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SOME CLASSES OF SURFACES IN $\mathbb{R}^3$ AND M_3 ARISING FROM SOLITON THEORY AND A VARIATIONAL PRINCIPLE

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Abstract. In this paper, modified Korteweg-de Vries (mKdV) and Harry Dym (HD) surfaces are considered which are arisen from using soliton surface technique and a variational principle. Some of these surfaces belong to Willmore-like and Weingarten surfaces, and surfaces that solve the generalized shape equation classes. Moreover, parameterized form of these surfaces are found for given solutions of the mKdV and HD equations.

1. **Introduction.** Surface theory in three dimensional Euclidean space is widely used in different branches of science, particularly mathematics (differential geometry, topology, Partial Differential Equations (PDEs)), theoretical physics (string theory, general theory of relativity), and biology [19]-[6]. There are some special subclasses of 2-surfaces which arise in the branches of science aforementioned. For the classification of surfaces, there are some conditions which are sometimes given as algebraic relations between Gaussian and mean curvatures and sometimes given as differential equations for these two curvatures. Here are some examples of some subclasses of 2-surfaces; Minimal surfaces, Surfaces with constant Gaussian curvature, Weingarten surfaces, Willmore surfaces, Surfaces that solve the shape equation of lipid membrane. Some examples and details about these surfaces can be found in [7]-[13].

Although numerous scientists have studied minimal surface theory, Joseph Plateau considerably contributed to minimal surface studies with his experiments on soap films and soap bubbles. Plateau worked on the free energy of films. In 1805 and 1839, T. Young and P. S. Laplace, respectively, considered the free energy of closed soap bubbles. In 1833, Poisson considered the free energy of a solid shell as $F = \oint_S H^2 dA$. In 1982, T. J. Willmore [28] found the Euler-Lagrange equation arising from Poisson's free energy F as $\nabla^2 H + 2H(H^2 - K) = 0$, where ∇^2 is the Laplace-Beltrami operator and K is the Gaussian curvature of the surface. Solutions of this equation are called *Willmore surfaces*. In 1973, Helfrich proposed the curvature energy per unit area of the bilayer $E_{lb} = (k_c/2)\,(2H + c_0)^2 + \bar{k}K$, where k_c and $\bar{k}$ are elastic constants, and c_0 is spontaneous curvature of the lipid bilayer. Using the Helfrich curvature energy, the free energy functional of the lipid vesicle

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is written as

$$F = \oiint_S (E_{lb} + \omega)\, dA + p \iiint dV. \tag{1}$$

Taking the first variation of F, Ou-Yang and Helfrich [16] obtained the shape equation of the bilayer, $p - 2\omega H + k_c \nabla^2(2H) + k_c(2H + c_0)(2H^2 - c_0 H - 2K) = 0$. Later Ou-Yang *et al.* considered the general energy functional

$$F = \oiint_S E(H,K)\, dA + p \iiint_V dV \tag{2}$$

which arises both in red blood cells and liquid crystals [18], [26]-[13]. Here E is function of H and K, p is a constant, and V is the volume enclosed within the surface S. For open surfaces, we let $p = 0$. The first variation (Euler-Lagrange) of F gives a highly nonlinear PDE of K and H on surface S. It is given by [18], [26]-[27]

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial E}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial E}{\partial K} - 4HE + 2p = 0, \tag{3}$$

where ∇^2 and $\nabla \cdot \bar{\nabla}$ will be defined latter part of the paper.

As it is shown, certain subclasses of surfaces arise as solutions of some differential equations. That is there are some relations between surfaces and PDEs. Since these equations are high order nonlinear PDEs, these equations are not so easy to solve. For this reason some indirect methods [20]-[5] have been developed for the construction of 2-surfaces in $\mathbb{R}^3$ and three dimensional Minkowskian geometry (M_3). One of these methods is arisen from Soliton theory. We call this technique as *Soliton surfaces technique*. Soliton surface theory was first developed by Sym [20]-[22]. Then it was studied by Fokas and Gel'fand [8].

Here, Fokas and Gel'fand approach is considered to construct surfaces by using soliton equations. Let G be a Lie group and $\mathfrak{g}$ be the corresponding Lie algebra. We give the theory for dim $\mathfrak{g} = 3$, it is possible to generalize it for finite dimension n. Assume that there exists an inner product $\langle , \rangle$ on $\mathfrak{g}$ such that for g_1, $g_2 \in \mathfrak{g}$ as $\langle g_1, g_2 \rangle$. Let $\{e_1, e_2, e_3\}$ be the orthonormal basis in $\mathfrak{g}$ such that $\langle e_i, e_j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta. In this paper, we use Einstein's summation convention on repeated indices over their range.

Theorem 1.1 (Fokas and Gel'fand [8]). *Let U, V, A, and B be $\mathfrak{g}$ valued differentiable functions of x, t, and λ for every $(x,t) \in \mathcal{O} \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. Assume that U, V, A, and B satisfy the following equations*

$$U_t - V_x + [U,V] = 0 \text{ and } A_t - B_x + [A,V] + [U,B] = 0. \tag{4}$$

Then the following equations

$$\Phi_x = U\,\Phi,\ \ \Phi_t = V\,\Phi \text{ and } F_x = \Phi^{-1}\,A\,\Phi,\ \ F_t = \Phi^{-1}\,B\,\Phi, \tag{5}$$

define surfaces $\Phi \in G$ and $F \in \mathfrak{g}$, respectively. The first and second fundamental forms of the surface F are of the following form, respectively

$$(ds_I)^2 \equiv g_{ij} dx^i\, dx^j = \langle A, A\rangle\, dx^2 + 2\langle A, B\rangle\, dx\, dt + \langle B, B\rangle\, dt^2, \tag{6}$$

$$(ds_{II})^2 \equiv h_{ij} dx^i\, dx^j = \langle A_x + [A,U], C\rangle\, dx^2 + 2\langle A_t + [A,V], C\rangle\, dx\, dt + \langle B_t + [B,V], C\rangle\, dt^2 \tag{7}$$

where $i,j = 1,2$, $x^1 = x$, $x^2 = t$. Here $C = [A,B]/\|[A,B]\|$, $[A,B]$ denotes the usual commutator, and $\|X\| = \sqrt{\langle X, X\rangle}$. The Gaussian and mean curvatures of *the surface are, respectively, shown by*

$$K = det(g^{-1})h,\ H = \frac{1}{2} trace(g^{-1}h), \tag{8}$$

where g and h denote the matrices (g_{ij}) and (h_{ij}), respectively, and g^{-1} stands for the inverse of the matrix g.

As it is seen in Theorem 1.1, we need to know the fundamental forms and curvatures to characterize a surface . In order to calculate them, it is sufficient to know U, V, A, and B. Since the main aim is to find a class of surfaces, which corresponds to integrable equations, we need here to find A and B from (4). But in general, solving this equation is difficult. However, there are some deformations that provide A and B directly. They are spectral parameter, symmetries of the (integrable) differential equations, the Gauge symmetries of the Lax equation deformations, and the deformation of parameters for solution of integrable equation. The first three were given by Sym [20]-[22], Fokas and Gel'fand [8], Fokas *et al.* [9] and Cieśliński [5]. The last one is introduced in [24]. In this paper, we consider the following deformations,

1. Spectral parameter λ invariance of the equation:
$$A = \mu_1 \frac{\partial U}{\partial \lambda},\ B = \mu_1 \frac{\partial V}{\partial \lambda},\ F = \mu_1 \Phi^{-1} \frac{\partial \Phi}{\partial \lambda}, \tag{9}$$
where μ_1 is an arbitrary function of λ [20]-[22].
2. The deformation of parameters for solution of integrable equation:
$$A = \mu_2 \left(\partial U/\partial \xi_i\right),\ B = \mu_2 \left(\partial V/\partial \xi_i\right),\ F = \mu_2 \Phi^{-1} \left(\partial \Phi/\partial \xi_i\right), \tag{10}$$
where $i = 0, 1$ and ξ_i are parameters of the solution $u(x, t, \xi_0, \xi_1)$ of the PDEs, μ_2 is constant. Here x and t are independent variables. [24]

On the other hand, there are some surfaces that arise from a variational principle for a given Lagrange function free energy, which is a polynomial of degree less than or equal to two in the mean curvature of the surfaces. Examples of this type are minimal surfaces, constant mean curvature surfaces, linear Weingarten surfaces, Willmore surfaces, and surfaces solving the shape equation for the Lagrange functions. Taking more general Lagrange function of the mean and Gaussian curvatures of the surface, we may find more general surfaces that solve the generalized shape equation (3). Examples for this type of surfaces can be found in [11] and [23].

Let S be a 2-surface (either in M_3 or in $\mathbb{R}^3$) with the mean and Gaussian curvatures H and K, respectively.

Definition 1.2. A free energy F of S is defined by (2), where E is some function of H and K, p is a constant and V is the volume enclosed within the surface S. For open surfaces, we let $p = 0$.

The following proposition gives the first variation of the functional F.

Proposition 1. *Let E be a twice differentiable function of H and K. Then the Euler-Lagrange equation for F reduces* [18], [26]-[27] *to (3), where ∇^2 and $\nabla\cdot\bar{\nabla}$ are defined as*

$$\nabla^2 = \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial x^i} \left(\sqrt{\tilde{g}} g^{ij} \frac{\partial}{\partial x^j} \right),\ \nabla\cdot\bar{\nabla} = \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial x^i} \left(\sqrt{\tilde{g}} K h^{ij} \frac{\partial}{\partial x^j} \right), \tag{11}$$

and $\tilde{g} = \det(g_{ij})$*,* g^{ij} *and* h^{ij} *are inverse components of the first and second fundamental forms, respectively, and* $i, j = 1, 2$*, where* $x^1 = x$*,* $x^2 = t$*. Equation (3) is called generalized shape equation.*

Some of the subclasses of the surfaces can be derived from a variational principle for a suitable E. These are given as: Minimal surfaces, $E = 1,\ p = 0$; Surfaces with constant mean curvature, $E = 1$; Linear Weingarten surfaces, $E = aH + b$, where a and b are some constants; Willmore surfaces, $E = H^2$ [28], [29]; Surfaces that solve the shape equation of lipid membrane, $E = (H - c)^2$, where c is a constant [18], [26]-[13]; Shape equation of closed lipid bilayer, $E = (k_c/2)\,(2H + c_0)^2 + \bar{k}K$, where k_c and $\bar{k}$ are elastic constants, and c_0 is the spontaneous curvature of the lipid bilayer [16].

Definition 1.3. Surfaces that solve the equation

$$\nabla^2 H + aH^3 + bH\,K = 0, \tag{12}$$

are called *Willmore-like* surfaces, where a and b are arbitrary constants.

Remark 1. $a = 2,\ b = -2$ case corresponds to the Willmore surfaces which arise from a variational problem. For other values of a and b Willmore-like surfaces cannot be derived from a variational problem.

For compact 2-surfaces, the constant p in (2) may be different than zero, but for noncompact surfaces we assume it to be zero. For the latter, asymptotic conditions are required, so K goes to a constant and H goes to zero asymptotically. For this purpose, we shall use the Euler-Lagrange equation (3) for surfaces obtained by mKdV, HD equations and search for solutions (surfaces) of the Euler-Lagrange equation [(3)].

The principal purpose of this paper is to find new classes of 2-surfaces using the deformations of Lax equations for mKdV and HD equations and to obtain solutions of the generalized shape equation (3) for polynomial Lagrange functions of the curvatures H and K

$$E = a_{N0}\,H^N + \ldots + a_{11}\,H\,K + a_{21}\,H^2\,K + \ldots + a_{01}\,K + ... \tag{13}$$

For each N, we find the constants a_{nl} in terms of the parameters of the surface, where $n, l = 0, 1, 2, \ldots$ and $N = 3, 4, 5, \ldots$

2. **2-surfaces in $\mathbb{R}^3$.**

In this section, in order to construct 2-surfaces in $\mathbb{R}^3$, we use Lie group SU(2) and its Lie algebra $\mathfrak{su}(2)$ with basis $e_j = -i\,\sigma_j,\ j = 1, 2, 3$, where σ_j denote the usual Pauli sigma matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

Define an inner product on $\mathfrak{su}(2)$ as $\langle X, Y\rangle = -(1/2)\,\mathrm{trace}(XY)$, for $X,\ Y \in \mathfrak{su}(2)$.

2.1. **mKdV surfaces.**

Let $u(x, t)$ satisfy the mKdV equation $u_t = u_{3x} + (3/2)u^2 u_x$. Substituting the travelling wave ansatz $u_t - \alpha\,u_x = 0$ in the previous equation, we get

$$u_{2x} = \alpha u - \frac{u^3}{2}, \tag{15}$$

where α is an arbitrary real constant and integration constant is taken to be zero. (15) can be obtained from Lax pairs U and V, where

$$U = \frac{i}{2}\begin{pmatrix} \lambda & -u \\ -u & -\lambda \end{pmatrix}, \tag{16}$$

$$V = -\frac{i}{2}\begin{pmatrix} \frac{1}{2}u^2 - (\alpha + \alpha\lambda + \lambda^2) & (\alpha+\lambda)u - iu_x \\ (\alpha+\lambda)u + iu_x & -\frac{1}{2}u^2 + (\alpha + \alpha\lambda + \lambda^2) \end{pmatrix}, \tag{17}$$

and λ is a spectral parameter.

In [23], we considered spectral parameter deformation and combination of spectral and Gauge deformations. In this paper, we consider the mKdV surfaces arising from deformations of parameters of the integrable equations' solution.

Consider the one soliton solution of mKdV equation (15) as

$$u = k_1\, sech\, \xi, \tag{18}$$

where $\alpha = k_1^2/4$, $\xi = k_1(k_1^2 t + 4x)/8 + \xi_0$, ξ_0 and k_1 are arbitrary constants.

Remark 2. mKdV surfaces constructed by ξ_0 deformation are sphere, where Gaussian and mean curvatures are

$$K = \frac{16\lambda^2}{k_1^2\mu^2},\ H = -\frac{4\lambda}{k_1\mu}. \tag{19}$$

Here ξ_0 and k_1 are parameters of the solution which are arbitrary constants.

Another parameter of the solution is k_1. It is also possible to use k_1 parameter to construct new mKdV surfaces. These classes of surfaces are given by the following proposition.

Proposition 2. *Let u, given by (18), satisfy the equation (15). The corresponding $\mathfrak{su}(2)$ valued Lax pairs U and V of the mKdV equation are given by (16) and (17), respectively. $\mathfrak{su}(2)$ valued matrices A and B are*

$$A = -\frac{i\mu}{2}\begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix},\ B = -\frac{i\mu}{2}\begin{pmatrix} u\,\phi & (k_1^2/4+\lambda)\phi - i\,\phi_x \\ (k_1^2/4+\lambda)\phi + i\,\phi_x & -u\,\phi \end{pmatrix}, \tag{20}$$

where $A = \mu\,(\partial U/\partial k_1)$, $B = \mu\,(\partial V/\partial k_1)$, and $\phi = \partial\, u/\partial k_1$; k_1 is a parameter of the one soliton solution u, and μ is a constant. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms $(j, k = 1, 2)$

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k,\ (ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k, \tag{21}$$

where
$g_{11} = (1/4)\mu^2\phi^2$, $g_{12} = g_{21} = (1/16)\,\phi(2\,k_1\,u + \phi[k_1^2 + 4\,\lambda])$, $g_{22} = (1/64)\mu^2\Big(4[k_1^2 + 4\,\phi^2]u^2 + 4\,k_1\,(k_1^2-4)u\,\phi + \phi_x^2 + (k_1^2+4\,\lambda)^2\,\phi^2 + 4\,k_1^2(\lambda+1)^2\Big)$, $h_{11} = (1/16)\Delta\,\mu^3\,\lambda\,\phi^2\big(k_1[\lambda+1] - 2\,u\,\phi\big)$, $h_{12} = (1/4)\Delta\,\mu^3\phi^2\Big(8\,\phi_x u_x + \Big[k_1(\lambda+1) - 2u\phi\Big]\Big[2(2\lambda^2 - u^2) + k_1^2(\lambda+1)\Big]\Big)$, $h_{22} = (1/256)\Delta\,\mu^3\,\phi\Big(8\,\phi_x\Big[2\,k_1\,u\,u_x + (k_1+4\,\lambda)(\phi\,u_x - u) + 4(u\phi)_t\Big] + \Big[k_1(\lambda+1) - 2\,u\,\phi\Big]\Big[16\,\phi_{xt} + \phi(k_1^2+4\,\lambda)(2[u^2+2\,\lambda] + k_1^2[\lambda+1]) - 4\,k_1\,u(u^2-2\,\lambda)\Big]\Big)$,

and after substitution u, given by (18), the corresponding Gaussian and mean curvatures are

$$K = \frac{1}{\mu^2\,\eta_0\,(4\,\eta_4^2+\eta_3^2)^2} Q_l\,(sech\,\xi)^l,\; H = \frac{1}{4\,\mu^2\,\eta_0^{1/2}\,(4\,\eta_4^2+\eta_3^2)^{3/2}} Z_m\,(sech\,\xi)^m, \quad (22)$$

where η_i, $i = 0,\ldots,4$, Z_m, $m = 1,\ldots,6$, and Q_l, $l = 1,\ldots,7$ are polynomial and trigonometric functions of x and t. Explicit form of them can be found in [24].

2.2. **The parameterized form of the four parameter family of mKdV surfaces.** In this section, we find position vector, $\overrightarrow{y} = (y_1(x,t), y_2(x,t), y_3(x,t))$, of mKdV surfaces by using soliton solution of mKdV equation and solution of the Lax equations for mKdV. Consider the one soliton solution of the mKdV equation

$$u = k_1\,sech\,\xi,\; \xi = \frac{k_1}{8}\left(k_1^2\,t + 4x\right), \quad (23)$$

where $\alpha = k_1^2/4$. The Lax pairs U and V are given by (16) and (17), respectively. By using these and solution u of mKdV equation, given by (23), we solve the Lax equations (5). By inserting Φ into the following equation

$$F = \mu\,\Phi^{-1}\frac{\partial\Phi}{\partial k_1}, \quad (24)$$

solving the resultant equation, and writing F in the form $F = -i(\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3)$, we obtain a four parameter (k_1, μ, A_1, B_1) family of surfaces parameterized by

$$y_1 = -\frac{1}{(e^{2\xi}+1)^2}\Big[R_{10}\,e^{\xi}\Big(\eta_{10}\sin G_2 + \eta_{11}\,\cos G_2\Big) + R_{12} + R_{13}\Big(\eta_6\left[e^{4\xi}+1\right] + \eta_7\,e^{2\xi}\Big)\Big], \quad (25)$$

$$y_2 = \frac{1}{(e^{2\xi}+1)^2} R_9\,e^{\xi}\Big(\eta_{11}\sin G_2 - \eta_{10}\,\cos G_2\Big), \quad (26)$$

$$y_3 = -\frac{1}{(e^{2\xi}+1)^2}\Big[R_{11}\,e^{\xi}\Big(\eta_{10}\sin G_2 + \eta_{11}\,\cos G_2\Big) - 2\,R_{10} + \frac{R_8}{\pi}\Big(\eta_{12}\left[e^{4\xi}+1\right] + \eta_{13}\,e^{2\xi}\Big)\Big], \quad (27)$$

where G_2 and η_i, $i = 5,\ldots,13$ are polynomial functions of x and t, and R_j, $j = 8,\ldots 13$ are constants in terms of μ, k_1, e, π, A_1, and B_1. Explicit form of them can be found in [24]. Thus the position vector $\overrightarrow{y} = (y_1(x,t), y_2(x,t), y_3(x,t))$ of the surface is given by (25)-(27).

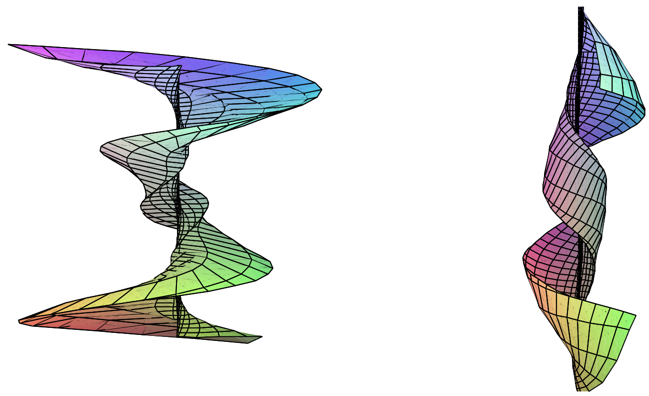
2.2.1. *The Analysis of the Four Parameter Family of Surfaces.*

Example 1. Taking $k_1 = -1/2, B_1 = 0$, and $\mu = 1$ in (25)-(27), we get the surface given by Fig. 1(a).

As ξ tends to $\pm\infty$, y_1 and y_2 approach zero, y_3 approaches $\mp\infty$. This can also be seen in Fig. 1(a). For small values of x and t, the surface has a twisted shape around a line.

Example 2. Taking $k_1 = 1/2, B_1 = 0$, and $\mu = 1$ in (25)-(27), we get the surface given by Fig. 1(b).

As ξ tends to $\pm\infty$, y_1 and y_2 approach zero, y_3 approaches $\mp\infty$.



(a) $(x,t) \in [-95,95] \times [-95,95]$ (b) $(x,t) \in [-60,60] \times [-60,60]$

FIGURE 1. mKdV surfaces.

3. **2-Surfaces in M_3.** In this section, we give a connection between integrable equation, HD equation, and surfaces in M_3. For the KdV case, this connection has been studied in [11]. In this chapter Lie group G is $SL(2,\mathbb{R})$ and the corresponding Lie algebra $\mathfrak{g}$ is $\mathfrak{sl}(2,\mathbb{R})$ with basis e_j, $j=1,2,3$, given by

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\; e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\; e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{28}$$

Define an inner product on $\mathfrak{sl}(2,\mathbb{R})$ as $\langle X,Y\rangle = (1/2)\,\mathrm{trace}(XY)$, for $X,Y \in \mathfrak{sl}(2,\mathbb{R})$.

3.1. **HD Surfaces from Spectral Deformations.** Let $u(x,t)$ satisfy the HD equation

$$u_t = -u^3\, u_{3x}. \tag{29}$$

Assuming the travelling wave ansatz $u_t - \alpha\, u_x = 0$ in the previous equation, we get

$$u_{2x} = \frac{\alpha}{2}\frac{1}{u} - C_1, \tag{30}$$

where α and C_1 are arbitrary constants. Equation (29) can be obtained from $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V where

$$U = \begin{pmatrix} 0 & 1 \\ \dfrac{\lambda^2}{u^2} & 0 \end{pmatrix},\; V = 2\lambda^2 \begin{pmatrix} u_x & -2u \\ u_{2x} - \dfrac{2\lambda^2}{u} & -u_x \end{pmatrix}, \tag{31}$$

were λ is a spectral constant. The Lax equations are given by (5), where the integrability of these equations are guaranteed by the HD equation or the zero curvature condition (4) for given U and V as (31). The following proposition gives HD surfaces from spectral deformations.

Proposition 3. *Let u satisfy the equation (29). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V of the HD equation are given by (31). $\mathfrak{sl}(2,\mathbb{R})$ valued matrices A and B are*

$$A = 2\mu\lambda \begin{pmatrix} 0 & 0 \\ \dfrac{1}{u^2} & 0 \end{pmatrix},\; B = 4\mu\lambda \begin{pmatrix} u_x & -2u \\ u_{2x} - \dfrac{4\lambda}{u} & -u_x \end{pmatrix} \tag{32}$$

where $A=\mu\,\partial U/\partial\lambda$, $B=\mu\,\partial V/\partial\lambda$, μ *is a constant and* λ *is a spectral parameter. Then the surface* S, *generated by* U,V,A *and* B, *has the following first and second fundamental forms* ($j,k=1,2$)

$$(ds_I)^2 \equiv g_{jk}\,dx^j\,dx^k - 16\,\mu^2\,\lambda^2\Big(\frac{1}{u}dx\,dt + [u_x^2 - 2\,u\,u_{2x} + 8\,\lambda^2]dt^2\Big), \quad (33)$$

$$(ds_{II})^2 \equiv h_{jk}\,dx^j\,dx^k = -\frac{2\,\mu\,\lambda}{u^2}\Big(dx^2 - 8\,\lambda^2\,u\,dx\,dt + 2\,u^2\left[2\,u^2\,u_x\,u_{3x} + u^3\,u_{4x} + 8\,\lambda^4\right]dt^2\Big), \quad (34)$$

and Gaussian and mean curvatures, where $x^1=x$ *and* $x^2=t$, *are*

$$K = -\frac{u^2}{8\,\mu^2\,\lambda^2}\left(2\,u_x\,u_{3x} + u\,u_{4x}\right),\ H = \frac{1}{4\,\mu\,\lambda}\left(u_x^2 - 2\,u\,u_{2x} + 4\,\lambda^2\right). \quad (35)$$

The following proposition gives Willmore-like HD surfaces.

Proposition 4. *Let* u *satisfy* $u_x^2=-\alpha/u-2\,C_1\,u+2\,C_2$. *Then the surface* S, *defined in Proposition 3, is a Willmore-like surface, i.e. the Gaussian and mean curvatures satisfy the equation (12), where* $a=-2$, $b=6$, $C_1=16\,\lambda^4/\alpha$, $C_2=-6\,\lambda^2$, *and* λ *is an arbitrary constant.*

In order to study the HD surfaces arising from variational principle, it is enough to know fundamental forms and curvatures for such surfaces. The following proposition gives a class of the HD surfaces that solve the equation (3).

Proposition 5. *Let* u *satisfy* $u_x^2=-\alpha/u-2\,C_1\,u+2\,C_2$. *Then there are HD surfaces, defined in Proposition 3, that satisfy the generalized shape equation (3) when* E *is a polynomial function of* H *and* K.

We have several examples:

Example 3.

Let $\deg(E)=N$, then

i) for $N=3: E=a_1\,H^3+a_2\,H^2+a_3\,H+a_4+a_5\,K+a_6\,K\,H$,

$$a_1=-\frac{11\,\mu\,a_2}{30\,\lambda}, a_3=-\frac{4\,\lambda\,a_2}{15\,\mu}, a_6=\frac{14\,\mu\,a_2}{15\,\lambda},\ a_4=0, C_1=p=0, C_2=2\,\lambda$$

where $\lambda\neq 0$, μ, and a_5 are arbitrary constants.

ii) for $N=4:\ E=a_1\,H^4+a_2\,H^3+a_3\,H^2+a_4\,H+a_5+a_6\,K+a_7\,K\,H+a_8\,K^2+a_9\,K\,H^2$,

$$a_1=-\frac{1}{64}\left(15\,a_8+34\,a_9\right),\ a_2=\frac{1}{480\,\mu\,\lambda}\left(\lambda^2\left[358\,a_9-7\,a_8\right]-176\,\mu^2\,a_3\right),$$

$$a_4=\frac{4\,\lambda}{15\,\mu^3}\left(\lambda^2\left[13\,a_8+8\,a_9\right]-\mu^2\,a_3\right),\ a_5=-\frac{3\,\lambda^4}{4\,\mu^4}\left(3\,a_8+2\,a_9\right),$$

$$a_7=\frac{1}{120\,\mu\,\lambda}\left(\lambda^2\left[359\,a_8+154\,a_9\right]+112\,\mu^2\,a_3\right),\ C_1=p=0, C_2=2\,\lambda,$$

where $\lambda\neq 0$, $\mu\neq 0$, and a_6 are arbitrary constants.

For general $N\geq 3$, from the above examples, the polynomial function E takes the form

$$E=\sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor (N-n)/2\rfloor} a_{nl}K^l,$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

3.2. **The Parameterized Form of the HD Surfaces.** In order to find the position vector $\overrightarrow{y}=(y_1(x,t),y_2(x,t),y_3(x,t))$, of the HD surfaces for a given solution of the HD equation and the corresponding Lax pairs, we use the similar technique that we use in section 2. Let $u=-(\alpha/2)\,18^{1/3}\,\xi^{2/3}$, $\xi=t+x/\alpha$, be a solution of the HD equation, where $\alpha\neq 0$ is a constant. We solved the Lax equations ($\Phi_x=U\Phi$ and $\Phi_t=V\Phi$) for given U, V and a solution u of the HD equation (29). By using solutions of the Lax equations, Φ, we solve the following equation

$$F=\mu\,\Phi^{-1}\,\frac{\partial\Phi}{\partial\lambda}, \tag{36}$$

and we get the family of the surfaces which are parameterized by

$$y_1=\zeta_4\Big(\zeta_5\,B_1\,B_2+\zeta_6\,A_1\,A_2+\frac{1}{2}\,\zeta_7\,(A_1\,B_2+A_2\,B_1)\Big), \tag{37}$$

$$y_2=\frac{\zeta_4}{2}\Big(\zeta_5(B_2^2-B_1^2)+\zeta_6\,(A_2^2-A_1^2)+\zeta_7\,(A_2\,B_2-A_1\,B_1)\Big), \tag{38}$$

$$y_3=\frac{\zeta_4}{2}\Big(\zeta_5(B_2^2+B_1^2)+\zeta_6\,(A_2^2+A_1^2)+\zeta_7\,(A_2\,B_2+A_1\,B_1)\Big), \tag{39}$$

where ζ_i, $i=4,\ldots,7$ are functions of x and t. Explicit form of them can be found in [24].

Proposition 6. *The HD surface, given by (37)-(39), is a quadratic Weingarten surface, i.e.*

$$3\,\mu^2\,H^2-6\,\mu\,\lambda\,H-4\,\mu^2\,K+3\,\lambda^2=0. \tag{40}$$

4. **Conclusion.** We constructed 2-surfaces in $\mathbb{R}^3$ and in M_3 by using the soliton surface technique. In $\mathbb{R}^3$, we found the mKdV surfaces using deformation of parameters of solution for mKdV equation. In M_3, we constructed the HD surfaces using spectral deformation. We found new HD surfaces that solve the generalized shape equation. Furthermore, we determined the parameterized form of the mKdV and HD surfaces.

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