DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **729–738**

EFFECTIVE ESTIMATES OF THE HIGHER SOBOLEV NORMS FOR THE KURAMOTO-SIVASHINSKY EQUATION

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Abstract. We consider the Kuramoto-Sivashinsky (KS) equation in finite domains of the form $[-L, L]$. Our main result provides effective new estimates for higher Sobolev norms of the solutions in terms of powers of L for the one-dimentional differentiated KS. We illustrate our method on a simpler model, namely the regularized Burger's equation. The underlying idea in this result is that *a priori* control of the L^2 norm is enough in order to conclude higher order regularity and in fact, it allows one to get good estimates on the high-frequency tails of the solution.

1. **Introduction.**

The Kuramoto-Sivashinsky equation

$$\left\{ \begin{array}{l} \phi_t + \Delta^2\phi + \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0, \quad -L \leq x \leq L, \quad \phi(t, x + 2L) = \phi(t, x) \\ \phi(0, x) := \phi_0(x) \end{array} \right. \tag{1}$$

where $L > 0$ models pattern formation in different physical contexts. It arises as a model of nonlinear evolution of linearly unstable interfaces in a variety of applications such as flame propagation (advocated by Sivashinsky [18]) and reaction-diffusion systems (derived by Kuramoto in [12]). It has been studied extensively in the last two decades. The main results in the periodic case are on the global existence of the solutions, their stability and long-time behavior.

It is often convenient to consider the differentiated Kuramoto-Sivashinsky equation. That is, set $u = \phi_x$ and differentiate (1) with respect to x to get a closed form equation for u

$$\left\{ \begin{array}{l} u_t + u_{xxxx} + u_{xx} + uu_x = 0, \quad u(t, x + 2L) = u(t, x) \\ u(0, x) := u_0(x) \end{array} \right. \tag{2}$$

For (2), the global well-posedness , the existence of global attractor and its dimension were studied in [1, 9, 15, 4] and many others. We are interested in the existence of an attracting ball and the dependence of $\lim_{t\to\infty} \|u(t, \cdot)\|_{L^2}$ on the size of the domain L. The best possible current result $\lim_{t\to\infty} \|u(t, \cdot)\|_{L^2} = o(L^{3/2})$ is achieved by Giacomelli and Otto in [9], see also [1] for a somewhat more direct proof of the slightly weaker result $\lim_{t\to\infty} \|u(t, \cdot)\|_{L^2} \leq CL^{3/2}$. We would like to point

2000 *Mathematics Subject Classification.* 35B35, 35G30, 35B40, 35K30, 37K40.

Key words and phrases. Kuramoto-Sivashinsky equation, regularized Burger's equation, Gevrey regularity.

The first author is supported by NSF grants NSF-DMS 0508184 and NSF-DMS 0807894.

The second author is supported by NSF grant NSF-DMS 0701802.

out that this last bound applies as well to the solutions of the so-called destabilized KS equation $u_t + u_{xxxx} + u_{xx} - \eta u + u u_x = 0, \quad \eta > 0$ and moreover, such a result is *optimal* in this context.

Before we embark on our discussion on the optimality of these results, it is worth noting the following two conjectures. Namely, based on numerical simulations about the dimension of the attractor, it is conjectured that

$$\limsup_{t\to\infty} \|u(t,\cdot)\|_{L^2} \le C\sqrt{L}, \tag{3}$$

whereas for $\|u(t,\cdot)\|_{L^\infty}$

$$\limsup_{t\to\infty} \|u(t,\cdot)\|_{L^\infty} \le C. \tag{4}$$

If true, these would be the best possible estimates, since they are satisfied by the stationary solutions of the problem, see [14]. For a nice discussion about these conjectures the reader is referred to the introduction in [3].

The question of Gevrey class regularity for the Kuramoto-Sivashinsky equation is of interest because it can be used to improve the error estimates in the computation of the approximate inertial manifolds (see [10] and also [8], [20], [11]).

In the paper [5] the analyticity properties of the solutions are studied and numerical experiments are presented to show that the solutions are analytic in a strip around the real axis whose width is independent of L. The L^p analog is studied in [7]. In [13] the author studies the Gevrey class regularity for the odd solutions of the one dimensional Kuramoto-Sivashinsky equation with periodic boundary conditions and odd initial data. Theorem 1 in his paper should be compared with the estimates in Corollary 1 and Corollary 2 of the current paper, which we do in the remarks after Corollary 1. The methods that we use are different from the above mentioned papers in that we use Littlewood-Paley projections on the high frequencies and their estimates.

Our main results are Gevrey regularity theorems for the solutions of (2), but we will not emphasize that fact in our presentation. Instead, we will concentrate on the specific estimates that one can get for the high-frequency tails of the solutions of (2). In order to illustrate our methods on a somewhat simpler model, we will first consider the regularized Burger's equation. In it, we can actually take the regularization operator in the more general form $A_s = (-\Delta)^{s/2}$.Thus, our model is

$$u_t + A_s u + \sum_{j=1}^{d} u \frac{\partial u}{\partial x_j} = 0, \quad x \in [-L, L]^d \quad u(0,x) := u_0(x), \tag{5}$$

where the formal definition of A_s is given in Section 2.

In the next two theorems, we give estimates of the high-frequency tails of the solutions of (5) and (2) respectively. For this, we shall need the Littlewood-Paley projections. The Littlewood-Paley operators acting on $L^2([-L, L]^d)$ are defined for a function f, given in the form $f(x) = \frac{1}{(2L)^{d/2}} \sum_{k\in\mathcal{Z}^d} a_k e^{2\pi i k\cdot x/L}$ via

$$P_{\le N} f(x) = (2L)^{-d/2} \sum_{k:|k|\le N} a_k e^{2\pi i k\cdot x/L}.$$

That is, $P_{\le N}$ truncates the terms in the Fourier series expansion with frequencies $k : |k| > N$. Clearly $P_{\le N}$ is a projection operator. We have the following.

Theorem 1.1. *Let $d \ge 1$, $1 < s \le 2$ or $s > 1 + d/2$. The regularized Burger's equation (5) is a globally well-posed problem whenever the data belongs to L^2. In addition, in the case $1 < s \le 2$, assume $u_0 \in L^2 \cap L^\infty$. Then, for every $0 < \delta << 1$,* *there exists $C_{\delta,s}$, so that for any $j \geq 0$,*

$$\|P_{\geq 2^j L} u(t,\cdot)\|_{L^2}^2 \leq (C_{\delta,s} \max(1, \|u_0\|_{L^2 \cap L^\infty}^2))^{j+1} 2^{-\min(t,1)(1-\delta)(s-1)j^2}. \tag{6}$$

For $s > 1 + d/2$, one has a constant C_s so that

$$\|P_{\geq 2^j L} u(t,\cdot)\|_{L^2}^2 \leq (C_s \max(1, \|u_0\|_{L^2}^2))^{j+1} 2^{-\min(t,1)(s-1-d/2)j^2}. \tag{7}$$

Remarks:

- As an easy corollary, one can estimate $\sup_{\delta<t<\infty} \|u(t,\cdot)\|_{H^m}$ in terms of quantities, which are independent of the size of the domain L.
- In both cases, our results show that the solution belongs to a certain Gevrey class. In particular, for every fixed $t > 0$ the function $x \to u(t,x)$ is real-analytic.

Similar results hold for the one dimensional Kuramoto-Sivashinsky equation (2). The main difference with the regularized Burger's equation will be the unavailability of control of $\|u(t,\cdot)\|_{L^2}$ over the course of the evolution. In fact, the function $t \to \|u(t,\cdot)\|_{L^2}$ may grow to at least $C\sqrt{L}$ for the solutions of (2), see (3).

Theorem 1.2. *Let $u_0 \in L^2(-L,L)$ and $L >> 1$. Set $H = \sup_{0\leq s<\infty} \|u(s)\|_{L^2}$, where u is a solution of (2). There exist absolute constants C_0, C_1, so that for every $j \geq 0$, one has $\|u_{>C_0 2^j H^{2/5} L}(t,\cdot)\|_{L^2} \leq C_1^j 2^{-\frac{1}{2}\min(t,5/2)j^2} \sup_s \|u(s)\|_{L^2}$.*

Regarding H in the statement in Theorem 1.2, one may actually infer from the results of [1], [9] and the earlier papers on the subject that

$$H = \sup_s \|u(s,\cdot)\|_{L^2} \leq \|u_0\|_{L^2} o(1/t) + CL^{3/2}, \tag{8}$$

of which then $\limsup_{t\to\infty} \|u(t,\cdot)\|_{L^2} = O(L^{3/2})$ is a corollary. When $L >> 1$ we have that $H \leq CL^{3/2}$. This gives an estimate of $\|u_{\geq CL^{8/5}}(t,\cdot)\|_{L^2}$, which we prefer to formulate as estimates on the higher Sobolev norms.

Corollary 1. *Let $s \geq 0$, $L >> 1, \delta > 0, \|u_0\|_{L^2} << L$. There exists $C_{s,\delta}$, so that*

$$\sup_{\delta\leq t<\infty} \|u(t,\cdot)\|_{H^s} \leq C_{s,\delta} L^{3s/5} L^{3/2}. \tag{9}$$

Remarks:

- The estimate (9) may be stated in the form

$$\sup_{\delta\leq t<\infty} \|u(t,\cdot)\|_{H^s} \leq C_{s,\delta} H^{2s/5+1}. \tag{10}$$

  In other words, if one improves the bounds on H in (8), then one immediately gets an improvement of the results (9) in the form (10). Said differently, with the best current technology, namely $H \leq CL^{3/2}$, (9) is an instance of (10).
- The bounds (9) and (10) apply for solutions of the destabilized Kuramoto-Sivashinsky equation as well. As we have discussed previously, $H < L^{3/2}$ is optimal here in contrast with the standard KSE.
- The estimate (9) should be compared with $\sup_t \|u(t,\cdot)\|_{H^s} \leq CL^{4s+5/2}$ in [13], which follows from a similar Gevrey regularity estimate. The best available bound at the time was $\sup_t \|u(t,\cdot)\|_{L^2} \leq CL^{5/2}$. Even with the use of that bound however, our method would have produced an estimate of the form $\sup_t \|u(t,\cdot)\|_{H^s} \leq CL^{s+5/2}$, which is again superior to the results of [13].

Next, we present another estimate, which gives bounds on $\sup_t \|u(t,\cdot)\|_{H^s}$ in terms of $\sup_t \|u(t,\cdot)\|_{L^\infty}$. This follows essentially the same scheme of proof and it gives at least as good bounds[1] as (9), see the discussion after Corollary 2. The reason for the effectiveness of such an approach is that it almost avoids the use of Sobolev embedding, which is clearly ineffective in this context. We will use the quantities $K_p = \sup_{0<t<\infty} \|u(t,\cdot)\|_{L^p}$, where one should think of p as being very large[2] .

Corollary 2. *Let $s \geq 0$. Then, there exists a constant $C_{s,p,\delta}$, so that*

$$\sup_{\delta\leq t<\infty} \|u(t,\cdot)\|_{H^s} \leq C_{s,p,\delta} K_p^{s/(3-1/p)} H. \tag{11}$$

Roughly speaking, we get a factor of $K_\infty^{1/3}$ for every derivative of u.

Remark: We would like to point out that to the best of our knowledge, the best estimate currently available for K is obtainable through the Sobolev embedding theorem and the estimates for $\sup_t \|u(t)\|_{H^{1/2+}}$ from (10). This is certainly a very crude estimate, but let us use it anyways. By the bound $H \leq CL^{3/2}$ and assuming $\|u_0\|_{L^2} << L$, $L >> 1$, we have that for every $2 < p < \infty$ by[3] (10)

$$K_p \leq C_p \sup \|u(t,\cdot)\|_{H^{1/2-1/p}} \leq C_p H^{1/5+1-2/(5p)} \leq C_p L^{9/5-3/(5p)}.$$

Clearly, with this bound for K_p, (11) is only slightly worse than (9). However, if the conjecture (4) holds true or even an estimate of the form $K_\infty \leq CL^{9/5-}$ is established, then (11) gives a better result. Indeed, if (4) holds, then

$$\sup_{\delta<t<\infty} \|u(t,\cdot)\|_{H^s} \leq C_{s,\varepsilon,\delta} L^{\varepsilon s} H \tag{12}$$

for every $\varepsilon > 0$. This would one more time confirm the empirical observations, that the whole action in the evolution of the KS comes in the low frequencies.

2. Preliminaries.

On the interval $[-L, L]$, introduce the discrete Fourier transform $L^2([-L, L]) \to l^2(\mathcal{Z}^d)$, by setting $f \to \{a_k\}_{k\in\mathcal{Z}^d}$, where

$$a_k = (2L)^{-d/2} \int_{[-L,L]^d} f(x) e^{-2\pi i k\cdot x/L} dx.$$

The inverse Fourier transform is the familiar Fourier expansion

$$f(x) = \frac{1}{(2L)^{d/2}} \sum_{k\in\mathcal{Z}^d} a_k e^{2\pi i k\cdot x/L}. \tag{13}$$

and the Plancherel's identity is $\|f\|_{L^2([-L,L]^d)} = \|\{a_k\}\|_{l^2(\mathcal{Z}^d)}$. The Littlewood-Paley operators acting on $L^2([-L, L])$ are defined for a function f in the form of (13) via $P_{\leq N} f(x) = \frac{1}{(2L)^{d/2}} \sum_{|k|\leq N} a_k e^{2\pi i k\cdot x/L}$, where $P_{\leq N}$ is a projection operator. More generally, we may define for all $0 \leq N < M \leq \infty$ the projection

$$P_{N\leq\cdot\leq M} f(x) = \frac{1}{(2L)^{d/2}} \sum_{N\leq|k|\leq M} a_k e^{2\pi i k\cdot x/L}.$$

Clearly, we may take M, N to be nonintegers as well. A basic result in harmonic analysis on the torus is that Fourier series $P_{<N} f$ converge to f in $L^p, 1 < p < \infty$. This is equivalent to the uniform boundedness of the operators $P_{<N}$, i.e. $\|P_{<N} f\|_{L^p([-L,L]^d)} \leq C_{d,p} \|f\|_{L^p}$. Note that this estimate fails as $p = \infty$ and thus $C_{s,p} \to \infty$ as $p \to \infty$. We will also need the following *Bernstein inequality*.

Lemma 2.1. *Let N be an integer and $f : [-L, L]^d \to \mathcal{C}$. For $1 \leq p < q \leq \infty$,*

[1] and potentially much better bounds

[2] As it was pointed out already, there is the standing conjecture (4), which puts an uniform bound on K_∞.

[3] Here we are ignoring the minor issue for the bounds in the interval $0 < t < \delta$, but recall that our discussion is about global behavior.

$$\|P_{<N} f\|_{L^q} \leq C_{d,p,q} (N/L)^{d(1/p-1/q)} \|f\|_{L^p}.$$

Proof. The proof of this lemma is rather elementary and we sketch it. First, $\|P_{<N} f\|_{L^\infty} \leq (2L)^{-d/2} (\sum_{|k|<N} |a_k|) \leq C N^{d/2} L^{-d/2} (\sum_k |a_k|^2)^{1/2}$, which is the statement for $q = \infty, p = 2$. By duality and interpolation with the easy bound $\|P_{<N}\|_{L^2 \to L^2} = 1$, it is easy to see the claim for $q = \infty, 1 \leq p < \infty$. Finally,using the boundedness of $P_{<N}$ on $L^r, 1 < r < \infty$, we can see again by complex interpolation that $P_{<N} : L^p \to L^q$ for all $p < q$ with the required operator norms. □

Introduce $\dot{H}^s((-L,L)^d) = \{f : (-L,L)^d \to \mathcal{C} \quad (\sum_{k \in \mathcal{Z}^d} |a_k|^2 \left(\frac{|k|}{L}\right)^{2s})^{1/2} < \infty\}$. One may also find it convenient to work with the equivalent norm

$$\|f\|_{\dot{H}^s} \sim (\sum_{j \in \mathcal{Z}} 2^{2sj} (\sum_{|k| \sim 2^j L} |a_k|^2))^{1/2} \sim (\sum_{j \in \mathcal{Z}} 2^{2sj} \|P_{\sim 2^j L} f\|_{L^2}^2)^{1/2}, \tag{14}$$

Define the (fractional) differentiation operator $A_s = (-\Delta)^{s/2}$, via[4]

$$A_s[\sum_k a_k e^{2\pi i k \cdot x/L}] = \sum_k a_k \left(\frac{2\pi |k|}{L}\right)^s e^{2\pi i k \cdot x/L}.$$

Sometimes in the sequel, we will just use the notation $|\nabla|^s$ instead of A_s. A useful corollary of the representation (14) is $\|A_s P_{2^j L} f\|_{L^2} \sim 2^{js} \|P_{2^j L} f\|_{L^2}$, and its obvious generalization $\|A_s P_{>2^j L} f\|_{L^2} \geq C 2^{js} \|P_{2^j L} f\|_{L^2}$ for $s \geq 0$. The following simple orthogonality lemma is used frequently in the energy estimates presented below.

Lemma 2.2. *Let A, B, C be three subsets of $\mathcal{Z}^d$, so that $0 \notin A + B + C$. Then, for any three functions $f, g, h \in L^2([-L,L]^d)$,*

$$\int_{[-L,L]^d} (P_A f)(P_B g)(P_C h) dx = 0. \tag{15}$$

In particular, for every N,

$$\int_{[-L,L]^d} f_{>N} g_{<N/2} h dx = \int_{[-L,L]^d} f_{>N} g_{<N/2} h_{>N/2} dx \tag{16}$$

Proof. The proof of (15) follows by expanding in Fourier series

$$fgh(x) = (2L)^{-d/2} \sum_{k,m,n} f_k g_m h_n e^{2\pi i (k+m+n) \cdot x/L},$$

and then realizing that since $(k+m+n) \neq 0$, all the terms will upon integration in x result in zero. The proof of (16) follows by observing that the difference between the two sides is

$$\int_{[-L,L]^d} f_{>N} g_{<N/2} h_{\leq N/2} dx = 0,$$

by (15), since $0 \notin \{n : |n| > N\} + \{m : |m| < N/2\} + \{k : |k| \leq N/2\}$. □

3. **Estimates of the high-frequency tails for regularized Burger's equations.**

In this section, we prove Theorem 1.1. The classical theory guarantees global existence of classical solutions, so we proceed with the estimates.

For $M >> L$, so that $M/L \in 2^{\mathcal{Z}}$, take the projection $P_{>M}$ on both sides of (5). We then take a scalar product of the result with u. We have

$$\frac{1}{2} \partial_t \|u_{>M}(t,\cdot)\|_{L^2}^2 + \|P_{>M} A_s^{1/2} u(t,\cdot)\|_{L^2}^2 \leq |\int u_{>M} div(u^2) dx| \tag{17}$$

[4] The definition here makes sense only for sequences $\{a_k\}$ with enough decay, say in $l^2_\sigma, \sigma > s + d/2$. One may of course take $A_s f$ to represent a distribution for less decaying $\{a_k\}$.

Clearly, $\|P_{>M}A_s^{1/2}u(t,\cdot)\|_{L^2}^2 \geq (M/L)^s\|u_{>M}(t,\cdot)\|_{L^2}^2$, while since $\int u_{>M}div[(u_{>M})^2]dx = \frac{2}{3}\int div[(u_{>M})^3]dx = 0$, one has

$$\int u_{>M}div(u^2)dx = 2\int u_{>M}div[u_{>M}u_{\leq M}]dx + \int u_{>M}div[u_{\leq M}^2]dx \leq$$
$$\leq \|u_{>M}\|_{L^2}^2\|\nabla u_{\leq M}\|_{L^\infty} + 2\int u_{>M}u_{\leq M}div[u_{\leq M}]dx,$$

Furthermore, by Lemma 2.2 $\int u_{>M}u_{<M/2}div[u_{<M/2}]dx = 0$ and hence

$$\int u_{>M}u_{\leq M}div[u_{\leq M}]dx$$
$$= \int u_{>M}(u_{\leq M/2} + u_{M/2<\cdot\leq M})div[u_{\leq M/2} + u_{M/2<\cdot\leq M}]$$
$$= \int u_{>M}u_{\leq M/2}div[u_{M/2<\cdot\leq M}]dx + \int u_{>M}u_{M/2<\cdot\leq M}div[u_{\leq M}]dx.$$

The last identity allows us to estimate by Hölder's $|\int u_{>M}u_{\leq M}div[u_{\leq M}]dx| \leq$

$$\leq C\|u_{>M}\|_{L^2}\|\nabla u_{M/2<\cdot\leq M}\|_{L^2}\|u_{\leq M/2}\|_{L^\infty} + C\|u_{>M}\|_{L^2}\|u_{>M/2}\|_{L^2}\|\nabla u_{\leq M}\|_{L^\infty}$$
$$\leq C(M/L)\|u_{>M}\|_{L^2}\|u_{>M/2}\|_{L^2}(\|u_{\leq M/2}\|_{L^\infty}\| + \|u_{\leq M}\|_{L^\infty})$$

Inserting all the relevant estimates in (17) yields

$$\partial_t\|u_{>M}(t,\cdot)\|_{L^2}^2 + 2(M/L)^s\|u_{>M}(t,\cdot)\|_{L^2}^2 \leq$$
$$\leq C(M/L)\|u_{>M}\|_{L^2}\|u_{>M/2}\|_{L^2}(\|u_{\leq M/2}\|_{L^\infty} + \|u_{\leq M}\|_{L^\infty})$$

At this stage, the argument splits into the two cases, $1 < s \leq 2$ and $s > 1 + d/2$. Note that these cases overlap for $d = 1$.

3.1. **Estimates in the case** $1 < s \leq 2$**.** In this case, we use the following pointwise inequality

$$\int_{[-L,L]^d}|\psi|^{p-2}\psi A^s[\psi]dx \geq C_{L,p}\|A^{s/2}\psi^{p/2}\|_{L^2}^2. \tag{18}$$

for any $0 \leq s \leq 2$, and for any smooth function $\psi(x) : [-L,L]^d \to R^1$ (see [6] and [16]). One observes then that taking a scalar product of (5) with $|u|^{p-2}u$ yields

$$\partial_t\frac{1}{p}\|u\|_{L^p}^p \leq \partial_t\frac{1}{p}\|u\|_{L^p}^p + C_{L,p}\|A^{s/2}u^{p/2}\|_{L^2}^2$$
$$\leq \int u_t u|u|^{p-2}dx + \int [A_s u]u|u|^{p-2}dx = 0,$$

hence $\|u(t,\cdot)\|_{L^p}$ is a decreasing function for every $p \geq 2$. By Lemma 2.1 and the monotonicity of $t \to \|u(t,\cdot)\|_{L^p}$, for any $p : 2 < p < \infty$ we have

$$\|u_{\leq M/2}\|_{L^\infty} + \|u_{\leq M}\|_{L^\infty} \leq C_{d,p}(M/L)^{d/p}\|u_0\|_{L^p}. \tag{19}$$

Select $p : d/p = 2\delta(s-1)$, so that $0 < \delta << 1$ and $p > 2$. Insert this into the estimate for $\partial_t\|u_{>M}(t,\cdot)\|_{L^2}^2 + (\frac{M}{L})^s\|u_{>M}(t,\cdot)\|_{L^2}^2$. After Cauchy-Schwartz, we get a bound of

$$C_{\delta,s}(\frac{M}{L})^{2(1+2\delta(s-1))-s}\|u_0\|_{L^p}^2\|u_{>M/2}(t,\cdot)\|_{L^2}^2 + \frac{1}{2}(\frac{M}{L})^s\|u_{>M}(t,\cdot)\|_{L^2}^2, \tag{20}$$

where we can of course absorb the last term on the right hand side on the left side. Furthermore, $\|u_0\|_{L^p} \leq \|u_0\|_{L^2\cap L^\infty}$ and take $M = 2^jL$, as this is suitable for the forthcoming induction argument. We will show the bound (6) first for $0 \leq t \leq 1$ and then, we will extend the result to $t > 1$.

In the case $0 \le t \le 1$, we will show by induction that there exists a constant C_0, depending on δ and s, so that

$$\|u_{>2^j L}(t)\|_{L^2}^2 \le (C_0 \max(1, \|u_0\|_{L^2\cap L^\infty}^2))^{j+1} 2^{-t(1-\delta)(s-1)j^2}, \quad t\in[0,1]. \tag{21}$$

The first thing to observe is that for all $0<j\le 5$, we have by the monotonicity of $t\to\|u(t)\|_{L^2}$, $\|u_{>2^jL}(t)\|_{L^2}^2 \le \|u(t)\|_{L^2}^2 \le \|u_0\|_{L^2}^2$, whence (21) holds, as long as we select $C_0 > 2^{25(s-1)(1-\delta)}$. Thus, assuming the validity of (21) for some $j-1$, $j\ge 6$, we have by (20) $\partial_t \|u_{>2^jL}(t,\cdot)\|_{L^2}^2 + 2^{js}\|u_{>2^jL}(t,\cdot)\|_{L^2}^2 \le$

$$\begin{aligned}&\le C_{\delta,s} 2^{j(2+2\delta(s-1)-s)}\|u_0\|_{L^2\cap L^\infty}^2 \|u_{>2^{j-1}}(t,\cdot)\|_{L^2}^2 \le\\ &\le C_{\delta,s} 2^{j(2+2\delta(s-1)-s)}\|u_0\|_{L^2\cap L^\infty}^2 (C_0\max(1,\|u_0\|_{L^2\cap L^\infty}^2))^j 2^{-t(1-\delta)(s-1)(j-1)^2}.\end{aligned}$$

Apply Gronwall's inequality to the last equation, taking into account that $\int_0^t e^{z(2^{js}-(j-1)^2(1-\delta)(s-1))}dz \le 2^{-js+1}e^{t2^{js}}$, since $2^{js} > 2(j-1)^2$ for $j\ge 6, s>1$. This gives $\|u_{>2^jL}(t,\cdot)\|_{L^2}^2 \le$

$$\|P_{>2^jL}u_0\|_{L^2}^2 e^{-t2^{js}} + 2C_{\delta,s}C_0^j \max(1,\|u_0\|_{L^2\cap L^\infty}^2)^{j+1} 2^{-2j(1-\delta)(s-1)} 2^{-t(1-\delta)(s-1)(j-1)^2}.$$

The exponents that arise can be estimated in the following straightforward manner. We have $e^{-t2^{js}} \le 2^{-t(s-1)(1-\delta)j^2}$ for all $j\ge 6, 1<s\le 2, 1>\delta>0$. Also, since $t\in[0,1]$, we have $-2j(1-\delta)(s-1) - t(1-\delta)(s-1)(j-1)^2 < -t(s-1)(1-\delta)j^2$. Thus, selecting $C_0 = 4C_{\delta,s} + 2^{25(s-1)(1-\delta)} + 2$ finishes the proof of (21).

The results of the previous case are easy to extend now to the case $t>1$. Namely, we will show that there exists a constant C_1, so that

$$\|u_{>2^jL}(t)\|_{L^2}^2 \le (C_1\max(1,\|u_0\|_{L^2\cap L^\infty}^2))^{j+1} 2^{-(1-\delta)(s-1)j^2}, \quad t>1 \tag{22}$$

Again, the case of $j=0,\dots,5$ is easy to be verified by the monotonicity of the L^2 norm. Assuming $j\ge 6$ and (22) for all $t>1$ and some $j-1$, we apply (20), where we insert the estimate (22) for the term $u_{>2^{j-1}L}$. We get

$$\begin{aligned}&\partial_t\|u_{>2^jL}(t,\cdot)\|_{L^2}^2 + 2^{js}\|u_{>2^jL}(t,\cdot)\|_{L^2}^2 \le\\ &\le C_{\delta,s}C_1^j 2^{j(2+2\delta(s-1)-s)}\max(1,\|u_0\|_{L^2\cap L^\infty}^2)^j 2^{-(1-\delta)(s-1)(j-1)^2}.\end{aligned}$$

Apply the Gronwall's inequality in the interval $(1,t)$.

$$\begin{aligned}&\|u_{>2^jL}(t,\cdot)\|_{L^2}^2 \le \|u_{>2^jL}(1,\cdot)\|_{L^2}^2 e^{-(t-1)2^{js}} +\\ &+C_{\delta,s}C_1^j\max(1,\|u_0\|_{L^2\cap L^\infty}^2)^{j+1}2^{-2j(1-\delta)(s-1)-(1-\delta)(s-1)(j-1)^2}.\end{aligned}$$

However, inserting the bound (21) for $\|u_{>2^jL}(1,\cdot)\|_{L^2}^2$ and realizing that again $-2j(1-\delta)(s-1)-(1-\delta)(s-1)(j-1)^2 \le -(1-\delta)(s-1)j^2$, we have for $t>1$,

$$\begin{aligned}&\|u_{>2^jL}(t,\cdot)\|_{L^2}^2 \le (C_0\max(1,\|u_0\|_{L^2\cap L^\infty}^2))^{j+1}2^{-(1-\delta)(s-1)j^2} +\\ &+C_{\delta,s}C_1^j\max(1,\|u_0\|_{L^2\cap L^\infty}^2)^{j+1}2^{-(1-\delta)(s-1)j^2} \le\\ &\le (C_1\max(1,\|u_0\|_{L^2\cap L^\infty}^2))^{j+1}2^{-(1-\delta)(s-1)j^2}, \text{ as long as } C_1 = 2C_0.\end{aligned}$$

This concludes the proof of (6).

3.2. **The case $s>1+d/2$.** The proof for $s>1+d/2$ goes almost identically to the case $1<s\le 2$. Note that the monotonicity of $t\to\|u(t)\|_{L^p}, p>2$ is unavailable[5] in this context, but we still have that $t\to\|u(t)\|_{L^2}$ is decreasing and therefore by Lemma 2.1 $\|u_{\le M/2}\|_{L^\infty}\| + \|u_{\le M}\|_{L^\infty} \le C(M/L)^{d/2}\|u_0\|_{L^2}$. Thus $\partial_t\|u_{>M}(t,\cdot)\|_{L^2}^2 + 2(M/L)^s\|u_{>M}(t,\cdot)\|_{L^2}^2 \le C(M/L)^{1+d/2}\|u_{>M}\|_{L^2}\|u_{>M/2}\|_{L^2}\|u_0\|_{L^2}$. Whence $\partial_t\|u_{>M}(t,\cdot)\|_{L^2}^2 + (M/L)^s\|u_{>M}(t,\cdot)\|_{L^2}^2 \le C(M/L)^{2+d-s}\|u_{>M/2}\|_{L^2}^2\|u_0\|_{L^2}^2$. This is similar to (20), except for the power of (M/L) on the right-hand side. One

[5] Or at least, we are not aware of such result.

can now perform an identical argument to show (7). This is done by systematically replacing the factor $(1-\delta)(s-1)$ by $s-1-d/2$, which is assumed to be positive.

4. **Estimates of the high-frequency tails for the Kuramoto-Sivashinsky equation.** In this section, we prove theorem 1.2. The approach that we take is very similar to the one in Section 3, except that now because of the destabilzing term u_{xx}, we do not have such a good control of $\|u(t)\|_{L^2}$.

We start as in Section 3 by taking the projection $P_{>M}$ in (2), with $M >> L$. After multiplication by u, integrating in x and integration by parts, we obtain

$$\partial_t \frac{1}{2}\|u_{>M}(t,\cdot)\|_{L^2}^2 + \|\partial_x^2 u_{>M}(t,\cdot)\|_{L^2}^2 - \|\partial_x u_{>M}(t,\cdot)\|_{L^2}^2 \leq |\int u_{>M} u u_x dx|$$

Now by the elementary properties of $P_{>M}$ in Section 2, we have
$\|\partial_x^2 u_{>M}(t,\cdot)\|_{L^2}^2 \geq C(M/L)^2\|\partial_x u_{>M}(t,\cdot)\|_{L^2}^2$ and
$\|\partial_x^2 u_{>M}(t,\cdot)\|_{L^2}^2 >> \|\partial_x u_{>M}(t,\cdot)\|_{L^2}^2$.
Moreover $\|\partial_x^2 u_{>M}(t)\|_{L^2}^2 \geq C(M/L)^4\|u_{>M}(t)\|_{L^2}^2$. On the other hand,

$$\begin{aligned} &|\int u_{>M} u u_x dx| \leq \frac{1}{2}\|u_{>M}\|_{L^2}^2\|\partial_x u_{\leq M}\|_{L^\infty} + \\ &+C(M/L)\|u_{>M}\|_{L^2}\|u_{>M/2}\|_{L^2}(\|u_{\leq M/2}\|_{L^\infty}\| + \|u_{\leq M}\|_{L^\infty}) \end{aligned}$$

For the second term on the right hand side, we further estimate via Cauchy-Schwartz

$$\begin{aligned} &(M/L)\|u_{>M}\|_{L^2}\|u_{>M/2}\|_{L^2}(\|u_{\leq M/2}\|_{L^\infty}\| + \|u_{\leq M}\|_{L^\infty}) \leq \\ &\leq \frac{1}{4}(M/L)^4\|u_{>M}\|_{L^2}^2 + C(M/L)^{-2}|u_{>M/2}\|_{L^2}^2(\|u_{\leq M/2}\|_{L^\infty}\| + \|u_{\leq M}\|_{L^\infty})^2. \end{aligned}$$

Putting all of these estimates together yields

$$\begin{aligned} \partial_t\|u_{>M}\|_{L^2}^2 + 2(M/L)^4\|u_{>M}\|_{L^2}^2 \leq C\|u_{>M}\|_{L^2}^2\|\partial_x u_{\leq M}\|_{L^\infty} + \\ +C(M/L)^{-2}\|u_{>M/2}\|_{L^2}^2(\|u_{\leq M/2}\|_{L^\infty} + \|u_{\leq M}\|_{L^\infty})^2. \end{aligned} \tag{23}$$

By Lemma 2.1, $\|u_{\leq M}\|_{L^\infty} + \|u_{\leq M/2}\|_{L^\infty} \leq C(M/L)^{1/2}\sup_t \|u(t,\cdot)\|_{L^2}$. Observe also that $\|\partial_x u_{\leq M}\|_{L^\infty} \leq C(\frac{M}{L})^{3/2}H$. All in all, (23), together with the previous two observations implies

$$\begin{aligned} &\partial_t\|u_{>M}\|_{L^2}^2 + 2(M/L)^4\|u_{>M}\|_{L^2}^2 \leq \\ &\leq C(M/L)^{3/2}\sup_s \|u(s,\cdot)\|_{L^2}\|u_{>M}\|_{L^2}^2 + C(M/L)^{-1}\sup_s \|u(s)\|_{L^2}^2\|u_{>M/2}\|_{L^2}^2. \end{aligned}$$

Let $M = 2^j L$, denote $H = \sup_s \|u(s,\cdot)\|_{L^2}$. Fix an integer j_0, so that $2^{5j_0} >> H^2$. Denote $I_j(t) := \|u_{>2^j L}(t,\cdot)\|_{L^2}^2$. We have

$$I_j'(t) + 2^{4j+1}I_j(t) \leq C2^{3j/2}HI_j(t) + C2^{-j}H^2 I_{j-1}(t). \tag{24}$$

Furthermore, since we are only interested in an estimate for $j \geq j_0$, it is easy to see that since $2^{-5j_0/2}H << 1$, one has $C2^{3j/2}HI_j \leq C2^{4j}2^{-5j_0/2}HI_j < 2^{4j}I_j$, which means that the first term on the right-hand side of (24) may be absorbed on the left-hand side. Thus, for all $j \geq j_0$,

$$I_j'(t) + 2^{4j}I_j(t) \leq C2^{-j}H^2 I_{j-1}. \tag{25}$$

We will apply the same idea as in the proof of (6). Namely, we run an induction argument based on (25) for $j \geq j_0$ for a short period of time $0 < t \leq 5/2$ and then we will extend to $t > 5/2$. In the case $0 \leq t \leq 5/2$, as above we show that there exists an absolute constant C_0, so that for all $0 < t < 5/2$, and all $j \geq j_0$,

$$I_j(t) \leq C_0^{j+1}2^{-t(j-j_0)^2}H^2. \tag{26}$$

For the case $t > 5/2$ we set our induction argument with the hypothesis

$$I_j(t) \leq C_1^{j+1} 2^{-\frac{5}{2}(j-j_0)^2} H^2. \tag{27}$$

We insert the induction hypothesis in (25) and then run a Gronwall's argument for the resulting inequality in the interval $[\frac{5}{2}, t]$. Thus we show (27) for all $j \geq j_0, t > \frac{5}{2}$.

5. **Estimates of the higher Sobolev norms for the KSE.** In this section, we show how to make use of the Gevrey regularity estimates for the solutions of KSE, provided by Theorem 1.2, to provide effective estimates on higher Sobolev norms.

5.1. **Proof of Corollary 1.** We actually show (10), which implies (9). By the equivalence of the norms in (14),

$$\|u(t,\cdot)\|_{\dot{H}^s} \leq C^s[\|u_{<C_0H^{2/5}L}\|_{L^2} H^{2s/5} + \left(\sum_{j=0}^{\infty} (2^j C_0 H^{2/5})^{2s} \|u_{\sim 2^j C_0 H^{2/5} L}\|_{L^2}^2 \right)^{1/2}],$$

where C is an absolute constant. For the first term, we have $\|u_{<C_0H^{2/5}L}\|_{L^2} \leq H$. For the second term, we estimate $\sup_{\delta \leq t} \|u_{\sim C_0 2^j H^{2/5} L}(t,\cdot)\|_{L^2} \leq C_1^j 2^{-\delta j^2/2} H$, which we insert in the sum above. We get

$$\sum_{j=0}^{\infty} (2^j C_0 H^{2/5})^{2s} \|u_{\sim 2^j C_0 H^{2/5}}\|_{L^2}^2 \leq C^s H^{4s/5+2} \sum_{j=0}^{\infty} C_1^{2j} 2^{2sj - \delta j^2} \leq C_{\delta,s} H^{4s/5+2}.$$

Taking square roots yields (10).

5.2. **Proof of Corollary 2.** The proof of corollary 2 requires us to revisit the proof of Theorem 1.2. Namely, starting again with (23), we estimate this time (by Lemma 2.1) $\|u_{\leq M}\|_{L^\infty} + \|u_{\leq M/2}\|_{L^\infty} \leq C_p (M/L)^{1/p} \sup_t \|u(t,\cdot)\|_{L^p}$. Thus, we get

$$\begin{aligned} &\partial_t \|u_{>M}\|_{L^2}^2 + 2(M/L)^4 \|u_{>M}\|_{L^2}^2 \\ \leq & C(M/L)^{1+1/p} \sup_s \|u(s,\cdot)\|_{L^p} \|u_{>M}\|_{L^2}^2 + C(M/L)^{-2+2/p} \sup_s \|u(s)\|_{L^p}^2 \|u_{>M/2}\|_{L^2}^2. \end{aligned}$$

Setting $M = 2^j L$ and denoting with $I_j(t) = \|u_{>2^j L}(t,\cdot)\|_{L^2}^2$, we obtain the inequality

$$I_j' + 2^{4j+1} I_j \leq C 2^{j(1+1/p)} K_p I_j + C 2^{j(-2+2/p)} K_p^2 I_{j-1}. \tag{28}$$

Setting again j_0 such that $2^{j_0(3-1/p)} = 100 \max(1, C^2) K_p$, we obtain $C 2^{j(1+1/p)} K_p I_j \leq 2^{4j} I_j$. Therefore one can absorb the first term on the right-hand side of (28), as long as $j \geq j_0$. The result is $I_j' + 2^{4j} I_j \leq C 2^{j(-2+2/p)} K_p^2 I_{j-1}$. An induction argument similar to the one needed for (26) applies again. We get for all[6] $t : 0 < t < 3 - \frac{1}{p}$

$$I_j(t) \leq C_0^{j+1} 2^{-t(j-j_0)^2} H^2. \tag{29}$$

In the case $t > 3 - 1/p$, we apply an induction, similar to the one needed for (27). We get for all $j \geq j_0$ and all $t > 3 - 1/p$ the estimate $I_j(t) \leq C_1^{j+1} 2^{-(3-1/p)(j-j_0)^2} H^2$. Combining the two estimates yields the Gevrey bound

$$I_j(t) \leq C^{j+1} 2^{-\min(t, 3-1/p)(j-j_0)^2} H^2. \tag{30}$$

Similarly to the proof of Corollary 1 (see Section 5.1), the Gevrey estimate (30) can be turned into estimates for higher Sobolev norms. Indeed, by (30) and since $2^{j_0} \sim K_p^{1/(3-1/p)}$, we obtain $\sup_{\delta \leq t} \|u_{\sim C_0 2^j K_p^{1/(3-1/p)} L}(t,\cdot)\|_{L^2} \leq C_1^j 2^{-\delta j^2/2} H$.

[6] Note that in the previous argument, we have been using $p = 2$.

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Received June 2008; revised May 2009.

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