. (Supplements for T-monotonicity) Let Φ be a proper l.s.c. and convex function on an abstract Hilbert space H, let $\mathcal{C}$ be any closed and convex subset in H, and let $\pi_\mathcal{C}$ be the orthogonal projection onto $\mathcal{C}$. Then, as one of consequences of T-monotonicity, we see the equivalency of the following two inequalities:

$$(z_1^* - z_2^*, (z_1 - z_2) - \pi_\mathcal{C}(z_1 - z_2))_{L^2(\Omega)} \ge 0, \quad \text{for all } [z_k^*, z_k] \in \partial\Phi \ (k = 1, 2); \tag{5}$$

$$\begin{gathered}\Phi(z_1 - \pi_\mathcal{C}(z_1 - z_2)) + \Phi(z_2 + \pi_\mathcal{C}(z_1 - z_2)) \le \Phi(z_1) + \Phi(z_2), \\ \text{for all } z_k \in D(\Phi) \ (k = 1, 2).\end{gathered} \tag{6}$$

Indeed, the item (ii) of Lemma 2.1 just corresponds to (5) in the case of $\Phi = \Phi_f$ and $\mathcal{C} = \{z \in H \,|\, z \le 0 \text{ a.e. in } \Omega\}$. Also, the inequality (5), under $\Phi = \Phi_f$ and $\mathcal{C} = \{z \in H \,|\, |z| \le 1 \text{ a.e. in } \Omega\}$, acts a key-role of the proof of item (iii) of Lemma 2.1. In either case, the validity of (5) is guaranteed by that of the equivalent form (6), which is relatively easy to check.

Now, let us look toward the evolution equation $(E;\theta)_f$.

Definition 2.2. (Solutions of $(E;\theta)_f$) Let $\theta \in L^2_{\mathrm{loc}}([0,+\infty); L^2(\Omega))$ be a fixed function. Then, a function $u : [0,+\infty) \longrightarrow L^2(\Omega)$ is called a solution of $(E;\theta)_f$, iff $u \in W^{1,2}_{\mathrm{loc}}([0,+\infty); L^2(\Omega)) \cap L^\infty([0,+\infty); BV(\Omega))$, and

$$\begin{gathered}((u + \theta - u_t)(t), z - u(t))_{L^2(\Omega)} \le \kappa\Phi_f(z) - \kappa\Phi_f(u(t)), \\ \text{a.e. } t > 0, \text{ for all } z \in D(\Phi_f).\end{gathered} \tag{7}$$

Remark 2. Let us note that the assertion (iii) of Lemma 2.1 guarantees the conformity of the above definition with the mathematical formulation of $(E;\theta)_f$.

In recent years, evolution equations, kindred to $(E;\theta)_f$, have been studied by several authors from various viewpoints (e.g. [4, 6, 12]). So, referring to some of them, we immediately have the following proposition, concerned with the key-properties of the Allen-Cahn equation $(E;\theta)_f$.

Proposition 1. *(Key-properties of $(E;\theta)_f$) Let us fix any $\theta \in L^2_{\rm loc}([0,+\infty);L^2(\Omega))$. Then, the following two statements hold.*

(I) (cf. [6, Chapter III]) For any initial value $u_0 \in D(\Phi_f)$, the Cauchy problem for $(E;\theta)_f$ admits a unique solution.

(II) (cf. [12, Theorem 9.1]) Let us fix a constant $-1 < \theta_* < 1$, and assume that

$$\theta - \theta_* \in L^2(0,+\infty;L^2(\Omega)) \quad \text{and} \quad \theta(t) \to \theta_* \text{ in } L^2(\Omega) \text{ as } t \to +\infty. \tag{8}$$

Then, for any solution u of $(E;\theta)_f$, the ω-limit set:

$$\omega(u) := \left\{ \; w \in L^2(\Omega) \; \middle| \; \begin{array}{l} \text{there is a sequence } \{t_i\} \subset (0,+\infty) \text{ such that} \\ t_i \nearrow +\infty \text{ and } u(t_i) \to w \text{ in } L^2(\Omega) \text{ as } i \to +\infty \end{array} \right\};$$

is nonempty, connected and compact in $L^2(\Omega)$, and any $w \in \omega(u)$ solves the following inclusion:

$$(E_\infty;\theta_*)_f \qquad \kappa\partial\Phi_f(w) \ni w + \theta_* \text{ in } L^2(\Omega);$$

having the equivalent variational formula:

$$(w+\theta_*, z-w)_{L^2(\Omega)} \le \kappa\Phi_f(z) - \kappa\Phi_f(w) \text{ for all } z \in D(\Phi_f). \tag{9}$$

Remark 3. In this paper, the phrases "steady-state problem" and "steady-state solutions" respectively mean the inclusion $(E_\infty;\theta_*)_f$ (or the equivalent formula (9)) and its solutions, under the convergence condition (8).

Finally, we briefly mention two key-lemmas in this study. The first key-lemma is concerned with the representation of the solutions of $(E;\theta)_f$, and it plays a useful role in the structural analysis for solutions.

Lemma 2.3. *(Representation of solutions) For any $\theta \in L^2_{\rm loc}([0,+\infty);L^2(\Omega))$, a function $u \in C([0,+\infty);L^2(\Omega))$ is a solution of $(E;\theta)_f$ iff there exists a vectorial function $\nu_{\rm CH} \in L^\infty(Q;\mathbb{R}^2)$, called "Cahn-Hoffmann vector field", such that:*

(a) $\operatorname{div}\nu_{\rm CH} \in L^2_{\rm loc}([0,+\infty);L^2(\Omega))$ and $f(\nu_{\rm CH}) \le 1$ a.e. in Q;

(b) $$-\kappa\operatorname{div}\nu_{\rm CH} = \begin{cases} \le u+\theta-u_t, & \text{if } u = 1, \\ = u+\theta-u_t, & \text{if } -1 < u < 1, \\ \ge u+\theta-u_t, & \text{if } u = -1, \end{cases} \quad \text{a.e. in } Q;$$

(c) $$-\int_\Omega \operatorname{div}\nu_{\rm CH}(t)u(t)\,dx = V_f(u(t)) \text{ and } -\int_\Omega \operatorname{div}\nu_{\rm CH}(t)\,\varphi\,dx = \int_\Omega \nu_{\rm CH}(t)\cdot\nabla\varphi\,dx$$ *for all $\varphi \in W^{1,1}(\Omega)\cap L^2(\Omega)$ and a.e. $t > 0$.*

Proof. This lemma can be proved just as in [18, Proposition 4.2]. But then, the application part of the theory of [2, Sections 4-5], based on the approximation by p-Laplacians, must be replaced by that of [14, Section 4], based on the approximation by Yosida-regularizations. □

Remark 4. Let us note that the restriction to time-independent situations enables us to apply Lemma 2.3 to the structural analysis for steady-state solutions.

The second key-lemma is the so-called comparison principle of solutions, that is often utilized in the stability analysis for steady-state solutions.

Lemma 2.4. *(Comparison principle) Let us fix two functions* $\theta_k \in L^2_{\rm loc}([0,+\infty); L^2(\Omega))$ $(k = 1, 2)$ *satisfying* $\theta_1 \leq \theta_2$ *a.e. in* Q*, and let us take two solutions* u_k *of* $(E; \theta_k)_f$ $(k = 1, 2)$ *satisfying* $u_1(0) \leq u_2(0)$ *a.e. in* Ω*. Then,* $u_1(t) \leq u_2(t)$ *a.e. in* Ω*, for any* $t \geq 0$.

Proof. This lemma is proved standardly, with the help from (ii) of Lemma 2.1. □

3. **Statement of the main results.** In this section, two theorems are stated as the main results of this paper. In either theorem, the geometric shapes of the interfaces will be characterized by means of a class of open subsets in Ω, prescribed as follows.

Definition 3.1. For any fixed constant $-1 < \theta_* < 1$, we define a class $\mathcal{D}_*(\theta_*) \subset \mathcal{B}(\Omega)$ of open subsets in Ω, by putting:

$$\mathcal{D}_*(\theta_*) := \{\emptyset, \Omega\} \cup \mathcal{D}_0(\theta_*);$$

with the use of a class $\mathcal{D}_0(\theta_*)$ of all open subsets in Ω, fulfilling the following five conditions (see also Figure 1 to get general ideas):

(D1) $\partial D \cap \Omega$ consists of at most a finite number of Jordan curves, included in Ω;

(D2) there exists $n_D \in \mathbb{N}$, such that $\partial D \cap \Omega$ coincides with the union $\bigcup_{k=1}^{n_D} \Gamma_k$ of C^2-curves Γ_k $(k = 1, \cdots, n_D)$. Hereafter, for any index $1 \leq k \leq n_D$, the C^2-curve Γ_k is supposed to be expressed as a graph of a vectorial function $\gamma_k \in C^2(J_k; \mathbb{R}^2)$, defined on a compact interval J_k of the arc-length parameter on Γ_k, and furthermore, the images $\Gamma_k^\circ := \gamma_k(J_k^\circ)$ of interiors J_k° of J_k $(k = 1, \cdots, n_D)$ are supposed to be pairwise disjoint;

(D3) for any index $1 \leq k \leq n_D$, there exists $\ell_k \in \mathbb{Z}$ realizing one of the following two situations:

(d1) $\gamma_k^\perp(s) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_k'(s) = \nu_{\ell_k}$, for all $s \in J_k$,

(d2) $f(\gamma_k^\perp(s)) = \max\{\nu_{\ell_k} \cdot \gamma_k^\perp(s),\ \nu_{\ell_k+1} \cdot \gamma_k^\perp(s)\} < 1$, for all $s \in J_k^\circ$;

(D4) there exists a constant $r_* > 2\kappa/(1 - |\theta_*|)$, such that

$$D = \bigcup_{\substack{x \in \Omega \\ W_f(x; r_*) \subset D}} W_f(x; r_*) \quad \text{and} \quad D^{\rm ex} = \bigcup_{\substack{x \in \Omega \\ W_f(x; r_*) \subset D^{\rm ex}}} W_f(x; r_*).$$

(D5) for any index $1 \leq k \leq n_D$ and any $0 \leq r \leq r_*$, let $\partial D_f(r)$ be the neighborhood of $\partial D \cap \Omega$, defined as

$$\partial D_f(r) := \begin{cases} \left\{ \; x \in \Omega \;\middle|\; \inf_{y \in \partial D \cap \Omega} f(y - x) < r \; \right\}, & \text{if } \partial D \cap \Omega \neq \emptyset \text{ and } r > 0, \\ \emptyset, \text{ otherwise.} & \end{cases}$$

Then, there exists a class of connected sets $\Gamma_k(r)$ $(0 < r \leq r_*,\ k = 1, \cdots, n_D)$, such that

(i) $\partial D_f(r) = \displaystyle\bigcup_{k=1}^{n_D} \Gamma_k(r)$,

(ii) for any $0 < r \leq r_*$, the n_D-interiors $\Gamma_k^\circ(r)$ of $\Gamma_k(r)$ $(k = 1, \cdots, n_D)$ are pairwise disjoint,

(iii) $\Gamma_k = \displaystyle\bigcap_{0 < r \leq r_*} \Gamma_k(r)$ and $\Gamma_k^\circ = \displaystyle\bigcap_{0 < r \leq r_*} \Gamma_k^\circ(r)$, for all $k = 1, \cdots, n_D$.

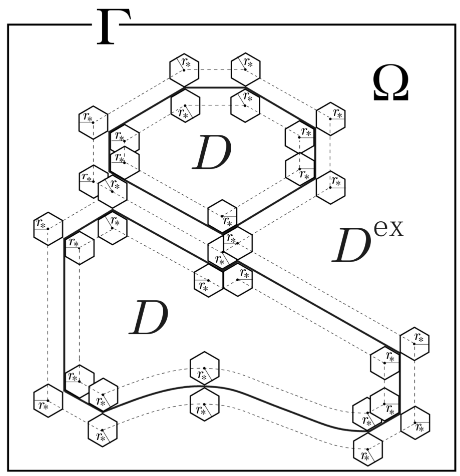

FIGURE 1. Profile of an open set $D \in \mathcal{D}_0(\theta_*)$ (in case of $m = 3$)

Remark 5. With regard to the connected sets $\Gamma_k(r)$ $(0 < r \le r_*,\ k = 1, \cdots, n_D)$, it is further seen that they can be taken to have some special geometric characteristics. In fact, for any index $1 \le k \le n_D$, we find an affine transform Λ_k, consisting of an orthogonal matrix Θ_k and a translation, and then the images $\Lambda_k\Gamma_k$ and $\Lambda_k\Gamma_k(r)$ $(0 < r \le r_*)$ correspond to one of the following two cases (see also FIGURES 2-3 to get general ideas):

(case 1) $f(\Theta_k^{-1}\xi) = f(\xi)$ for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, and there exists a constant $R_k \ge r_*$, such that:

$$\begin{cases} \Lambda_k\Gamma_k = \left\{ \ \xi = (\xi_1,\xi_2) \in \mathbb{R}^2 \ \middle|\ f(\xi) = \nu_0 \cdot \xi \text{ and } \xi_1 = R_k \ \right\}, \\ \Lambda_k\Gamma_k(r) = ((R_k + r)\mathcal{W}_f \setminus (R_k - r)\overline{\mathcal{W}_f}) \cap \left\{ \ \xi \in \mathbb{R}^2 \ \middle|\ f(\xi) = \nu_0 \cdot \xi \ \right\}; \end{cases}$$

(case 2) $f(\Theta_k^{-1}\nu_0^\perp) = \cos\frac{\pi}{2m}$ for a unit vector $\nu_0^\perp := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\nu_0 = (0, 1)$, and there exist a compact interval $I_k := [a_k, b_k]$, with $-\infty < a_k < b_k < +\infty$, and a function $\beta_k \in C^2(I_k)$, such that:

$$\begin{cases} |\beta_k'| \le \tan\frac{\pi}{2m} \text{ on } I_k, \ \ \Lambda_k\Gamma_k = \left\{ \ \xi \in \mathbb{R}^2 \ \middle|\ \xi = (\tau, \beta_k(\tau)),\ \tau \in I_k \ \right\}, \\ \Lambda_k\Gamma_k(r) = \left\{ \ \xi \in \mathbb{R}^2 \ \middle|\ \xi = (\tau, \rho + \beta_k(\tau)),\ \tau \in I_k,\ |\rho| < r\sec\frac{\pi}{2m} \ \right\}, \\ \Lambda_k(\Gamma_k(r) \cap D) = \left\{ \ \xi = (\xi_1,\xi_2) \in \Lambda_k\Gamma_k(r) \ \middle|\ \xi_2 > \beta_k(\xi_1) \ \right\}. \end{cases}$$

Now, in the first theorem, concrete profiles of some steady-state solutions are presented, on the basis of the above notations.

Main Theorem 1. *(Structural theorem for steady-state solutions) Under the same notations as in Definition 3.1, let us set a class $\mathcal{S}_*(\theta_*)$ of piecewise constant BV-functions, by putting:*

$$\mathcal{S}_*(\theta_*) := \left\{ \ w = (1+\theta_*)\chi_D + (-1+\theta_*)\chi_{D^{\mathrm{ex}}} - \theta_* \ \middle|\ D \in \mathcal{D}_*(\theta_*) \ \right\}. \tag{10}$$

Then, $\mathcal{S}_*(\theta_*)$ is a subset of the solution class of the steady-state problem $(E_\infty;\theta_*)_f$.

Remark 6. The class $\mathcal{S}_*(\theta_*)$, given in (10), includes two constant functions 1 and -1 on Ω, that respectively correspond to the cases of $D = \Omega$ and $D = \emptyset$, in the expressions as in (10). In the meantime, let us note that the class $\mathcal{S}_*(\theta_*)$ is just a proper subset of the solution class of $(E_\infty;\theta_*)_f$. In fact, it is easily checked that the constant function $-\theta_*$ on Ω solves (9), while $-\theta_* \notin \mathcal{S}_*(\theta_*)$. Besides, the class

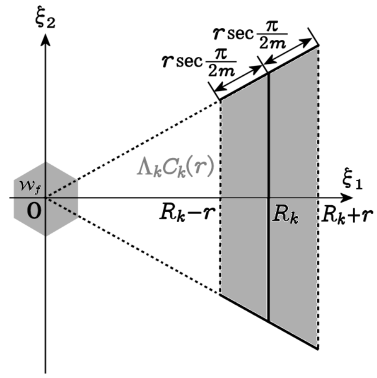

FIGURE 2. $\Lambda_k C_k(r)$ in (case 1) (in case of $m = 3$)

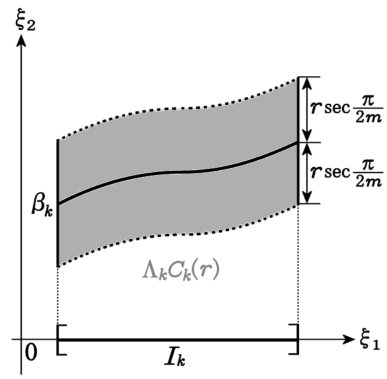

FIGURE 3. $\Lambda_k C_k(r)$ in (case 2) (in case of $m = 3$)

$\mathcal{S}_*(\theta_*)$ does not include any nonconstant solution of $(E_\infty;\theta_*)_f$ that represents interfaces having contacts with the boundary Γ, as in [18, Example 4.3].

The second theorem is concerned with the stability analysis for steady-state solutions, exemplified in the above.

Main Theorem 2. *(Stability analysis for steady-state solutions) Under the same notations as in Definition 3.1, Remark 5 and Main Theorem 1, let us fix any steady-state solution $w_* = (1+\theta_*)\chi_D + (-1+\theta_*)\chi_{D^{\rm ex}} - \theta_* \in \mathcal{S}_*(\theta_*)$ with $D \in \mathcal{D}_*(\theta_*)$, and let us fix positive constants $\bar{\varepsilon}$ and $\bar{\delta}$, to satisfy:*

$$\begin{cases} \bullet\ 2\bar{\varepsilon} + \bar{\delta} \le 1 - |\theta_*|, \text{ if } w_* \in \mathcal{S}_*(\theta_*) \text{ is given in constant cases,} \\ \bullet\ 0 < \dfrac{2\kappa}{1 - |\theta_*| - 3\bar{\varepsilon}} < r_* - \bar{\delta}, \text{ if } w_* \in \mathcal{S}_*(\theta_*) \text{ is given in nonconstant cases.} \end{cases} \tag{11}$$

Then, any steady-state solution $w_* \in \mathcal{S}_*(\theta_*)$ shows the stability (restoring force for oscillations), in the following sense.

(∗) *For any $0 < \varepsilon < \bar{\varepsilon}$ and $0 < \delta < \bar{\delta}$, there exists a finite time $t_* = t_*(\varepsilon,\delta)$, depending on ε and δ, such that $u(t) = w_*$ a.e. in $\Omega \setminus \partial D_f(\delta)$, for any $t \ge t_*$, any function $\theta \in L^2_{\rm loc}([0,+\infty); L^2(\Omega))$ and any solution u of $(E;\theta)_f$, satisfying*

$$|\theta - \theta_*|_{L^\infty(Q)} \le \bar{\varepsilon} \quad \text{and} \quad |u(0) - w_*|_{L^\infty(\Omega\setminus\partial D_f(\delta))} \le \varepsilon. \tag{12}$$

In particular, when w_* is constant on Ω, the value $t_*(\varepsilon,\delta)$ is determined independently on δ, and hence the stability (∗) actually asserts the realization of the uniform convergence of $u(t)$ to w_* on Ω, at the finite time $t_* = t_*(\varepsilon)$.

4. **Proof of Main Theorem 1.**

In this section, we prove Main Theorem 1.

Let us take any $w_* \in \mathcal{S}_*(\theta_*)$. Then, as is easily seen, the main difficulty is in the nonconstant (but piecewise constant) situation of w_*. In fact, when w_* is constant, namely when $w_* \equiv 1$ on Ω or $w_* \equiv -1$ on Ω, the variational inequality (9) will immediately checked for w_*, since:

$$\Phi_f(w_*) = 0 \le \Phi_f(z) \text{ and } |z| \le |w_*| = 1 \text{ a.e. in } \Omega, \text{ for all } z \in D(\Phi_f).$$

When w_* is nonconstant, namely when w_* is expressed as a step function $(1+\theta_*)\chi_D + (-1+\theta_*)\chi_{D^{\rm ex}} - \theta_*$ with the use of an open set $D \in \mathcal{D}_0(\theta_*)$, we first prepare Lipschitz functions $\lambda^{(r)} \in C^{0,1}(\mathbb{R}) \cap C_{\mathrm{c}}(\mathbb{R})$ $(0 < r \leq r_*)$ by putting:

$$\lambda^{(r)}(\rho) := \begin{cases} -\dfrac{1}{r}\,\rho + 1, \text{ if } 0 \leq \rho < r, \\ \dfrac{1}{r}\,\rho + 1, \text{ if } -r < \rho < 0, \\ 0, \text{ otherwise,} \end{cases} \quad \text{for all } \rho \in \mathbb{R} \text{ and all } 0 < r \leq r_*;$$

to define vectorial functions $\nu_*^{(r)} \in C^{0,1}(\overline{\Omega}; \mathbb{R}^2)$ $(0 < r \leq r_*)$, as follows.

$$\nu_*^{(r)}(x) := \begin{cases} \sigma_k \Theta_k^{-1} \left(\dfrac{\lambda^{(r)}(f(\Lambda_k x) - R_k)}{f(\Lambda_k x)}\, \Lambda_k x \right), \\ \qquad \text{if } x \in C_k(r) \text{ in (case 1), for some index } 1 \leq k \leq n_D, \\ \sec \frac{\pi}{2m}\, \Theta_k^{-1} \left(\lambda^{(r \sec \frac{\pi}{2m})}(\Lambda_k x \cdot \nu_0^{\perp} - \beta_k(\Lambda_k x \cdot \nu_0))\nu_0^{\perp} \right), \\ \qquad \text{if } x \in C_k(r) \text{ in (case 2), for some index } 1 \leq k \leq n_D, \\ 0 \ (= (0,0)), \text{ otherwise;} \end{cases} \tag{13}$$

for all $x \in \Omega$ and all $0 < r \leq r_*$, where σ_k $(k = 1, \cdots, n_D)$ are sign constants (just made for (case 1)), prescribed as:

$$\sigma_k := \begin{cases} 1, \text{ if } \Theta_k^{-1}\nu_0 \cdot \frac{Dw_*^s}{|Dw_*^s|} \geq 0, \mathcal{H}^1\text{-a.e. on } C_k(r) \cap \partial D, \\ -1, \text{ if } \Theta_k^{-1}\nu_0 \cdot \frac{Dw_*^s}{|Dw_*^s|} < 0, \mathcal{H}^1\text{-a.e. on } C_k(r) \cap \partial D, \end{cases} \quad k = 1, \cdots, n_D.$$

Then, with regard to the vectorial function $\nu_*^{(r_*)}$ under $r = r_*$, we have

$$f(\nu_*^{(r_*)}) \leq 1, \ -\kappa \operatorname{div} \nu_*^{(r_*)} \begin{cases} \leq \dfrac{2\kappa}{r_*} \leq 1 - |\theta_*| \leq w_* + \theta_*, \\ \qquad \text{if } w_* = 1, \\ \geq -\dfrac{2\kappa}{r_*} \geq -(1 - |\theta_*|) \geq w_* + \theta_*, \\ \qquad \text{if } w_* = -1, \end{cases} \text{a.e. in } \Omega, \tag{14}$$

$$\text{and } \ \nu_*^{(r_*)} \in \partial f^\circ(\tfrac{Dw_*^s}{|Dw_*^s|}), \ |Dw^s|\text{-a.e. in } \Omega \text{ (actually } \mathcal{H}^1\text{-a.e. on } \partial D); \tag{15}$$

by fundamental calculations with helps from (4), (D1)-(D5) and Remark 5.

Here, let us set that

$$u(t) = w_* \text{ in } L^2(\Omega) \text{ and } \nu_{\mathrm{CH}}(t) = \nu_*^{(r_*)} \text{ in } L^\infty(\Omega; \mathbb{R}^2), \ \text{ for all } t \geq 0. \tag{16}$$

Then, it is seen from (14) that the vectorial function ν_{CH} $(\equiv \nu_*^{(r_*)})$ satisfies conditions (a)-(b) as in Lemma 2.3. Furthermore, taking account of [3, Theorem 1.9 and Proposition 2.3] and (15), we obtain that

$$\begin{aligned} -\int_\Omega \operatorname{div} \nu_*^{(r_*)} w\, dx &= \int_\Omega \nu_*^{(r_*)} \cdot \tfrac{Dw_*}{|Dw_*|}\, |Dw_*| = \int_\Omega \nu_*^{(r_*)} \cdot \tfrac{Dw_*^s}{|Dw_*^s|}\, |Dw_*^s| \\ &= \int_\Omega f^\circ(\tfrac{Dw_*^s}{|Dw_*^s|})\, |Dw_*^s| = \int_\Omega f^\circ(Dw_*) = V_f(w_*). \end{aligned}$$

This implies that the function $\nu_*^{(r_*)}$ also satisfies the condition (c) as in Lemma 2.3, since the remaining variational inequality (equality) immediately follows from the condition that $\nu_*^{(r_*)} \equiv 0$ on Γ.

Thus, applying Lemma 2.3 under (16), we conclude that w_* is a solution of the steady-state problem $(E_\infty; \theta_*)_f$. □

5. **Proof of Main Theorem 2.** This section is devoted to the proof of Main Theorem 2.

First, we see the proof of nonconstant cases of the steady-state solution $w_* \in \mathcal{S}_*(\theta_*)$. Let us fix any nonconstant steady-state solution $w_* = (1+\theta_*)\chi_D + (-1+\theta_*)\chi_{D^{\rm ex}} - \theta_* \in \mathcal{S}_*(\theta_*)$ with the open set $D \in \mathcal{D}_0(\theta_*)$ and the constants $\bar{\varepsilon}$ and $\bar{\delta}$ as in (11). Also, for any $0 < \delta < \bar{\delta}$, let us set two subsets $D_\delta^{(+)}, D_\delta^{(-)} \subset \Omega$, and two functions $u_\delta^{(+)}, u_\delta^{(-)} \in W^{1,2}_{\rm loc}([0,+\infty); L^2(\Omega))$, as follows.

$$D_\delta^{(\pm)} := \left\{ \ x \in \Omega \setminus \overline{\partial D_f(\delta)} \ \middle| \ w_*(x) = \pm 1 \ \right\}, \text{ and}$$

$$u_\delta^{(\pm)}(t) := \begin{cases} \{(\pm 1 + \theta_* \mp 2\bar{\varepsilon}) \pm \bar{\varepsilon}e^t\}\chi_{D_\delta^{(\pm)}} + (\mp 1 + \theta_*)\chi_{(D_\delta^{(\pm)})^{\rm ex}} - \theta_* \\ \qquad \text{in } L^2(\Omega), \text{ if } 0 \le t < \log 2, \\ (\pm 1 + \theta_*)\chi_{D_\delta^{(\pm)}} + (\mp 1 + \theta_*)\chi_{(D_\delta^{(\pm)})^{\rm ex}} - \theta_* \\ \qquad \text{in } L^2(\Omega), \text{ if } t \ge \log 2. \end{cases} \tag{17}$$

Then, due to (11), both $D_\delta^{(+)}$ and $D_\delta^{(-)}$ belong to the class $\mathcal{D}_0(\theta_*)$, however in either case, the replacement of the constant r_* by $r_* - \delta$ ($> 2\kappa/(1-|\theta_*|)$) is needed to check the condition (D4). Therefore, by similar arguments as in the previous section, we further see that the functions $u_\delta^{(+)}$ and $u_\delta^{(-)}$ respectively solve evolution equations $(E;\theta_\delta^{(+)})_f$ and $(E;\theta_\delta^{(-)})_f$, where $\theta_\delta^{(\pm)} \in L^\infty([0,+\infty); L^\infty(\Omega))$ are given functions, prescribed as:

$$\theta_\delta^{(\pm)}(t) := \begin{cases} \theta_* \mp \bar{\varepsilon} - \{(\pm 1 + \theta_* \mp 3\bar{\varepsilon}) + \kappa \,{\rm div}\, \nu_*^{(r_*-\delta)}\}\chi_{D_\delta^{(\pm)}} \text{ in } L^2(\Omega), \\ \qquad \text{if } 0 \le t < \log 2, \\ \theta_* \mp \bar{\varepsilon} \text{ in } L^2(\Omega), \text{ if } t \ge \log 2, \end{cases}$$

with the use of the vectorial function $\nu_*^{(r_*-\delta)}$, as in (13), under $r = r_* - \delta$. Incidentally, for the functions $u_\delta^{(\pm)}$, the setting that:

$$\nu_{\rm CH}(t) := \nu_*^{(r_*-\delta)} \text{ in } L^\infty(\Omega;\mathbb{R}^2), \text{ for all } t \ge 0;$$

will provide the exact expressions of the vectorial functions, denoted by $\nu_{\rm CH}$, to check the conditions (a)-(c) of Lemma 2.3.

Now, let us take any solution u of $(E;\theta)_f$, under the setting of (11)-(12). Then, since:

$$\begin{cases} u_\delta^{(+)}(0) \le u(0) \le u_\delta^{(-)}(0) \ \text{ a.e. in } \Omega, \\ \theta_\delta^{(+)}(t) \le \theta_* - \bar{\varepsilon} \le \theta(t) \le \theta_* + \bar{\varepsilon} \le \theta_\delta^{(-)}(t) \ \text{ a.e. in } \Omega, \ \text{ for all } t \ge 0; \end{cases}$$

applying Lemma 2.4 under:

$$\theta_1 = \theta_\delta^{(+)}, \ \theta_2 = \theta, \ u_1 = u_\delta^{(+)} \text{ and } u_2 = u$$
$$(\text{resp. } \theta_1 = \theta, \ \theta_2 = \theta_\delta^{(-)}, \ u_1 = u \text{ and } u_2 = u_\delta^{(-)});$$

yields that:

$$u_\delta^{(+)}(t) \le u(t) \ (\text{resp. } u(t) \le u_\delta^{(-)}(t)) \text{ a.e. in } \Omega, \ \text{ for all } t \ge 0. \tag{18}$$

This implies the asserted stability $(*)$, and then the time $\log 2$, as in (17), actually indicates an upper bound for the finite time $t_* = t_*(\varepsilon, \delta)$.

Next, let us consider the constant case of $w_* \in \mathcal{S}_*(\theta_*)$. The proof of this case will be a simplified version of that of nonconstant case. More precisely, when $w_* \equiv 1$ (resp. $w_* \equiv -1$), we can show $(*)$ by using the following two functions:

$$u^{(+)}(t) := \min\{1, (1+\theta_*)e^t/2 - \theta_*\} \quad \text{and} \quad u^{(-)}(t) := w_*$$
$$\big(\text{resp. } u^{(+)}(t) := w_* \quad \text{and} \quad u^{(-)}(t) := \max\{-1, (-1+\theta_*)e^t/2 - \theta_*\}\big) \quad \text{a.e. in } \Omega.$$

In fact, we easily see that three functions u, $u^{(+)}$ and $u^{(-)}$ fulfill the variational inequality (7), under $\theta \equiv \theta_*$ in Q, namely they are all the solutions of the Allen-Cahn equation $(E;\theta_*)_f$, satisfying $u^{(+)}(0) \le u(0) \le u^{(-)}(0)$ a.e. in Ω. Hence, just as in the derivation of (18), we conclude $(*)$, by using the comparison functions $u^{(+)}$ and $u^{(-)}$, instead of $u_\delta^{(+)}$ and $u_\delta^{(-)}$. □

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