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BLOW-UP ESTIMATES OF POSITIVE SOLUTIONS OF A REACTION-DIFFUSION SYSTEM

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Abstract. This paper is concerned with positive solutions of the reaction-diffusion system

$$u_t - \Delta u = u^{m_1} v^{n_1},$$
$$v_t - \Delta v = u^{m_2} v^{n_2},$$

which blow up at $t = T$. We obtain the following estimates on the blow-up rates:

$$c(T-t)^{-(n_1-n_2+1)/\gamma} \le \max_{x\in\Omega} u(x,t) \le C(T-t)^{-(n_1-n_2+1)/\gamma},$$

$$c(T-t)^{-(m_2-m_1+1)/\gamma} \le \max_{x\in\Omega} v(x,t) \le C(T-t)^{-(m_2-m_1+1)/\gamma},$$

for some positive constants c, C and $\gamma = m_2 n_1 - (1-m_1)(1-n_2)$.

1. **Introduction.** The purpose of this paper is to derive some estimates of blow-up rates of positive solutions of the following nonlinear reaction-diffusion system

$$u_t - \Delta u = u^{m_1} v^{n_1}, \quad x \in \Omega,\ t > 0, \tag{1}$$
$$v_t - \Delta v = u^{m_2} v^{n_2}, \quad x \in \Omega,\ t > 0, \tag{2}$$

with initial and boundary conditions

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{3}$$
$$u(x,0) = u_0(x),\ v(x,0) = v_0(x), \quad x \in \Omega, \tag{4}$$

where Ω is a bounded convex domain in R^n with smooth boundary $\partial\Omega$, $m_1, n_2 \ge 0$ and $m_2, n_1 > 0$. The latter implies that the system (1-2) is completely coupled. The functions $u_0(x), v_0(x)$ are nonnegative, continuous, and bounded. System (1-2) is usually used as a model to describe heat propagation in a two-component combustible mixture. In this case u and v represent the temperatures of the interacting components, thermal conductivity is supposed constant and equal for both substances, and a volume energy release given by some powers of u and v is assumed. For more details on combustion theory, see [1].

The mechanism of the blow-up of solutions for nonlinear scalar equations of the type

$$u_t = \Delta u + f(u) \tag{5}$$

has been investigated over the past two decades (cf. [2] for a recent survey). For example, if $f(u) = u^p$ $(p > 1)$ and T is the blow-up time for a solution $u(x,t)$ of (5),

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Friedman and McLeod [3] proved that if $\Delta u_0 + u_0^p \geq 0$, then there exists a constant $C > 0$ such that

$$u(x,t) \leq C(T-t)^{-1/(p-1)}, \quad (1 < p < \infty). \tag{6}$$

Giga and Kohn [4, 5], and Weissler [6] established the estimate (6) by replacing $\Delta u_0 + u_0^p \geq 0$ by $p < (n+2)/(n-2)$, The estimate (6) allows one to characterize the asymptotic behavior of positive solutions of (5) near the blow-up point and to derive a complete classification of blow-up patterns [5, 7]. The basic motivation for this paper is to obtain an analogue of the blow-up estimates for system (1)–(4), where the mechanism of singularity formation is much less understood than in the scalar case. If $\gamma = m_2 n_1 - (1-m_1)(1-n_2) > 0$, it is proved in [8] that for sufficiently "large" initial data, the solution $(u(x,t), v(x,t))$ of (1)–(4) blows up in finite time simultaneously. For the special case $m_1 = n_2 = 0$ and $\Omega = B(0,R)$, Caristi and Mitidieri [9], under very restricted assumptions, established the following estimates for the radial solution of (1)–(4),

$$u(r,t) \leq C(T-t)^{-(n_1+1)/(m_2 n_1 - 1)}, \tag{7}$$

$$v(r,t) \leq C(T-t)^{-(m_2+1)/(m_2 n_1 - 1)}. \tag{8}$$

Their proof is based on Weissler's approach for the scalar case [6]. Recently, Andreucci et.al [10] obtained the same estimates (7), (8) for Cauchy problems by using a parabolic Liouville theorem. In this paper, we will establish the following upper bound estimates

Theorem 1 Suppose $(u(x,t), v(x,t))$ is a positive solution of (1)–(4) which blows up at time T. Assume, as well, that $m_2 - m_1 + 1 > 0, n_1 - n_2 + 1 > 0$ and $\gamma = m_2 n_1 - (1-m_1)(1-n_2) > 0$. If $n \leq 2$ or

$$\frac{\gamma}{n_1 - n_2 + 1} < \frac{4}{n-2}, \quad \frac{\gamma}{m_2 - m_1 + 1} < \frac{4}{n-2}, \tag{9}$$

then there exists a positive constant C such that

$$u(x,t) \leq C(T-t)^{-(n_1-n_2+1)/\gamma}, \tag{10}$$

$$v(x,t) \leq C(T-t)^{-(m_2-m_1+1)/\gamma}. \tag{11}$$

This generalizes the estimates (7), (8). Our method is based on Theorem 2.3 in [8] and a comparison argument, which enable us to reduce the system to the scalar case. Moreover, in the course of analysing (1)–(4), some new and interesting facts emerge even for the scalar equation (5). Precisely, we obtain the following lower bound estimates of the blow-up rate for the scalar equation.

Theorem 2 Suppose $u(x,t)$ is a positive solution of (5) with $f(u) = u^p$, which blows up at time T. Assume, as well, that $u_t(x,t) \geq 0$. If $n \leq 2$ or $p < (n+2)/(n-2)$, then there exists a positive constant c such that

$$\max_{x \in \Omega} u(x,t) \geq c(T-t)^{-1/(p-1)}. \tag{12}$$

This lower bound seems new and has independent interest. Following Giga and Kohn [4], setting

$$w(y,s) = (T-t)^{1/(p-1)} u(a+x,t),$$

where a is the blow-up point and

$$x = (T-t)^{1/2} y, \quad s = -\ln(T-t)$$

we get

$$w_s - \Delta w + \frac{1}{2}\, y \cdot \nabla w + \frac{w}{p-1} - w^p = 0. \tag{13}$$

The asymptotic behavior of $w(y,s)$ as $s \to \infty$ gives the asymptotic behavior of $u(x,t)$ near (a,T). It is shown in [4] that $w(y,s)$ approaches a state-steady solution of (13) and the only nonnegative bounded state-steady solution is either $w_s = 0$ or $w_p = (1/(p-1))^{1/(p-1)}$. The lower bound (12) rules out the case $w \to 0$. Thus, as a corollary we regain the main result of [7] about the asymptotic behavior of $u(x,t)$ near the blow-up point a.

Corollary Suppose the conditions in Theorem 2 hold. Then

$$\lim_{t\to T} (T-t)^{1/(p-1)} u(a + y(T-t)^{1/2}, t) = \left(\frac{1}{p-1}\right)^{1/(p-1)}. \tag{14}$$

As a final goal of this paper, we prove the following analogue of (12) for the system (1)–(4).

Theorem 3 Suppose the conditions in Theorem 1 hold. Assume, as well, that $u_t \geq 0, v_t \geq 0$. Then there exists a positive constant c such that

$$\max_{x\in\Omega} u(x,t) \geq c(T-t)^{-(n_1-n_2+1)/\gamma}, \tag{15}$$

$$\max_{x\in\Omega} v(x,t) \geq c(T-t)^{-(m_2-m_1+1)/\gamma}. \tag{16}$$

The remaining part of this paper is organized as follows. In Section 2 we collect some blow-up related properties of the solution of (1)–(4). We then prove Theorems 1–3 in Section 3.

2. **Preliminaries.**

In this section we collect some results of [8, 11] adapted to our interests. Set

$$D_T = \{(x,t) : \ x \in \Omega,\ 0 < t < T\}.$$

By a solution (u,v) of (1)–(4), we mean a classical solution, that is, $u, v \in C(\bar{\Omega} \times [0,T)) \cap C^{2,1}(D_T)$ and (u,v) satisfies (1)–(4). The following local existence of a solution to (1)–(4) is well-known (cf. Chapter 11 in [11] and Chapter 7 in [12]).

Lemma 2.1 Let $m_i \geq 0, n_i \geq 0$ (i = 1, 2) and assume that $u_0(x)$ and $v_0(x)$ are nonnegative, continuous and bounded. Then there exists a nonnegative solution to (1)–(4) in D_T for some $0 < T \leq +\infty$. Moreover, if $u_0(x) \not\equiv 0$ or $v_0(x) \not\equiv 0$, the solution is unique.

Let the potential energy E be defined by

$$E[u](t) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{p+1}\int_\Omega u^{p+1}\, dx.$$

Chen proved the following blow-up result (see [8], Theorem 3.6).

Lemma 2.2 Suppose $\gamma > 0$. Then the solution of (1)–(4) blows up simultaneously in a finite time if $E[u_0] \leq 0$ and $E[v_0] \leq 0$.

We will henceforth assume that the solution (u, v) blows up at a finite time $t = T$. In view of Lemmas 2.1 and 2.2, this is equivalent to

$$\lim_{t\to T} \sup \| u \|_\infty = \lim_{t\to T} \sup \| v \|_\infty = +\infty.$$

T is called the blow-up time. Next we give the definition of upper and lower solutions of (1)–(4).

Definition. A pair of functions $(\tilde{u}(x,t), \tilde{v}(x,t))$ is called an upper solution of (1)–(4) in D_T if both $\tilde{u}$ and $\tilde{v} \in C(\bar{\Omega} \times [0,T)) \cap C^{2,1}(D_T)$, and

$$L\tilde{u} \geq \tilde{u}^{m_1}\tilde{v}^{n_1}, \;\; L\tilde{v} \geq \tilde{u}^{m_2}\tilde{v}^{n_2}, \;\; \text{in } \; D_T,$$

$$\tilde{u}(x,t) \geq 0, \;\; \tilde{v}(x,t) \geq 0, \;\; \text{on } \; \partial\Omega \times (0,T),$$

$$\tilde{u}(x,0) \geq u_0(x), \; \tilde{v}(x,0) \geq v_0(x), \;\; \text{in } \; \Omega.$$

A lower solution $(\hat{u}(x,t), \hat{v}(x,t))$ is defined similarly with the reversed inequalities. Since the reaction functions here are quasi-monotone increasing, using a suitable monotone iterative scheme and the maximum principle we have the following existence-comparison lemma.

Lemma 2.3 Let $(\tilde{u}(x,t), \tilde{v}(x,t))$ and $(\hat{u}(x,t), \hat{v}(x,t))$ be a pair of upper-lower solutions of (1)–(4) in D_T such that $(\hat{u}(x,t), \hat{v}(x,t)) \leq (\tilde{u}(x,t), \tilde{v}(x,t))$. Then the problem (1)–(4) has a unique solution $(u(x,t), v(x,t))$ and

$$(\hat{u}, \hat{v}) \leq (u, v) \leq (\tilde{u}, \tilde{v}) \;\; \text{in} \;\; D_T.$$

We end this section with a comparison result between the solution u and v due to [8], which allows us to reduce the system to a scalar equation and which will play a crucial role later on.

Lemma 2.4 Let $(u(x,t), v(x,t))$ be the positive solution of (1)–(4) and assume that $m_2 - m_1 > -1, n_1 - n_2 > -1$. Then there exist constants C_1, C_2 such that

(i): If $m_1 + n_1 \geq m_2 + n_2$, then

$$u(x,t) \geq C_1 v^\alpha(x,t) \quad \text{where} \;\; \alpha = \frac{n_1 - n_2 + 1}{m_2 - m_1 + 1}. \tag{17}$$

(ii): If $m_1 + n_1 \leq m_2 + n_2$, then

$$v(x,t) \geq C_2 u^\beta(x,t) \quad \text{where} \;\; \beta = \frac{m_2 - m_1 + 1}{n_1 - n_2 + 1}. \tag{18}$$

3. **Blow up Estimates for the System.** We assume that $m_2 - m_1 > -1$ and $n_1 - n_2 > -1$ throughout this section, and we now prove Theorem 1.

Proof. If $m_1 + n_1 \geq m_2 + n_2$, using the estimate (17) in (1), we have

$$u_t - \Delta u \leq C_3 u^{m_1 + n_1/\alpha} = C_3 u^p,$$

where $C_3 = (1/C_1)^{n_1/\alpha}$ and

$$p \equiv m_1 + n_1/\alpha = 1 + \frac{\gamma}{n_1 - n_2 + 1}.$$

The assumptions $\gamma > 0$ and (9) lead to $1 < p < (n+2)/(n-2)$ if $n \geq 3$. In view of the upper bound estimate (6) for the scalar equation, there exists a constant $C_4 > 0$ such that

$$u(x,t) \leq C_4(T-t)^{-1/(p-1)} = C_4(T-t)^{-(n_1-n_2+1)/\gamma}. \tag{19}$$

Using this upper bound for u and the inequality (17), we obtain

$$\begin{aligned} v(x,t) &\leq \left(\frac{u}{C_1}\right)^{1/\alpha} \leq \left[\frac{C_4(T-t)^{-(n_1-n_2+1)/\gamma}}{C_1}\right]^{1/\alpha} \\ &= \left(\frac{C_4}{C_1}\right)^{1/\alpha}(T-t)^{-(m_2-m_1+1)/\gamma}. \end{aligned}$$

This, together with (19), leads to the estimates (10), (11) with $C = \max\{C_4, (C_4/C_1)^{1/\alpha}\}$. For the case $m_1+n_1 \leq m_2+n_2$, the corresponding estimates are obtained in a similar way by using the estimate (18) in (2). This proves Theorem 1. □

Let $G(x,t)$ be the Green's function for the heat equation with homogeneous Dirichlet boundary condition. Then the solution of (5) is given by

$$\begin{aligned} u(x,t) &= \int_\Omega G(x-y,t-s)u(y,s)\,dy \\ &\quad + \int_s^t d\tau \int_\Omega G(x-y,t-s-\tau)f(u(y,\tau))\,dy \end{aligned} \tag{20}$$

for $0 < s < t < T$. Now we can prove Theorem 2.

Proof. Let $u(x,t)$ be the solution of (5) with $f(u) = u^p$ which blows up at $t = T$. Let

$$U(t) = \max_{x\in\Omega} u(x,t).$$

Since $u_t \geq 0$, $U(t)$ is nondecreasing in $(0,T)$. In view of (20), we have

$$U(t) \leq U(s) + (T-s)U^p(t), \text{ for } s < t.$$

It then follows that

$$\begin{aligned} U(t) &\leq U(s) + (T-s)[U(s) + (T-s)U^p(t)]^p \\ &\leq U(s) + 2^p(T-s)U^p(s) + 2^p(T-s)^{p+1}U^{p^2}(t), \end{aligned} \tag{21}$$

where the inequality

$$(a+b)^p \leq 2^p(a^p+b^p), \quad \text{for } a,\ b,\ p\ \geq 0,$$

has been used. Since

$$\lim_{t\to T} u(x,t) = \infty,$$

the Intermediate Value Theorem implies that for any $s \in (0,T)$, there exists $t \in (s,T)$ such that

$$U(t) = 2(U(s) + 2^p(T-s)U^p(s)).$$

Substituting this into (21), we obtain

$$\begin{aligned} U(s) + 2^p(T-s)U^p(s) &\leq 2^p(T-s)^{p+1}\,2^{p^2}(U(s) + 2^p(T-s)U^p(s))^{p^2} \\ &= 2^{p(p+1)}(T-s)^{p+1}(U(s) + 2^p(T-s)U^p(s))^{p^2}. \end{aligned}$$

This is equivalent to

$$U(s) + 2^p(T-s)U^p(s) \geq 2^{-p/(p-1)}(T-s)^{-1/(p-1)}. \tag{22}$$

In the following we will prove (12) by contradiction. Suppose that (12) is false, then there exist sequences $\{t_n\} \to T$ and $\{\epsilon_n\} \to 0$ such that

$$U(t_n) \leq \epsilon_n (T-t_n)^{-1/(p-1)}.$$

Using this and the inequality (22), we have

$$\begin{aligned} U(t_n) &= (U(t_n) + 2^p(T-t_n)U^p(t_n)) - 2^p(T-t_n)U^p(t_n) \\ &\geq 2^{-p/(p-1)}(T-t_n)^{-1/(p-1)} - 2^p(T-t_n)\epsilon_n^p(T-t_n)^{-p/(p-1)} \\ &= (2^{-p/(p-1)} - 2^p\epsilon_n^p)(T-t_n)^{-1/(p-1)}. \end{aligned} \tag{23}$$

On the other hand, let $a > \max\{1, 2^pC\}$ be a constant and $s = T - a(T-t_n)$. Since $t_n \to T$, we have $s \in (0, t_n)$ for large n. From (21) and the upper bound (6), we obtain

$$\begin{aligned} U(t_n) &\leq U(T - a(T-t_n)) + a(T-t_n)U^p(t_n) \\ &\leq Ca^{-1/(p-1)}(T-t_n)^{-1/(p-1)} + a\epsilon_n^p(T-t_n)^{-1/(p-1)} \\ &= (Ca^{-1/(p-1)} + a\epsilon_n^p)(T-t_n)^{-1/(p-1)}. \end{aligned}$$

Combining this inequality with (23), we get

$$2^{-p/(p-1)} - 2^p\epsilon_n^p \leq Ca^{-1/(p-1)} + a\epsilon_n^p.$$

This is a contradiction to the fact that $a > 2^pC$. Therefore (12) holds. This proves Theorem 2. □

Now we apply the lower bound (12) and Lemmas 2.3 and 2.4 to prove Theorem 3.

Proof. If $m_1 + n_1 \geq m_2 + n_2$, let $(\hat{u}(x,t), \hat{v}(x,t))$ be a solution of

$$\begin{aligned} \hat{u}_t - \Delta\hat{u} &= \hat{u}^{m_1}\hat{v}^{n_1}, \\ \hat{v}_t - \Delta\hat{v} &= C_1^{m_2}\hat{v}^{m_2\alpha+n_2} \end{aligned}$$

subject to the initial and boundary conditions (3), (4). Using the estimate (17) and Lemma 2.3, we see that $(\hat{u}, \hat{v})$ is a lower solution of (1)–(4). Noting the exponent of $\hat{v}$ is equal to

$$p \equiv m_2\alpha + n_2 = 1 + \frac{\gamma}{m_2 - m_1 + 1},$$

and observing that the assumptions of Theorem 2 are satisfied, we get from (12)

$$\max_{x\in\Omega} \hat{v} \geq C_5(T-t)^{-(m_2-m_1+1)/\gamma}. \tag{24}$$

for some positive constant C_5. Using this lower bound for $\hat{v}$ and the inequality (17), we obtain

$$\begin{aligned} \max_{x\in\Omega} \hat{u} &\geq C_1(\max_{x\in\Omega} \hat{v})^\alpha \geq C_1(C_5(T-t)^{-(m_2-m_1+1)/\gamma})^\alpha \\ &= C_1C_5^\alpha(T-t)^{-(n_1-n_2+1)/\gamma}. \end{aligned} \tag{25}$$

Taking $c = \min\{C_5, C_1(C_5)^\alpha\}$, the estimates (15), (16) follow from (24), (25) and $(u(x,t), v(x,t)) \geq (\hat{u}, \hat{v})$.

For the case $m_1 + n_1 \leq m_2 + n_2$, the corresponding estimates are obtained in a similar way by using the estimate (18) in (1). This proves Theorem 3. □

Based on Giga and Kohn's [7] ideas, the upper bounds (10), (11) and lower bounds (15), (16) enable us to characterize the asymptotic behavior of the solution of (1)–(4) near a blow-up point. In fact, set

$$u(x,t) = (T-t)^{-(n_1-n_2+1)/\gamma}\Phi(y,\tau),\ v(x,t) = (T-t)^{-(m_2-m_1+1)/\gamma}\Psi(y,\tau)$$

where $y = x(T-t)^{-1/2}, \tau = -\ln(T-t)$. We can show an analogue of (14) to system (1)–(4) as follows:

$$\Phi \to a,\ \ \Psi \to b \text{ as } \tau \to \infty$$

uniformly on compact sets $|y| \le R < \infty$, and $a, b > 0$ satisfy the system of equations

$$a^{m_2} = \frac{m_2 - m_1 + 1}{\gamma}\, b^{1-n_2},\ b^{n_1} = \frac{n_1 - n_2 + 1}{\gamma}\, a^{1-m_1}.$$

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