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ASYMPTOTICAL DYNAMICS OF THE MODIFIED SCHNACKENBERG EQUATIONS

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Abstract. The existence of a global attractor in the L^2 product phase space for the solution semiflow of the modified Schnackenberg equations with the Dirichlet boundary condition on a bounded domain of space dimension $n \leq 3$ is proved. This reaction-diffusion system features two pairs of oppositely-signed nonlinear terms so that the dissipative sign-condition is not satisfied. The proof features two types of rescaling and grouping estimation in showing the absorbing property and the uniform smallness in proving the asymptotical compactness by the approach of a new decomposition.

1. **Introduction.** The system of Schnackenberg equations originally formulated in [23] serves as a simplified model of some trimolecular autocatalytic biochemical or chemical reactions with diffusion. Applications of this model of equations include pattern formations in embryogenesis and skin analysis [3, 25]. Simulations and mathematical analysis have found that Gray–Scott equations [11, 16, 18, 21, 27] and Schnackenberg equations [28] exhibit some interesting patterns, chaos, and structures such as self-oscillations and self-replicating spikes and stripes for space dimensions one and two.

Let $\Omega \subset \mathbb{R}^n$ $(n \leq 3)$ be a bounded domain that has a locally Lipschitz continuous boundary and lies locally on one side of its boundary. In this paper, we shall study the asymptotical dynamics of the following modified Schnackenberg equations (MSE) on Ω,

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \rho - au + u^2 v - Fu^3, \quad t > 0,\ x \in \Omega, \tag{1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + b - u^2 v + Fu^3, \quad t > 0,\ x \in \Omega, \tag{2}$$

where the parameters d_1, d_2, ρ, a, b and F are positive constants, with the homogeneous Dirichlet (non-slip) boundary conditions

$$u(t,x) = 0,\ v(t,x) = 0, \quad t > 0,\ x \in \partial\Omega, \tag{3}$$

and initial conditions

$$u(0,x) = u_0(x),\ v(0,x) = v_0(x), \quad x \in \Omega. \tag{4}$$

We do not assume initial data u_0 and v_0, nor $u(t,x)$ and $v(t,x)$, are nonnegative.

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A key autocatalytic reaction scheme underlying the MSE is shown by

$$\mathrm{S} + 2\mathrm{P} \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} 3\mathrm{P},$$

where S and P are two reactants. When the reaction is non-reversible, $k_{-1} = 0$, then by the law of mass action we have $F = 0$ so that the model is the diffusive Schnackenberg equations [23]. The modified Schnackenberg equations with additional terms $\pm Fu^3$ can also be called *reversible* Schnackenberg equations, which is not purely a mathematical extension, but a more realistic and precise model for such a reversible autocatalytic reaction-diffusion system.

Earlier results [8,22] on the global existence and uniform boundedness of solutions for Schnackenberg equations and similar reaction-diffusion systems were proved under the condition that initial data or solutions are nonnegative. As far as the author knows, no results are reported concerning global dynamics or temporal-spatial patterns of the modified Schnachenberg equations on 3D (three-dimensional) domains either in numerical simulations or by analytic methods. Recently in [30, 31] it is proved that for the 3D Brusslator equations and Gray-Scott equations there exists a global attractor of finite Hausdorf dimension and fractal dimension.

Schnackenberg equations have been used as an exemplar reaction-diffusion model for biochemical kinetics to study biological and cellular pattern formations. Here are some recently published, interesting results. In [2, 7], the numerical simulation and analysis of solution behavior on the 1D growing domain showed that mode-doubling and peak splitting generate a sequence of Turing patterns whose instability and final selection depend on rapid or slow domain growing rates and on the incorporated gene expression time delay. In [12,13], two novel computational schemes of moving-grid finite element method were developed to investigate the effect of continuously deforming 2D regular and irregular domains on Turing pattern selection and pattern transitions by numerically solving the Schnackenberg model equations and some other models.

In [10], the dynamical behavior of multi-spot solutions in a 2D domain for a limiting form of the singularly perturbed Schnackenberg equations is analyzed through asymptotic expansions and derivation of a number of ordinary differential algebraic systems (DAE). A combination of numerical and analytical techniques is then used to determine the stability thresholds of the quasi-equilibrium patterns and to make explicit predictions for the onset (strengths and locations) of spot-splitting events.

Concerning the global dynamics of other reaction-diffusion systems, we mention the following rigorous results. For singularly perturbed Hodgkin–Huxley systems, the existence of a global attractors together with finite dimensionality and upper semicontinuity has been shown in [5]. The existence of a global attractor and an exponential attractor has been proved for the nonnegative solutions of chemotaxis models in [29] and [17], respectively. The existence of a global attractor and its upper semicontinuity for the nonnegative solutions of Gierer–Meinhardt equations as an activator-inhibitor model with the nonlinear terms u^p/v^q have been proved in [15] based on the estimates shown in [14] and through reduction to a shadow system and construction of a Lyapunov functional.

For 1D autocatalytic reaction-diffusion systems of cubic-quadratic mixed order, the existence and stability of traveling waves were proved in [6]. Recently in [4, 20], sharp estimates on minimum traveling wave speed of a particular Schnackenberg system on a 1D domain were obtained. For that particular Schnackenberg system on a multi-dimensional bounded domain, under certain conditions on the parameters, the precise asymptotic dynamics was shown in [9] in terms of the convergence to one of the at most three steady states in the long run by studying the exact multiplicity of nonnegative solutions to the nonlinear elliptic problem. In [19, 33] the renormalization group method was used to study global dynamics of Gray-Scott reaction-diffusion systems.

For most reaction-diffusion systems consisting of two or more equations arising from the scenarios of autocatalytic chemical reactions or biochemical activator-inhibitor reactions, such as the Schnackenberg equations and modified Schnackenberg equations here, the asymptotically dissipative sign condition in vector version

$$\lim_{|s|\to\infty} f(s)\cdot s \le 0$$

is usually or inherently not satisfied by the opposite-signed and coupled nonlinear terms, see (7) later.

Beside this common difficulty, there are more serious technical difficulties in analyzing the modified Schnackenberg equations, which do not occur in treating Brusslater equations [30], Gray–Scott equations [31], and Selkov equations [32]. The challenge is that due to the $+Fu^3$ term in the v-equation we can no longer make a dissipative *a priori* estimate on the v-component by using the v-equation separately and then use the sum $y(t,x) = u(t,x) + v(t,x)$ to deal with the u-component in the part of proving absorbing property and in the part of proving asymptotical compactness of the solution semiflow as we did in [30–32]. The novel feature in this paper is to overcome this obstacle and make the *a priori* estimates by a new method of *rescaling and grouping estimation*, as shown in Sections 2 and 3.

Define the product Hilbert spaces

$$H = L^2(\Omega)\times L^2(\Omega), \quad E = H_0^1(\Omega)\times H_0^1(\Omega). \tag{5}$$

The norm and inner-product of H or $L^2(\Omega)$ will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. We use $|\cdot|$ to denote an absolute value or a vector norm in a Euclidean space. One can check that [24] the densely defined, sectorial, linear operator

$$A = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix} : D(A)\to H, \tag{6}$$

where

$$D(A) = \{(\varphi,\psi)\in H^2(\Omega)\times H^2(\Omega) : \varphi = 0,\ \psi = 0 \text{ on } \partial\Omega\},$$

is the generator of an analytic C_0-semigroup $\{e^{At}\}_{t\ge 0}$, on the Hilbert space H. By the fact that the injection mapping $H^1(\Omega)\hookrightarrow L^6(\Omega)$ is a continuous embedding for $n\le 3$ and since

$$\|u^2 v\| \le \|u\|_{L^6}^2\|v\|_{L^6}, \quad \text{for } u,v\in L^6(\Omega),$$

one can check that the nonlinear mapping

$$f(u,v) = \begin{pmatrix} \rho - au + u^2v - Fu^3 \\ b - u^2 v + Fu^3 \end{pmatrix} : E \longrightarrow H \tag{7}$$

is a locally Lipschitz continuous mapping defined on E. Then the initial-boundary value problem (1)–(4) is formulated into an initial value problem of the evolutionary equation:

$$\frac{dw}{dt} = Aw + f(w), \quad t>0, \tag{8}$$

$$w(0) = w_0 = (u_0, v_0) \in H,$$

where $w(t) = (u(t,\cdot), v(t,\cdot))$, $t \geq 0$.

By conducting *a priori* estimates on the Galerkin approximate solutions of the IVP (8) and the weak convergence, we can prove the local existence and uniqueness of the weak solution $w(t)$ of (8) in the sense of J. M. Ball specified in [1], which is shown to be a local mild solution [1] and further turns out to be a local strong solution [24, Theorem 46.2]. Moreover, by taking the H-inner-product of (8) with this strong solution $w(t)$ itself and using Gronwall inequality, we can prove the continuous dependence of the solutions on the initial data w_0. This together with the boundedness shown in (13) later in Section 2 shows that the solution operator generates a dynamical system (semiflow) on H. By the theory of evolutionary equations [24], the solution w of (8) satisfies the property

$$w \in C([0, T_{\max}); H) \cap C^1((0, T_{\max}); H) \cap L^2([0, T_{\max}); E). \tag{9}$$

We refer to [24, 26] for the related concepts of and facts in the theory of infinite dimensional dynamical systems. Here is the main result of this work.

Theorem 1. *For any positive parameters d_1, d_2, ρ, a, b and F, there exists a global attractor $\mathscr{A}$ in H for the solution semiflow $\{S(t)\}_{t\geq 0}$ generated by the modified Schnackenberg evolutionary equation (8).*

2. **Bounded Absorbing Set.**

First we show the existence of a bounded absorbing set in H.

Lemma 1. *For any $w_0 = (u_0, v_0) \in H$, the unique strong solution $w(t) = (u(t), v(t))$ of the modified Schnackenberg evolutionary equation (8) exists for $t \in [0, \infty)$. Moreover, there exists a bounded absorbing set B_0 in H for the solution semiflow $\{S(t)\}_{t\geq 0}$,*

$$B_0 = \{w \in H : \|w\|^2 \leq K_0\}, \tag{10}$$

where K_0 is a uniform constant.

Proof. Taking the inner-products $\langle (1), Fu(t)\rangle$ and $\langle (2), v(t)\rangle$ and summing up the resulting equalities, by (3) and the Poincaré inequality $\|\nabla \varphi\|^2 \geq \gamma \|\varphi\|^2$ in $H_0^1(\Omega)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[F\|u\|^2 + \|v\|^2\right] + d_1 F \|\nabla u\|^2 + d_2 \|\nabla v\|^2 \\
=& \int_\Omega (\rho F u + bv)\, dx - aF\|u\|^2 - \int_\Omega \left(Fu^2 - uv\right)^2 dx \\
\leq& \rho F |\Omega|^{1/2} \|u\| + b|\Omega|^{1/2}\|v\| - aF\|u\|^2 \\
\leq& \frac{\rho^2 F|\Omega|}{2a} + \frac{b^2|\Omega|}{2d_2\gamma} + \frac{1}{2} d_2\gamma \|v\|^2 - \frac{1}{2}\, aF\|u\|^2,
\end{aligned} \tag{11}$$

so that

$$\frac{d}{dt}\left[F\|u\|^2 + \|v\|^2\right] + \left(d_1 F\|\nabla u\|^2 + d_2\|\nabla v\|^2\right) + aF\|u\|^2 \leq \frac{\rho^2 F|\Omega|}{a} + \frac{b^2|\Omega|}{d_2\gamma}. \tag{12}$$

Let $d_0 = \min\{d_1, d_2\}$. Then there is a uniform constant $C_0 > 0$ such that

$$F\|u(t)\|^2 + \|v(t)\|^2 \leq e^{-d_0\gamma t}\left(F\|u_0\|^2 + \|v_0\|^2\right) + C_0|\Omega|, \quad \text{for } \ t \in [0, T_{max}). \tag{13}$$

This inequality shows that the strong solution $w(t) = (u(t), v(t))$ of (8) will never blow up at any finite time, so that $T_{max} = \infty$ for any initial data. The family of global strong solutions $\{w(t; w_0), t \geq 0, w_0 \in H\}$ defines a semiflow on H,

$$S(t) : w_0 \mapsto w(t; w_0), \quad w_0 \in H, \ t \geq 0. \tag{14}$$

Then (13) also proves that a bounded absorbing set B_0 shown in (10) exists, where the uniform constant K_0 can be chosen as $K_0 = (F^{-1} + 1)(C_0|\Omega| + 1)$.

Moreover, since (12) also implies

$$\int_0^t e^{d_2\gamma s} \frac{d}{ds}(F\|u(s)\|^2 + \|v(s)\|^2)\, ds + \int_0^t e^{d_2\gamma s} d_1 F \|\nabla u(s)\|^2 ds \\ \leq \frac{1}{d_2\gamma} e^{d_2\gamma t} \left(\frac{\rho^2 F}{a} + \frac{b^2}{d_2\gamma} \right) |\Omega|,$$

we have

$$\begin{aligned} \int_0^t e^{d_2\gamma s} \|\nabla u(s)\|^2 ds &\leq \frac{1}{d_1 F}(F\|u_0\|^2 + \|v_0\|^2) + \frac{e^{d_2\gamma t}}{d_1 d_2 \gamma F} \left(\frac{\rho^2 F}{a} + \frac{b^2}{d_2\gamma} \right) |\Omega| \\ &+ \frac{d_2\gamma}{d_1 F} \int_0^t e^{d_2\gamma s} \left[e^{-d_0\gamma s}(F\|u_0\|^2 + \|v_0\|^2) + C_0|\Omega| \right] ds, \end{aligned} \tag{15}$$

which is for later use. The proof is completed. □

3. **Decomposition for Asymptotical Compactness.** Next we show that the solution semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact on H. Due to the two pairs of opposite signed nonlinear terms in (1)–(2), it is challenging for this attempt. In this section, we first show a generic result that provides a new decomposition approach for proving that the semiflow generated by a system of two coupled reaction-diffusion equations to be asymptotically compact.

We shall use the notation

$$\Omega_M^u = \Omega(|u(t)| \geq M) = \{x \in \Omega : |u(t,x)| \geq M\},$$
$$\Omega_{u,M} = \Omega(|u(t)| < M) = \{x \in \Omega : |u(t,x)| < M\}.$$

Lebesgue measure of a subset Ω_s of Ω is denoted by $m(\Omega_s)$ or $|\Omega_s|$. The Kuratowski measure of noncompactness for a bounded set B in a Banach space X is defined by

$$\kappa(B) \stackrel{\text{def}}{=} \inf \{\delta : B \text{ has a finite cover by open sets in } X \text{ of diameters} < \delta\}.$$

If B is an unbounded set, then define $\kappa(B) = \infty$. The basic properties of the Kuratowski measure can be found in [24, Lemma 22.2].

Theorem 2. *For the Schnackenberg semiflow* (14)*, there exists a global attractor* $\mathscr{A}$ *in* H *if and only if the following two conditions are satisfied*:

(i) *There exists a bounded absorbing set* B_0 *in* H *for this semiflow.*

(ii) *For any* $\varepsilon > 0$*, there are positive constants* $M = M(\varepsilon)$*,* $T = T(\varepsilon)$ *and a uniform constant* $C > 0$ *such that*

$$\int_{\Omega(|u(t)|\geq M)} |(S(t)w_0)(x)|^2\, dx < C\varepsilon, \quad \textit{for any } t \geq T, \ w_0 \in B_0, \tag{16}$$

and

$$\kappa\left((S(t)B_0)_{\Omega(|u(t)|<M)}\right) \longrightarrow 0, \quad \textit{as } t \to \infty, \tag{17}$$

where

$$(S(t)B_0)_{\Omega(|u(t)|<M)} \stackrel{def}{=} \{(S(t)w_0)(\cdot)\theta_M(\cdot\,; t, w_0) : \textit{for } w_0 \in B_0\}, \tag{18}$$

in which $\theta_M(x;t,w_0)$ *is the characteristic function of the subset* $\Omega(|u(t)| < M)$, *and* $u(t)$ *is the* u*-component of the solution of the modified Schnackenberg equations* (1)–(2).

The proof of Theorem 2 is similar to the corresponding result [30, 31, Theorem 2] and is omitted. But it is different from the approach in [30, 31]. Here the decomposition is in terms of the truncation of the u-component instead of the v-component, that makes a nontrivial difference in view of the $\pm Fu^3$ terms in the modified Schnackenberg equations but not in the Brusselator equations [30] nor in the Gray–Scott equations [31]. More substantially, the process of checking the first condition (16) and the second condition (17) in part (ii) of Theorem 2 is quite different from what was done for the similar but non-modified systems. In this paper the key is a rescaling technique followed by the grouped *a priori* estimates.

Let $\tilde{u}(t,x) = u(t,x)$ and $\tilde{v}(t,x) = v(t,x)/F$. Then the original system (1)–(2) becomes

$$\frac{\partial \tilde{u}}{\partial t} = d_1 \Delta \tilde{u} + \rho - a\tilde{u} + F\tilde{u}^2\tilde{v} - F\tilde{u}^3, \quad t > 0,\ x \in \Omega, \tag{19}$$

$$\frac{\partial \tilde{v}}{\partial t} = d_2 \Delta \tilde{v} + \frac{b}{F} - \tilde{u}^2\tilde{v} + \tilde{u}^3, \quad t > 0,\ x \in \Omega, \tag{20}$$

Lemma 2. *For any* $\varepsilon > 0$, *there exists positive constants* $M_1 = M_1(\varepsilon)$ *and* $T_1 = T_1(\varepsilon)$ *such that the* u*-component* $u(t) = u(t,x,w_0)$ *of the solution of the MSE* (1)–(2) *satisfies*

$$\int_{\Omega^u_{M_1}} |u(t)|^2 dx < C_1 \varepsilon, \quad \text{for } t > T_1,\ w_0 = (u_0, v_0) \in B_0, \tag{21}$$

where B_0 *is the absorbing set shown in Lemma* 1, *and* $C_1 = C_1(K_0)$ *is a uniform constant.*

Proof. Since B_0 attracts itself, there is a uniform time $T_0 = T_0(B_0) \geq 0$ such that $\{S(t)w_0 : t \geq T_0, w_0 \in B_0\} \subset B_0$. Thus $\|u(t)\|^2 \leq \|S(t)w_0\|^2 \leq K_0$. For any given $\varepsilon > 0$, there is a sufficiently large $M = M(\varepsilon) > 0$, such that

$$m(\Omega(|u(t)| \geq M)) \leq \frac{K_0}{M^2} < \varepsilon, \quad \text{for } t \geq T_0,\ w_0 \in B_0. \tag{22}$$

Taking the inner-product $\langle (19), (u(t) - M)_+ \rangle_{\Omega^u_M}$, where M is given in (22) and

$$(\varphi - M)_+ = \begin{cases} \varphi(x) - M, & \text{if } \varphi(x) \geq M, \\ 0, & \text{if } \varphi(x) < M, \end{cases}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\tilde{u} - M)_+\|^2_{\Omega^u_M} + d_1 \|\nabla(\tilde{u} - M)_+\|^2_{\Omega^u_M} \\
\leq &\int_{\Omega(|u(t)| \geq M)} \rho(\tilde{u} - M)_+ dx - a\|(\tilde{u} - M)_+\|^2_{\Omega^u_M} \\
&+ \int_{\Omega(|u(t)| \geq M)} F\tilde{u}^2(\tilde{v} - M)_+(\tilde{u} - M)_+ dx + MF \int_{\Omega(|u(t)| \geq M)} \tilde{u}^2(\tilde{u} - M)_+ dx \\
&- F \int_{\Omega(|u(t)| \geq M)} \tilde{u}^2(\tilde{u} - M)^2_+ dx - MF \int_{\Omega(|u(t)| \geq M)} \tilde{u}^2(\tilde{u} - M)_+ dx.
\end{aligned}$$

Taking the inner-product $\langle(20), F(\tilde{v}(t)-M)_+\rangle_{\Omega_M^u}$, where M is the same as above, we get

$$\frac{F}{2}\frac{d}{dt}\|(\tilde{v}-M)_+\|^2_{\Omega_M^u} + Fd_2\|\nabla(\tilde{v}-M)_+\|^2_{\Omega_M^u} \le \int_{\Omega(|u(t)|\ge M)} b(\tilde{v}-M)_+dx$$
$$- \int_{\Omega(|u(t)|\ge M)} F\tilde{u}^2(\tilde{v}-M)^2_+dx - MF\int_{\Omega(|u(t)|\ge M)}\tilde{u}^2(\tilde{v}-M)_+dx$$
$$+ F\int_{\Omega(|u(t)|\ge M)}\tilde{u}^2(\tilde{u}-M)_+(\tilde{v}-M)_+dx + MF\int_{\Omega(|u(t)|\ge M)}\tilde{u}^2(\tilde{v}-M)_+dx.$$

Let $d_0=\min\{d_1,d_2\}$. Add up the above two inequalities (in which the two integrals with coefficient MF are cancelled out in each of them) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|(\tilde{u}-M)_+\|^2_{\Omega_M^u} + F\|(\tilde{v}-M)_+\|^2_{\Omega_M^u}\right) \\
&\quad + d_0\left(\|\nabla(\tilde{u}-M)_+\|^2_{\Omega_M^u} + F\|\nabla(\tilde{v}-M)_+\|^2_{\Omega_M^u}\right) \\
&\le \int_{\Omega(|u(t)|\ge M)}\rho(\tilde{u}-M))_+dx + \int_{\Omega(|u(t)|\ge M)} b(\tilde{v}-M)_+dx - a\|(\tilde{u}-M)_+\|^2_{\Omega_M^u} \\
&\quad - F\int_{\Omega(|u(t)|\ge M)}\left[\tilde{u}(\tilde{u}-M)_+ + \tilde{u}(\tilde{v}-M)_+\right]^2dx \\
&\le \int_{\Omega(|u(t)|\ge M)}\rho(\tilde{u}-M))_+dx + \int_{\Omega(|u(t)|\ge M)} b(\tilde{v}-M)_+dx - a\|(\tilde{u}-M)_+\|^2_{\Omega_M^u} \\
&\le \frac{1}{2}d_0F\gamma\|(\tilde{v}-M)_+\|^2_{\Omega_M^u} + \left(\frac{\rho^2}{2a}+\frac{b^2}{2d_0F\gamma}\right)|\Omega_M^u|.
\end{aligned} \tag{23}$$

Since $|\Omega_M^u|<\varepsilon$ by (22), it follows that

$$\begin{aligned}
&\frac{d}{dt}\left(\|(\tilde{u}-M)_+\|^2_{\Omega_M^u} + F\|(\tilde{v}-M)_+\|^2_{\Omega_M^u}\right) \\
&\quad + d_0\left(\|\nabla(\tilde{u}-M)_+\|^2_{\Omega_M^u} + F\|\nabla(\tilde{v}-M)_+\|^2_{\Omega_M^u}\right) \le C_2\,\varepsilon,
\end{aligned} \tag{24}$$

where

$$C_2 = \frac{\rho^2}{a} + \frac{b^2}{d_0F\gamma}$$

is a uniform positive constant independent of M. Then by the Poincaré inequality and the Gronwall inequality, we get

$$\begin{aligned}
&\|(\tilde{u}-M)_+\|^2_{\Omega_M^u} + F\|(\tilde{v}-M)_+\|^2_{\Omega_M^u} \\
&\qquad \le e^{-d_0\gamma t}\left(\|(\tilde{u}_0-M)_+\|^2_{\Omega_M^u} + F\|(\tilde{v}_0-M)_+\|^2_{\Omega_M^u}\right) + \frac{C_2\,\varepsilon}{d_0\gamma} \\
&\qquad \le (1+F)K_0e^{-d_0\gamma t} + \frac{C_2\,\varepsilon}{d_0\gamma}.
\end{aligned}$$

Hence, there exists a time $T_+(\varepsilon)$ such that for any $t>T_+$ and any $w_0\in B_0$, one has

$$\|(\tilde{u}-M)_+\|^2_{\Omega_M^u} \le \|(\tilde{u}-M)_+\|^2_{\Omega_M^u} + F\|(\tilde{v}-M)_+\|^2_{\Omega_M^u} < \frac{2C_2\,\varepsilon}{d_0\gamma}. \tag{25}$$

Similarly there exists a time $T_-(\varepsilon)$ such that for any $t>T_-$ and any $w_0\in B_0$, one has

$$\|(\tilde{u}+M)_-\|^2_{\Omega_M^u} \le \|(\tilde{u}+M)_-\|^2_{\Omega_M^u} + F\|(\tilde{v}+M)_-\|^2_{\Omega_M^u} < \frac{2C_2\,\varepsilon}{d_0\gamma}, \tag{26}$$

where

$$(\varphi + M)_- = \begin{cases} \varphi(x) + M, & \text{if } \varphi(x) \le -M, \\ 0, & \text{if } \varphi(x) > -M. \end{cases}$$

By (25) and (26), for any $t > T_1(\varepsilon) = \max\{T_+, T_-\}$ and any $w_0 \in B_0$ it holds that

$$\int_{\Omega(|u(t)| \ge M)} (|\tilde{u}(t)| - M)^2 dx < \frac{4C_2\, \varepsilon}{d_0 \gamma}. \tag{27}$$

Then

$$\begin{aligned} \int_{\Omega(|u(t)| \ge \ell M)} |\tilde{u}(t)|^2 dx &\le 2 \int_{\Omega(|u(t)| \ge M)} (|\tilde{u}(t)| - M)^2 dx + 2M^2 m(\Omega(|u(t)| \ge \ell M)) \\ &< \frac{8C_2\, \varepsilon}{d_0 \gamma} + \frac{2M^2 K_0}{\ell^2 M^2} < \frac{8C_2\, \varepsilon}{d_0 \gamma} + \varepsilon \end{aligned} \tag{28}$$

for a sufficiently large $\ell = \ell(\varepsilon)$. Finally, since the rescaling preserves $\tilde{u}(t) = u(t)$, (21) is proved with $C_1 = 1 + 8C_2/(d_0\gamma)$ and $M_1(\varepsilon) = \ell M$. ☐

Although $\|(\tilde{v} - M)_+\|^2_{\Omega^u_M}$ and $\|(\tilde{v} + M)_-\|^2_{\Omega^u_M}$ are involved in the grouped estimates in the proof of Lemma 2, we cannot draw a conclusion on $\tilde{v}(t,x) = v(t,x)/F$ similar to (28) and (21), because unlike $(\tilde{u} - M)_+$ on Ω^u_M,

$$\int_{\Omega(|u(t)| \ge M)} (\tilde{v} - M)^2_+ \, dx \quad \text{and} \quad \int_{\Omega(|u(t)| \ge M)} (\tilde{v} - M)^2 \, dx$$

may not be equal, and same for the integrals of $(\tilde{v} + M)^2_-$ and $(\tilde{v} + M)^2$ on Ω^u_M. Thus we need a different treatment for the v-component in order to prove the following lemma.

Lemma 3. *For any $\varepsilon > 0$, there exists positive constants $M_2 = M_2(\varepsilon)$ and $T_2 = T_2(\varepsilon)$ such that the v-component $v(t) = v(t,x,w_0)$ of the solution of the MSE (1)–(2) satisfies*

$$\int_{\Omega^u_{M_2}} |v(t)|^2 dx < C_3\, \varepsilon, \quad \text{for } t > T_2,\ w_0 = (u_0, v_0) \in B_0, \tag{29}$$

where B_0 is the absorbing set shown in Lemma 1, and $C_3 = C_3(K_0)$ is a uniform constant.

Proof. We work on the sum $y(t) = u(t,x,w_0) + v(t,x,w_0)$, where (u,v) is the solution of (1)–(2) with the initial data $w_0 \in B_0$. Indeed, $y(t)$ satisfies the equation

$$\frac{\partial y}{\partial t} = d_2 \Delta y + (d_1 - d_2)\Delta u + (\rho + b) - au. \tag{30}$$

Taking the inner-product $\langle (30), y(t) \rangle_{\Omega^u_M}$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{\Omega^u_M} + d_2 \|\nabla y(t)\|^2_{\Omega^u_M} &= -(d_1 - d_2) \int_{\Omega(|u(t)| \ge M)} \nabla u(t) \nabla y(t)\, dx \\ &\quad + (\rho + b) \int_{\Omega(|u(t)| \ge M)} y(t)\, dx - a \int_{\Omega(|u(t)| \ge M)} u(t) y(t)\, dx \\ &\le \frac{d_2}{2}\|\nabla y(t)\|^2_{\Omega^u_M} + \frac{3|d_1 - d_2|^2}{2d_2}\|\nabla u(t)\|^2_{\Omega^u_M} + \frac{3(\rho + b)^2}{2d_2\gamma}|\Omega^u_M| + \frac{3a^2}{2d_2\gamma}\|u(t)\|^2_{\Omega^u_M}. \end{aligned}$$

It follows that

$$\begin{aligned}&\frac{d}{dt}\|y(t)\|^2_{\Omega^u_M} + d_2\|\nabla y(t)\|^2_{\Omega^u_M}\\ \le&\frac{3|d_1-d_2|^2}{d_2}\|\nabla u(t)\|^2_{\Omega^u_M} + \frac{3(\rho+b)^2}{d_2\gamma}|\Omega^u_M| + \frac{3a^2}{d_2\gamma}\|u(t)\|^2_{\Omega^u_M}.\end{aligned} \tag{31}$$

By exponential multiplication and integration we get

$$\begin{aligned}&\frac{d}{dt}\left(e^{d_2\gamma t}\|y(t)\|^2_{\Omega^u_M}\right)\\ \le&\frac{3|d_1-d_2|^2}{d_2}e^{d_2\gamma t}\|\nabla u(t)\|^2_{\Omega^u_M} + \frac{3e^{d_2\gamma t}}{d_2\gamma}\left(a^2\|u(t)\|^2_{\Omega^u_M} + (\rho+b)^2|\Omega^u_M|\right)\end{aligned}$$

and by using (13) we have

$$\begin{aligned}\|y(t)\|^2_{\Omega^u_M} \le\ & e^{-d_2\gamma t}\|u_0+v_0\|^2_{\Omega^u_M} + \frac{3|d_1-d_2|^2}{d_2}e^{-d_2\gamma t}\int_0^t e^{d_2\gamma s}\|\nabla u(s)\|^2_{\Omega^u_M}\,ds\\ &+\frac{3}{d_2\gamma}e^{-d_2\gamma t}\beta(t)a^2(\|u_0\|^2_{\Omega^u_M} + F^{-1}\|v_0\|^2_{\Omega^u_M}) + \frac{3}{d_2^2\gamma^2}\left(C_0F^{-1}a^2 + (\rho+b)^2\right)|\Omega^u_M|,\end{aligned} \tag{32}$$

where

$$\beta(t) = \int_0^t e^{(d_2-d_0)\gamma s}\,ds \le \begin{cases} t, & \text{if } d_2 = d_0;\\ \frac{e^{(d_2-d_0\gamma)t}}{(d_2-d_0)\gamma}, & \text{if } d_2 > d_0;\\ \frac{1}{(d_0-d_2)\gamma}, & \text{if } d_2 < d_0.\end{cases}$$

Note that $e^{-d_2\gamma t}\beta(t) \to 0$ as $t\to\infty$. For the integral term in (32), similar to (15) we have

$$\begin{aligned}\int_0^t e^{d_2\gamma s}\|\nabla u(s)\|^2_{\Omega^u_M}\,ds \le\ & \frac{1}{d_1d_2\gamma F}\,e^{d_2\gamma t}\left(\frac{\rho^2F}{a} + \frac{b^2}{d_2\gamma}\right)|\Omega^u_M|\\ &+\frac{1}{d_2F}(F\|u_0\|^2_{\Omega^u_M} + \|v_0\|^2_{\Omega^u_M})\\ &+\frac{d_2\gamma}{d_1F}\int_0^t e^{d_2\gamma s}\left[e^{-d_0\gamma s}\left(F\|u_0\|^2_{\Omega^u_M} + \|v_0\|^2_{\Omega^u_M}\right) + C_0|\Omega^u_M|\right]ds.\end{aligned} \tag{33}$$

Substitute (33) into (32) to get

$$\begin{aligned}\|y(t)\|^2_{\Omega^u_M} \le\ & e^{-d_2\gamma t}\left(\|u_0+v_0\|^2_{\Omega^u_M} + \frac{3}{d_2\gamma}\beta(t)a^2(\|u_0\|^2_{\Omega^u_M} + F^{-1}\|v_0\|^2_{\Omega^u_M})\right)\\ &+\frac{3|d_1-d_2|^2}{d_2}e^{-d_2\gamma t}\left(\frac{1}{d_2F} + \frac{d_2\gamma}{d_1F}\beta(t)\right)\left(F\|u_0\|^2_{\Omega^u_M} + \|v_0\|^2_{\Omega^u_M}\right)\\ &+\frac{3}{d_2^2\gamma^2}\left(C_0F^{-1}a^2 + (\rho+b)^2\right)|\Omega^u_M|\\ &+\frac{3|d_1-d_2|^2}{d_1d_2F}\left[\frac{1}{d_2\gamma}\left(\frac{\rho^2F}{a} + \frac{b^2}{d_2\gamma}\right) + C_0\right]|\Omega^u_M|.\end{aligned} \tag{34}$$

Therefore, there exists two uniform positive constant $C_4 = C_4(K_0)$ and $C_5 = C_5(C_0)$, where K_0 is determined by the absorbing set B_0 in (10), such that

$$\|y(t)\|^2_{\Omega^u_M} \le C_4\,e^{-d_2\gamma t} + C_5|\Omega^u_M|, \tag{35}$$

which implies that there exist sufficiently large $T_2(\varepsilon)$ and $M_2(\varepsilon)$ by (22), such that

$$\int_{\Omega_M^u} |y(t)|^2 dx < (C_4 + C_5)\,\varepsilon. \tag{36}$$

Finally, since $v(t) = y(t) - u(t)$, combining (21) and (36), the inequality (29) is valid with $C_3 = 2(C_1 + C_4 + C_5)$. Thus the lemma is proved. □

By Lemma 2 and Lemma 3, we have proved that the condition (16) in Theorem 2 is satisfied with

$$C = C_1 + C_3, \quad M = \max\{M_1(\varepsilon), M_2(\varepsilon)\}, \quad \text{and} \quad T = \max\{T_1(\varepsilon), T_2(\varepsilon)\},$$

by the solution semiflow of the modified Schnackenberg equations.

4. **The κ-Contraction Property.** In this section we shall check that the κ-contraction condition (17) in Theorem 2 is also satisfied by the solution semiflow of the modified Schnackenberg equations.

Lemma 4. *For any given $M > 0$, the solution semiflow of the MSE (1)–(2) satisfies the κ-contracting property (17):*

$$\kappa\left((S(t)B_0)_{\Omega(|u(t)|<M)}\right) \longrightarrow 0, \quad \text{as } t \to \infty,$$

where B_0 is the absorbing set in (10) and $(S(t)B_0)_{\Omega(|u(t)|<M)}$ is defined in (18).

Proof. Taking the inner-product $\langle (1), -\Delta u\rangle_{\Omega_{u,M}}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{\Omega_{u,M}} + d_1\|\Delta u(t)\|^2_{\Omega_{u,M}} &= -\int_{\Omega_{u,M}} \rho\Delta u\,dx - a\|\nabla u(t)\|^2_{\Omega_{u,M}} \\ &\quad - \int_{\Omega_{u,M}} u^2 v\Delta u\,dx + F\int_{\Omega_{u,M}} u^3\Delta u\,dx \\ &\le (FM^3+\rho)\int_{\Omega_{u,M}} |\Delta u(t)|\,dx - a\|\nabla u(t)\|^2_{\Omega_{u,M}} + M^2\int_{\Omega_{u,M}} |v(t)||\Delta u(t)|\,dx.\end{aligned}$$

By Cauchy-Schwarz inequality it follows that

$$\begin{aligned}\frac{d}{dt}\|\nabla u(t)\|^2_{\Omega_{u,M}} + a\|\nabla u(t)\|^2_{\Omega_{u,M}} &\le \frac{(FM^3+\rho)^2}{d_1}|\Omega_{u,M}| + \frac{M^4}{d_1}\|v(t)\|^2_{\Omega_{u,M}} \\ &\le \frac{(FM^3+\rho)^2}{d_1}|\Omega| + \frac{M^4K_0}{d_1},\end{aligned} \tag{37}$$

for $w_0 \in B_0$ and $t > T_0$, where $T_0 = T_0(B_0)$ is given in the beginning of the proof of Lemma 2.

From (12) we can get that, for $t \ge 0$ and any $w_0 = (u_0, v_0) \in H$,

$$\begin{aligned}\int_t^{t+1} \|\nabla u(t)\|^2_{\Omega_{u,M}}\,ds &\le \int_t^{t+1}\|\nabla u(t)\|^2\,ds \\ &\le \frac{1}{d_1 F}\left[(F\|u(t)\|^2 + \|v(t)\|^2) + \left(\frac{\rho^2F}{a} + \frac{b^2}{d_2\gamma}\right)|\Omega|\right].\end{aligned}$$

Here for any initial data $(u_0, v_0) \in B_0$, it holds that

$$\int_t^{t+1}\|\nabla u(t)\|^2_{\Omega_{u,M}}\,ds \le \frac{1}{d_1F}\left[(F+1)K_0 + \left(\frac{\rho^2F}{a} + \frac{b^2}{d_2\gamma}\right)|\Omega|\right], \quad \text{for } t > T_0. \tag{38}$$

Therefore, by the uniform Gronwall inequality [24,26], from (37) we can assert that

$$\|\nabla u(t)\|^2_{\Omega_{u,M}} \le e^a(C_6 + C_7), \quad \text{for } t > T_0 + 1, \tag{39}$$

where

$$C_6 = \frac{1}{d_1 F}\left[(F+1)K_0 + \left(\frac{\rho^2 F}{a} + \frac{b^2}{d_2\gamma}\right)|\Omega|\right]$$

and

$$C_7 = \frac{1}{d_1}\left[(FM^3+\rho)^2)|\Omega| + M^4 K_0\right]$$

are uniform constants.

Next we can take inner-product $\langle (2), -\Delta v(t)\rangle_{\Omega_{u,M}}$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2_{\Omega_{u,M}} + d_2\|\Delta v(t)\|^2_{\Omega_{u,M}} \\
&\quad = -\int_{\Omega_{u,M}} b\Delta v\,dx + \int_{\Omega_{u,M}} u^2 v\Delta v\,dx - F\int_{\Omega_{u,M}} u^3\Delta v\,dx \\
&\quad \le (FM^3+b)\int_{\Omega_{u,M}} |\Delta v(t)|\,dx + M^2\int_{\Omega_{u,M}} |v(t)||\Delta v(t)|\,dx.
\end{aligned}$$

Similarly we can prove that there exists a uniform constant $C_8 = C_8(K_0) > 0$ such that

$$\|\nabla v(t)\|^2_{\Omega_{u,M}} \le C_8, \quad \text{for } t > T_0 + 1. \tag{40}$$

Due to the compact Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ for space dimension $n \le 3$, (39) and (40) show that for any fixed $t > T_0 + 1$,

$$(S(t)B_0)_{\Omega(|u(t)|<M)} \text{ is a precompact set in } H.$$

Therefore, by the property of Kuratowski measure, in the product phase space H we have

$$\kappa\left((S(t)B_0)_{\Omega(|u(t)|<M)}\right) = 0, \quad \text{for } t \ge T_0 + 1.$$

The lemma is proved. □

We now finish the proof of the main result, Theorem 1, as follows.

Proof of Theorem 1. By Lemma 1, the solution semiflow $\{S(t)\}_{t\ge 0}$ of the modified Schnackenberg equations (1)–(2) has a bounded absorbing set B_0 in H and the condition (i) in Theorem 2 is satisfied. Then by Lemma 2 through Lemma 4, this solution semiflow $\{S(t)\}_{t\ge 0}$ satisfies the conditions (16) and (17), so that the condition (ii) in Theorem 2 is also satisfied. Thus by Theorem 2, there exists a global attractor $\mathscr{A}$ in H for the Schnackenberg semiflow $\{S(t)\}_{t\ge 0}$. □

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