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ERROR ANALYSIS OF A CONSERVATIVE FINITE-ELEMENT APPROXIMATION FOR THE KELLER-SEGEL SYSTEM OF CHEMOTAXIS

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Abstract. We are concerned with the finite-element approximation for the Keller-Segel system that describes the aggregation of slime molds resulting from their chemotactic features. The scheme makes use of a semi-implicit time discretization with a time-increment control and Baba-Tabata's conservative upwind finite-element approximation in order to realize the positivity and mass conservation properties. The main aim is to present error analysis that is an application of the discrete version of the analytical semigroup theory.

1. **Introduction.** The purpose of this paper is to study the finite-element method applied to a nonlinear parabolic system for the functions $u = u(x,t)$ and $v = v(x,t)$ of $(x,t) \in \overline{\Omega} \times [0,J]$:

$$u_t = \nabla \cdot (D_u \nabla u - \lambda u \nabla v) \qquad \text{in} \quad \Omega \times (0,J), \tag{1a}$$

$$k v_t = D_v \Delta v - k_1 v + k_2 u \qquad \text{in} \quad \Omega \times (0,J), \tag{1b}$$

$$\partial u/\partial \nu = 0, \quad \partial v/\partial \nu = 0 \qquad \text{on} \quad \partial\Omega \times (0,J), \tag{1c}$$

$$u|_{t=0} = u_0, \quad v|_{t=0} = v_0 \qquad \text{on} \quad \Omega, \tag{1d}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ is a bounded domain with the boundary $\partial\Omega$, ν is the outer unit normal vector to $\partial\Omega$, $\partial/\partial\nu$ denotes differentiation along ν on $\partial\Omega$, $u_0 = u_0(x)$, $v_0 = v_0(x)$ are initial values, and $\lambda, D_u, D_v, k, k_1, k_2, J$ are positive constants.

As is well-known, the system (1), which is called the Keller-Segel system, describes the aggregation of slime molds resulting from their chemotactic features (cf. [20]). Here, u is defined to be the density of the cellular slime molds, v the concentration of the chemical substance secreted by molds themselves, k the relaxation time, and $k_1 v - k_2 u$ the ratio of generation/extinction. There is a large number of works devoted to mathematical analysis of the Keller-Segel system; see [17], [18], [19] and [27]. A key feature of the solution u is the conservation of the L^1 norm:

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \qquad (t \in [0,J]), \tag{2}$$

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which plays an important role to study the Keller-Segel system. Equality (2) is a readily consequence of the conservation of positivity

$$u_0(x) \geq 0, \not\equiv 0 \text{ on } \Omega \quad \Rightarrow \quad u(x,t) > 0 \text{ in } \Omega \times (0, J]$$

and the conservation of total mass

$$\int_\Omega u(x,t)\ dx = \int_\Omega u_0(x)\ dx \qquad (t \in [0, J]).$$

Therefore, it is desired that numerical solutions enjoy the discrete analogues of these properties, when we solve the Keller-Segel system by numerical methods. Those conservation properties are simple to hold in a continuous problem, whereas some difficulties arise in a discrete one. (An elementary example to illustrate this issue is given in [26].)

In a previous paper [25], we considered the case $k = 0$, which is called a simplified Keller-Segel system, and proposed a conservative finite-element scheme. Our scheme made use of Baba and Tabata's upwind approximation combined with the mass lumping based on the barycentric domain and a semi-implicit time discretization with a time-increment control. That is, at every discrete time step $t_n = \tau_1 + \cdots + \tau_n$, we adjust the time-increment τ_n to obtain a positive solution. Consequently, our finite-element approximation has the conservation of positivity and total mass for an arbitrary $h > 0$, the granularity parameter of the spatial discretization, if the triangulation is of acute type (H1) described below. At this stage, we would like to point out that the conservation of total mass is satisfied by the standard finite-element method and this can be verified by taking the unity as the test function. The important point is, however, that we realize the positivity and mass conservation properties simultaneously.

Furthermore, in [25], we succeeded in establishing error estimates in $L^p \times W^{1,\infty}$ with a suitable $p > d$, where d is the dimension of a spatial domain. The main tool of our error analysis is the analytical semigroup theory in Banach spaces. Actually, if the triangulation is of acute type, the operator A_h which is a finite-element approximation of $-\Delta + 1$ of the lumped mass type becomes sectorial on a finite-element space $\mathcal{X}_{h,p}$ equipped with a modified L^p norm. In particular, $-A_h$ generates the analytic semigroup on $\mathcal{X}_{h,p}$. (The precise meaning of these symbols will be given in Section 3.) We then make use of Duhamel's principle, fractional powers of operators, and the smoothing property of the semigroup. Although semigroup theory is somewhat abstract, several L^p estimates can be derived in a quite formal manner. Moreover, our method of analysis is a discrete analogue of the standard approach for solving nonlinear evolution problems.

This paper is a continuation of [25], and we are going to extend our method and results to the Keller-Segel system (1). The main aim here is to prove the error estimate (Theorem 2.4), since the proof of conservation properties (Theorems 2.1, 2.2 and 2.3) is the same as that of [25]. To this end, we basically follow the method of [25]; we, however, need new devices described in subsequent sections. Out finite-element scheme has already presented in a previous paper [26], and the validity of the scheme is confirmed by several numerical examples; this paper includes no numerical results.

It is well-known that the solution of (1) may blow up in finite time (cf. [17], [18], [19] and [27]) and many researchers would like to elucidate the behavior of blow up solutions. Hence, it is worth-while noting that our convergence result (Theorem

2.4) is valid for $0 \leq t \leq J$ for an arbitrary $J \in (0, J_{\max})$, where $J_{\max}$ is the blow up time.

Recently, Efendiev et al. [11] has succeeded in obtaining an estimate of the fractal dimension of the global attractor in terms of D_u, D_v, k, k_1, k_2 and h for a semidiscrete (in space) version of our finite-element scheme applied to a generalization of (1). The estimate of the fractal dimension has exactly the same order as that for the original system. On the other hand, they described that we can only obtain a poorer estimate of the fractal dimension for the standard finite-element scheme. This means that our conservative finite-element scheme preserves the structure of dynamical systems governed by (1) from the viewpoint of attractor dimension.

Before concluding this Introduction, we briefly discuss some other results that are related to numerical methods for the Keller-Segel system. In [22], Nakaguchi and Yagi present finite-element/Runge-Kutta approximations for a generalization of (1) without any numerical results. They also establish error estimates in the $H^{1+\varepsilon}$ norm, $\varepsilon \in (0, 1/2)$, for a sufficiently small J, though they devote little attention to conservation of the L^1 norm of approximate solutions. Marrocco [21] discusses mixed finite-element approximations for the simplified Keller-Segel system and offered various numerical examples, but a convergence analysis is not undertaken. The aim of Filbet [12] is similar as ours. He proposes a fully-implicit/finite-volume method for the simplified system, and derives the L^1 conservation under some condition on a (fixed) time-increment. Moreover, a convergence result without any convergence rate is also proved if the L^1 norm of an initial datum is sufficiently small. It should be kept in mind that, as far as the spatial discretization is concerned, our finite-element scheme is equivalent to Filbet's finite-volume scheme if we take the mass lumping based on the circumcentric domain instead of the barycentric domain. Higher order approximations using the discontinuous Galerkin finite-element method (DGFEM) are studied by Epshteyn and Kurganov [10] and Epshteyn and Izmirlioglu [9]. They consider the Keller-Segel system in the two dimensional rectangle and use Cartesian grids to apply an interior penalty DGFEM. Convergence theorems of the schemes are established for both semidiscrete in space ([10]) and fully discrete cases ([9]) under some appropriate assumptions on the regularity of solutions. Positivity of solutions is confirmed by several numerical experiments. A finite-volume scheme of Chertock and Kurganov [5] is regarded as the lower order version of [10] and [9]. The CFL type conditions to ensure the positivity of solutions are given. Moreover, they confirm high resolution, stability and robustness of the schemes by many numerical examples. The approach of Haškovec and Schmeiser [16] differs from these works. They consider the measure-valued global in time solutions of the simplified system in $\mathbb{R}^2$ and propose a stochastic particle approximation. The advantage of their method is that it reproduces the dichotomy in the qualitative behavior of the system and captures the solution even after the (possible) blow up events. However, the complete proof of convergence is a rather difficult task and they postpone it to their future work.

The organization of this paper is as follows. In Section 2, we state our conservative finite-element scheme and formulate theorems about conservation laws (Theorems 2.1–2.3) and error estimates (Theorem 2.4). The proof of the main result (Theorem 2.4) is described in Section 4, after having prepared some preliminary results in Section 3. We conclude this paper by giving a few remarks in Section 5.

Notation. We follow the notation of [1]. We write as $W^{m,p} = W^{m,p}(\Omega)$, $H^m = W^{m,2}$, $L^p = L^p(\Omega)$, $\| \cdot \|_{m,p} = \| \cdot \|_{W^{m,p}}$, $\| \cdot \|_p = \| \cdot \|_{L^p}$ for $m \in \mathbb{N}$ and $p \in [1, \infty]$.

The standard inner product in L^2 is denoted by $(\cdot,\cdot)$. We set, for $p \in [1,\infty)$,

$$\mathcal{W}_p = \left\{ v \in W^{2,p} \;\middle|\; \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

We set $[a]_\pm = \max\{0, \pm a\}$ for $a \in \mathbb{R}$. The d-dimensional Lebesgue measure of $\mathcal{O} \subset \mathbb{R}^d$ is denoted by $\text{meas}\,(\mathcal{O}) = \text{meas}_{\,d}(\mathcal{O})$. For a Banach space X, its dual space is denoted as X'. Generic positive constants depending on Ω are denoted as C, C', and so forth. In particular, C does not depend on the discretization parameters h and τ described below. If it is necessary to specify the dependence on other parameters, say α, β, then we write them as $C_{\alpha,\beta}$ or $C(\alpha,\beta)$. However, if the contribution of those parameters is not necessary for our argument, we omit indicating them. We shall use the same symbol I to indicate the identity operator on any space. Finally, $\mathcal{D}(B)$ represents the domain of the definition of an operator B.

2. **Finite element scheme and theorems.**

Throughout this paper, Ω is assumed to be a bounded polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$. We first convert the system (1) into a weak form as follows:

$$(u_t, \chi) + (D_u \nabla u, \nabla \chi) + \lambda b(v, u, \chi) = 0 \qquad (\forall \chi \in H^1), \tag{3a}$$

$$(k v_t, \chi) + (D_v \nabla v, \nabla \chi) + (k_1 v - k_2 u, \chi) = 0 \qquad (\forall \chi \in H^1), \tag{3b}$$

$$u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \tag{3c}$$

where

$$b(v, u, \chi) = -\int_\Omega u \; \nabla v \cdot \nabla \chi \; dx.$$

Let $\{\mathcal{T}_h\} = \{\mathcal{T}_h\}_{h\downarrow 0}$ be a family of triangulations $\mathcal{T}_h$ of Ω:

1. $\mathcal{T}_h$ is a set of closed d-simplices (elements), and $\overline{\Omega} = \bigcup\{T \mid T \in \mathcal{T}_h\}$;
2. any two elements of $\mathcal{T}_h$ meet only in entire common faces or sides or in vertices.

We set

$$\begin{aligned} h_T &= \text{the diameter of the circumscribed ball of } T, \\ \rho_T &= \text{the diameter of the inscribed ball of } T, \\ \kappa_T &= \text{the minimal perpendicular length of } T, \\ h &= \max\{h_T \mid T \in \mathcal{T}_h\}, \\ \kappa_h &= \min\{\kappa_T \mid T \in \mathcal{T}_h\}. \end{aligned}$$

We assume that $\{\mathcal{T}_h\}_h$ is regular in the sense that there is a positive constant γ_1 satisfying

$$h_T \le \gamma_1 \rho_T \qquad (\forall T \in \mathcal{T}_h \in \{\mathcal{T}_h\}_h).$$

Let $\{P_i\}_{i=1}^N$ be the set of all vertices of $\mathcal{T}_h$, $N = N_h$ being a positive integer. With P_i, we associate $\hat{\phi}_i \in C(\overline{\Omega})$ such that $\hat{\phi}_i$ is an affine function on each $T \in \mathcal{T}_h$ and $\hat{\phi}_i(P_j) = \delta_{ij}$, where δ_{ij} denotes Kronecker's delta. We define as

$$X_h = \text{ the vector space spanned by } \{\hat{\phi}_i\}_{i=1}^N$$

and regard it as a closed subspace of H^1. We also consider the space X_h, which is equipped with the topology induced from L^2, and express it using the same symbol X_h. With P_i, we associate the barycentric domain D_i; see [25] for the definition.

Let $\overline{\phi}_i \in L^\infty$ be the characteristic function of D_i. We introduce a Hilbert space $\overline{X}_h \subset L^2$ spanned by $\{\overline{\phi}_i\}_{i=1}^N$. The operator $M_h : X_h \to \overline{X}_h$ is defined by

$$M_h v_h = \sum_{i=1}^{N} v_h(P_i)\overline{\phi}_i \qquad (v_h \in X_h),$$

and it is called the lumping operator. We define

$$(v_h, \chi_h)_h = (M_h v_h, M_h \chi_h) \qquad (v_h, \chi_h \in X_h).$$

Thereby, $(\cdot,\cdot)_h^{1/2}$ is equivalent to $\|\cdot\|_2$ on X_h (see (27) below).

Our results are formulated under the following conditions on $\{\mathcal{T}_h\}$:

(H1) *Acuteness.* It is assumed that

$$\max\{\cos(\nabla\phi_i^T, \nabla\phi_j^T) \mid 1 \le i, j \le d+1\} \le 0 \qquad (\forall T \in \mathcal{T}_h \in \{\mathcal{T}_h\}),$$

where $\{\phi_i^T\}_{i=1}^{d+1}$ represent the barycentric coordinates of T with respect to the vertices of T.

(H2) *Inverse assumption.* There exists a positive constant γ_2 such that

$$\gamma_2 h \le h_T \qquad (\forall T \in \mathcal{T}_h \in \{\mathcal{T}_h\}).$$

Remark 2.1. As is well-known, the condition (H1) guarantees the non-positivity of $(\nabla\hat{\phi}_i, \nabla\hat{\phi}_j)$ for $i \neq j$. For $d = 2$, (H1) is equivalent to a statement that each triangle of $\mathcal{T}_h$ is a right-angle or an acute triangle. For $d = 3$, (H1) is satisfied if, and only if, all angles spanned by two faces of each tetrahedron of $\mathcal{T}_h$ are less than or equal to $\pi/2$.

The time variable is discretized as

$$t_n = \tau_1 + \tau_2 + \cdots + \tau_n, \qquad \tau_n > 0.$$

Then, we consider the finite-element scheme to obtain an approximation (u_h^n, v_h^n) of the solution $(u(t_n), v(t_n))$ to (3): find $\{u_h^n\}_{n\ge 0} \subset X_h$ and $\{v_h^n\}_{n\ge 0} \subset X_h$ such that

$$\left(\frac{u_h^n - u_h^{n-1}}{\tau_n}, \chi_h\right)_h + (D_u \nabla u_h^n, \nabla\chi_h) + \lambda b_h(v_h^{n-1}, u_h^n, \chi_h) = 0 \qquad (\chi_h \in X_h,\ n \ge 1), \tag{4a}$$

$$\left(k\frac{v_h^n - v_h^{n-1}}{\tau_n}, \chi_h\right)_h + (D_v \nabla v_h^n, \nabla\chi_h) + (k_1 v_h^n - k_2 u_h^n, \chi_h)_h = 0 \qquad (\chi_h \in X_h,\ n \ge 1), \tag{4b}$$

$$u_h^0 = u_{0h},\ v_h^0 = v_{0h}. \tag{4c}$$

Here u_{0h} and v_{0h} denote suitable approximations of u_0 and v_0. Moreover, $b_h(v_h, u_h, \chi_h)$ is Baba and Tabata's approximation of $b(v, u, \chi)$ defined by

$$b_h(v_h, u_h, \chi_h) = \sum_{i=1}^{N} \chi_h(P_i) \sum_{j\in\Lambda_i} \left\{u_h(P_i)\beta_{ij}^+(v_h) - u_h(P_j)\beta_{ij}^-(v_h)\right\}$$

for $v_h, u_h, \chi_h \in X_h$, where

$$\begin{aligned}
\Lambda_i &= \{P_j \mid P_i \text{ and } P_j \text{ share an edge}\};\\
\beta_{ij}^{\pm}(v_h) &= \int_{\Gamma_{ij}} [\nabla v_h \cdot \nu_{ij}]_{\pm}\ dS;\\
\Gamma_{ij} &= \partial D_i \cap \partial D_j;\\
\nu_{ij} &= \text{the outer unit normal vector to } \Gamma_{ij} \text{ with respect to } D_i.
\end{aligned}$$

The solution (u_h^n, v_h^n) of the finite-element scheme (4) enjoys fine conservative properties. The first one is related to the discrete version of the conservation of total mass.

Theorem 2.1 (Conservation of total mass)**.** *Let $\{(u_h^n, v_h^n)\}_{n\geq 0} \subset X_h \times X_h$ be the solution of* (4)*. Then, we have $(u_h^n, 1)_h = (u_{0h}, 1)_h$ for $n \geq 0$.*

The second one is concerned with the unique solvability of (4) and conservation of positivity.

Theorem 2.2 (Unique solvability and conservation of positivity)**.** *Suppose that* (H1) *is satisfied. Assume that $u_{0h}, v_{0h} \in X_h$ are non-negative and are not identically zero. Take $\tau > 0$ and $\varepsilon \in (0, 1]$. Then, the finite-element scheme* (4) *with a time increment control*

$$\tau_n = \min\left\{\tau, \quad \frac{\varepsilon \kappa_h}{2d\lambda \|\nabla v_h^{n-1}\|_\infty}\right\} \tag{5}$$

admits a unique solution $\{(u_h^n, v_h^n)\}_{n\geq 0} \subset X_h \times X_h$ such that $u_h^n > 0$ and $v_h^n > 0$ for $n \geq 1$.

Remark 2.2. When v_{0h} is a constant function, the value of $\|\nabla v_{0h}\|_\infty^{-1}$ is formally understood as ∞.

As a readily obtainable consequence of these theorems, we deduce the following.

Theorem 2.3 (Conservation of the L^1 norm)**.** *Let $\{(u_h^n, v_h^n)\}_{n\geq 0} \subset X_h \times X_h$ be the solution of* (4) *as in Theorem* 2.2*. Then, we have $\|u_h^n\|_1 = \|u_{0h}\|_1$ for $n \geq 0$.*

The proof of Theorems 2.1 and 2.2 is the exactly same as that of [25, Theorems 2.1 and 2.2]; so we omit describing it.

Remark 2.3. We consider the solution (u_h^n, v_h^n) stated in Theorem 2.2. Substituting $\chi_h = 1$ into (4b) and using the conservation of the L^1 norm, we have

$$\|v_h^n\|_1 - \|v_h^{n-1}\|_1 \leq \frac{\tau_n k_2}{k}\|u_h^n\|_1 = \frac{\tau_n k_2}{k}\|u_{0h}\|_1,$$

and, hence,

$$\|v_h^n\|_1 \leq \|v_{0h}\|_1 + \frac{k_2 J}{k}\|u_{0h}\|_1.$$

On the other hand, since all norms are equivalent on X_h, there exists a positive constant c_h depending on h such that $\|\nabla v_h^n\|_\infty \leq c_h \|v_h^n\|_1$. Combining these inequalities, we obtain

$$\tau_n \geq \min\left\{\tau, \ \frac{\varepsilon \kappa_h k}{2dc_h(k\|v_{0h}\|_1 + k_2 J\|u_{0h}\|_1)}\right\}.$$

Thus, τ_n is bounded from below by a positive constant independent of n. This implies that τ_n never converges to zero as n increases, and therefore the time increment control (5) is always valid. Consequently, (u_h^n, v_h^n) actually exists for all $n \geq 1$.

We suppose that $A_p : L^p \to L^p$ is the L^p realization of $-\Delta + I$ with the Neumann boundary condition,

$$\mathcal{D}(A_p) = \mathcal{W}_p, \qquad A_p v = -\Delta v + v \quad (v \in \mathcal{D}(A_p)). \tag{6}$$

Then, we make the following conditions (see Remarks 2.6 and 2.7 below).

(A1) There exists $\mu \in (d, \infty)$ such that A_p is an isomorphism from $\mathcal{W}_p$ onto L^p for every $p \in (d, \mu)$.

(A2) $\mathcal{D}(A_p^{1/2}) = W^{1,p}$.

The closed linear operator A_p is sectorial in L^p under (A1). Therefore, its fractional powers A_p^α, $\alpha \in (0, 1)$, are defined in a natural way. See [23] for these facts. Below, we simply write A to express A_p, if there is no possibility of confusion.

Now we are in a position to state the main result of this paper.

Theorem 2.4 (Error estimate)**.** *Let* (A1) *and* (A2) *be satisfied with some $\mu \in (d, \infty)$ and for some $p \in (d, \mu)$, respectively. Assume that the system* (3) *admits a unique solution (u, v) satisfying the following regularity condition with some $J \in (0, \infty)$ and $\sigma \in (0, 1]$:*

$$u \in C^1([0, J] : W^{2,p}) \cap C^{1+\sigma}([0, J] : L^p), \tag{7a}$$

$$v \in C^1([0, J] : W^{2,p}) \cap C^{1+\sigma}([0, J] : W^{1,p}). \tag{7b}$$

Suppose that (H1) *and* (H2) *hold. Further, assume that $u_{0h}, v_{0h} \in X_h$ are taken as*

$$h\|u_0 - u_{0h}\|_{1,p} + \|v_0 - v_{0h}\|_p + h^{1+d/p}\|v_0 - v_{0h}\|_{1,\infty} \le \alpha_0 h^2 \tag{8}$$

with a constant $\alpha_0 > 0$. Let τ be chosen as

$$\tau = \frac{\varepsilon \kappa_h}{2d\ \|\nabla v_{0h}\|_\infty} \tag{9}$$

with some $\varepsilon \in (0, 1]$. Then, there exist positive constants h_1 and C_1 independent of h such that we have the error estimate

$$\sup_{0 \le t_n \le J} (\|u(t_n) - u_h^n\|_p + \|v(t_n) - v_h^n\|_{1,\infty}) \le C_1(h^{1-d/p} + \tau^\sigma) \tag{10}$$

for $h \in (0, h_1)$, where (u_h^n, v_h^n) is the solution of (4) *as in Theorem* 2.2.

Some remarks are in order.

Remark 2.4. Let $u_0 \in W^{2,p}$ and $v_0 \in W^{2,p}$ with $p > d$. Then the linear interpolations $u_{0h} = \pi_h u_0$ and $v_{0h} = \pi_h v_0$ of u_0 and v_0 satisfy

$$\|u_0 - u_{0h}\|_{1,p} \le Ch\|u_0\|_{2,p},$$

$$\|v_0 - v_{0h}\|_p \le Ch^2\|v_0\|_{2,p}, \quad \|v_0 - v_{0h}\|_{1,\infty} \le Ch^{1-d/p}\|v_0\|_{2,p}.$$

(The definition of π_h will be given in the beginning of the next section.) Thus, Assumption (8) is fulfilled with $\alpha_0 = C(\|u_0\|_{2,p} + \|v_0\|_{2,p})$.

Remark 2.5. Because the upwind approximation employed in this paper corresponds to a one-sided difference approximation, the rate of convergence with respect to spatial discretization is expected to be $O(h)$ at best. However, such a rate of convergence is not achieved in (10). That shortfall stems from the lack of regularity of solutions of a linear elliptic problem in a polygonal domain. Therefore, on considering (3) in a smooth domain, we can deduce a refined estimate. See Subsection 5.1 or [25, Section 7] for further discussions.

Remark 2.6. When $\Omega \subset \mathbb{R}^2$ is a convex polygon, (A1) is always satisfied. On the other hand, when $\Omega \subset \mathbb{R}^3$ is a convex polyhedron, it is satisfied, if the angles of the edges of Ω are sufficiently small that they do not produce singularities. See, for more complete descriptions, Theorems 8.2.1.2 and 8.2.2.8 of [15].

Remark 2.7. When Ω is a bounded smooth domain, (A2) holds true for every $p \in [1,\infty)$. More precisely, we have (cf. [14])

$$\mathcal{D}(A_p^\theta) = [L^p, W^{2,p}]_\theta \quad \text{for} \quad 0 < 2\theta < 1 + \frac{1}{p}, \tag{11}$$

where $[L^p, W^{2,p}]_\theta$ denotes the complex interpolation space between L^p and $W^{2,p}$ with the exponent θ. Because of $[L^p, W^{2,p}]_{1/2} = W^{1,p}$, we have (A2). When Ω is a convex polygonal domain in $\mathbb{R}^2$, we can obtain (11) by the method of [14]. However, the case of a polyhedral domain in $\mathbb{R}^3$ seems to be open at present.

Remark 2.8. In a previous paper [25], the condition (A2) was not explicitly stated. However, error estimates in [25] are valid under (A2).

The proof of Theorem 2.4 will be described in Section 4, after having prepared some preliminary results in Section 3.

3. **Preliminaries.**

3.1. **Some auxiliary operators.** The Lagrange interpolation operator $\pi_h : C(\overline{\Omega}) \to X_h$ is defined by $\pi_h v(P_i) = v(P_i)$ for all $P_i \in \mathcal{P}_h$. For $T \in \mathcal{T}_h$, we have

$$\|\pi_h v - v\|_{L^p(T)} \le C h_T^2 \|v\|_{W^{2,p}(T)} \quad (p \in (d/2,\infty],\ v \in W^{2,p}), \tag{12}$$

$$\|\nabla(\pi_h v - v)\|_{L^p(T)} \le C h_T \|v\|_{W^{2,p}(T)} \quad (p \in (d/2,\infty],\ v \in W^{2,p}), \tag{13}$$

$$\|\pi_h v - v\|_{L^\infty(T)} \le C h_T^{2-d/p} \|v\|_{W^{2,p}(T)} \quad (p \in (d,\infty],\ v \in W^{2,p}), \tag{14}$$

$$\|\nabla(\pi_h v - v)\|_{L^\infty(T)} \le C h_T^{1-d/p} \|v\|_{W^{2,p}(T)} \quad (p \in (d,\infty],\ v \in W^{2,p}). \tag{15}$$

For the proof of these inequalities, we refer to [2]. We note that (12) and (13) give

$$\|\pi_h v - v\|_p + h\|\nabla(\pi_h v - v)\|_p \le Ch\|v\|_{2,p} \quad (p \in (d/2,\infty],\ v \in W^{2,p}).$$

Moreover, (14) and (15) imply

$$\|\pi_h v - v\|_\infty + h\|\nabla(\pi_h v - v)\|_\infty \le Ch^{2-d/p}\|v\|_{2,p} \quad (p \in (d,\infty],\ v \in W^{2,p}).$$

We frequently use the L^2 and H^1 projection operators $P_h : L^2 \to X_h$ and $R_h : H^1 \to X_h$, which are defined as

$$(P_h v - v, \chi_h) = 0 \qquad (\forall \chi_h \in X_h), \tag{16}$$

$$(\nabla R_h v - \nabla v, \nabla \chi_h) + (R_h v - v, \chi_h) = 0 \qquad (\forall \chi_h \in X_h). \tag{17}$$

Under Assumption (H2), we have

$$\|P_h v\|_p \le C\|v\|_p \qquad (p \in [1,\infty],\ v \in L^p), \tag{18}$$

$$\|P_h v\|_{1,p} \le C\|v\|_{1,p} \qquad (p \in [1,\infty],\ v \in W^{1,p}), \tag{19}$$

$$\|P_h v - v\|_p \le Ch^2\|v\|_{2,p} \qquad (p \in (d/2,\infty],\ v \in W^{2,p}), \tag{20}$$

$$\|P_h v - v\|_{1,\infty} \le Ch^{1-d/p}\|v\|_{2,p} \qquad (p \in (d,\infty],\ v \in W^{2,p}). \tag{21}$$

Inequalities (18) and (19) are attributed to [8], [7] and [4]. To show (20), we note that $\|P_h v - v\|_p \le \|P_h v - \pi_h v\|_p + \|\pi_h v - v\|_p \le C\|\pi_h v - v\|_p$ by (18). Hence, (20) follows from (12). Similarly, (21) follows from (19), (14) and (15).

On the other hand, under Assumptions (H2) and (A1), we have

$$\|R_h v\|_{1,p} \le C\|v\|_{1,p} \qquad (p \in (1,\infty],\ v \in W^{1,p}), \tag{22}$$

$$\|R_h v - v\|_{1,p} \le Ch\|v\|_{2,p} \qquad (p \in (1,\infty],\ v \in W^{2,p}), \tag{23}$$

$$\|R_h v - v\|_{p} \le Ch^2\|v\|_{2,p} \qquad (p \in (\mu/(\mu-1),\infty),\ v \in W^{2,p}), \tag{24}$$

$$\|\nabla(R_h v - v)\|_{\infty} \le Ch^{1-d/p}\|v\|_{2,p} \qquad (p \in (d,\infty],\ v \in W^{2,p}). \tag{25}$$

In fact, the derivation of (22)–(24) is the same as that shown in [2, Chapter 8]. Therein, the case of the Dirichlet boundary condition was considered explicitly. The proof of (25) is the same as that of (21).

Let M_h^* be the adjoint operator of M_h in L^2, and set

$$K_h = M_h^* M_h.$$

Thereby, we have

$$C\|v_h\|_p \le \|M_h v_h\|_p \le C'\|v_h\|_p \qquad (p \in [1,\infty],\ v_h \in X_h). \tag{26}$$

Moreover,

$$C\|v_h\|_p \le \|K_h v_h\|_p \le C'\|v_h\|_p \qquad (p \in [1,\infty],\ v_h \in X_h), \tag{27}$$

and

$$\|M_h v_h - v_h\|_p \le Ch\|\nabla v_h\|_p \qquad (p \in [1,\infty],\ v_h \in X_h). \tag{28}$$

See [13] and [6] for these inequalities. Furthermore, in the same way as the proof of [24, Lemma 4], if Assumption (H2) is satisfied, we have

$$\|(K_h - I)v_h\|_p \le Ch\|\nabla v_h\|_p \qquad (p \in [1,\infty), v_h \in X_h). \tag{29}$$

3.2. **Discrete Laplace operator.** We introduce operators L_h and A_h of $X_h \to X_h$ defined as

$$L_h u_h = f_h \quad \Leftrightarrow \quad (\nabla u_h, \nabla \chi_h) + (u_h, \chi_h) = (f_h, \chi_h) \quad (\forall \chi_h \in X_h), \tag{30}$$

$$A_h u_h = f_h \quad \Leftrightarrow \quad (\nabla u_h, \nabla \chi_h) + (u_h, \chi_h)_h = (f_h, \chi_h)_h \quad (\forall \chi_h \in X_h). \tag{31}$$

Obviously, we have

$$K_h A_h - L_h = K_h - I \tag{32}$$

and, for $p \in [1,\infty)$,

$$L_h R_h v = P_h A v \qquad (v \in \mathcal{D}(A)), \tag{33}$$

where $A = A_p$ is the operator defined as (6).

Remark 3.1. In [25], we used the identity $K_h A_h = L_h$ that is incorrect. However, this can be replaced by (32) and then we can conclude the proof with some slight modifications.

To state operator theoretical properties of A_h, we regard any function space as a complex valued one, and propose a re-definition:

$$(u, v) = \int_\Omega u(x)\overline{v(x)}\ \mathrm{d}x \qquad \left(u \in L^p, v \in L^q,\ \tfrac{1}{p} + \tfrac{1}{q} = 1\right).$$

For $p \in [1,\infty)$, we introduce the discrete L^p norm:

$$\|v_h\|_{h,p} = \left(\int_\Omega M_h \pi_h |v_h(x)|^p\ \mathrm{d}x\right)^{1/p} \qquad (v_h \in X_h).$$

It is readily verifiable that

$$C\|v_h\|_{h,p} \le \|v_h\|_p \le C'\|v_h\|_{h,p} \qquad (v_h \in X_h), \tag{34}$$

$$|(v_h,\chi_h)_h| \le \|v_h\|_{h,p}\|\chi_h\|_{h,q} \qquad \left(v_h,\chi_h \in X_h,\ \tfrac{1}{p}+\tfrac{1}{q}=1\right), \tag{35}$$

$$\|v_h\|_{h,p} \le C \sup_{\chi_h\in X_h} \frac{(v_h,\chi_h)_h}{\|\chi_h\|_{h,q}} \qquad \left(v_h \in X_h,\ \tfrac{1}{p}+\tfrac{1}{q}=1\right). \tag{36}$$

We regard X_h as a Banach space equipped with the norm $\|\cdot\|_{h,p}$ and indicate it by $\mathcal{X}_{h,p}$. In particular, $\mathcal{X}_{h,2}$ forms a Hilbert space with respect to the inner product $(\cdot,\cdot)_h$. Furthermore, the operator norm in $\mathcal{X}_{h,p}$ is denoted by the same symbol $\|\cdot\|_{h,p}$. For instance,

$$\|A_h\|_{h,p} = \sup_{v_h\in X_h} \frac{\|A_h v_h\|_{h,p}}{\|v_h\|_{h,p}}. \tag{37}$$

Lemma 3.1 ([25, Lemma 4.3])**.** *Let $p \in (1,\infty)$, and suppose that* (H1) *is satisfied. Then,*

(i) *A_h is sectorial in $\mathcal{X}_{h,p}$, and its fractional powers A_h^α, $\alpha \in (0,1)$, are defined.*
(ii) *A_h and A_h^α, $\alpha\in(0,1)$, are positive and self-adjoint in $\mathcal{X}_{h,2}$.*
(iii) *if* (H2) *is also satisfied, for any $\theta\in[0,1]$ and $\{\tau_j\}_{j=1}^n$, $\tau_j>0$, we have*

$$\|r(\tau_n A_h)\cdots r(\tau_1 A_h)A_h^\theta\|_{h,p} \le C_\theta(\tau_n+\cdots+\tau_1)^{-\theta}, \tag{38}$$

where $r(\tau_j A_h) = (I+\tau_j A_h)^{-1}$.

Remark 3.2. Since A_h^θ and $r(\tau_n A_h)$ are commutative, Inequality (38) implies

$$\|A_h^\theta r(\tau_n A_h)\cdots r(\tau_1 A_h)\|_{h,p} \le C_\theta(\tau_n+\cdots+\tau_1)^{-\theta}. \tag{39}$$

Lemma 3.2. *Under Assumption* (H1)*, we have*

$$\|v_h\|_{h,p} \le C\|A_h^\theta v_h\|_{h,p} \qquad (p\in(1,\infty),\ \theta\in[0,1],\ v_h\in X_h). \tag{40}$$

Proof. The sectorialness of A_h implies

$$\|v_h\|_{h,p} \le C\|A_h v_h\|_{h,p} \qquad (p\in(1,\infty),\ v_h\in X_h).$$

Hence, by Heinz's inequality, we deduce (40). □

Lemma 3.3. *Let* (H1) *and* (H2) *be satisfied. Further we suppose that* (A1) *and* (A2) *hold, respectively, with some $\mu\in(d,\infty)$ and for some $p\in(d,\mu)$. Then we have*

$$\|A_h^\theta v_h\|_{h,p} \le C\|v_h\|_{1,p} \qquad (\theta\in[0,1/2),\ v_h\in X_h), \tag{41}$$

and

$$\|v_h\|_{1,p} \le C\|A_h^\theta v_h\|_{h,p} \qquad (\theta\in(1/2,1],\ v_h\in X_h). \tag{42}$$

(When $p=2$, we can take $\theta=1/2$ without (A1) *and* (A2)*.)*

Proof. It is described in Appendix A of [25]. □

Remark 3.3. If, in addition to the assumptions of Lemma 3.3, we suppose that (11) holds, we can prove

$$C\|v_h\|_{1,p} \le \|A_h^{1/2} v_h\|_{h,p} \le C_2\|v_h\|_{1,p} \qquad (v_h\in X_h).$$

Lemma 3.4. *Under the same assumption of Lemma 3.3, we have*

$$\|\nabla A_h^{-\theta} v_h\|_p \le \|A_h^{-\theta} v_h\|_{1,p} \le C\|v_h\|_{h,p} \qquad (\theta \in (1/2,1],\ v_h \in X_h). \tag{43}$$

Proof. The replacement v_h by $A_h^{-\theta} v_h$ in (42) implies (43). □

Lemma 3.5 ([25, Lemma 4.6])**.** *Under the same assumption of Lemma 3.3,*

$$\|A_h^{-\theta}(K_h^{-1} - I)v_h\|_{h,p} \le Ch^2\|\nabla v_h\|_p \qquad (\theta \in (1/2,1],\ v_h \in X_h). \tag{44}$$

Remark 3.4. Let $p \in (1,\infty)$ and (H2) be satisfied. Then, we have

$$\|(K_h^{-1} - I)v_h\|_{h,p} \le Ch\|\nabla v_h\|_p \qquad (v_h \in X_h) \tag{45}$$

without (H1), (A1) and (A2). To verify this, we note that by (29) and (34)

$$\begin{aligned} ((K_h^{-1} - I)v_h, \chi_h)_{h,p} &= ((I - K_h)v_h, \chi_h) \\ &\le \|(I - K_h)v_h\|_p \|\chi_h\|_q \\ &\le Ch\|\nabla v_h\|_p \|\chi_h\|_{h,q} \end{aligned}$$

for $\chi_h \in X_h$ and $1/p + 1/q = 1$. This, together with (37), implies (45).

Lemma 3.6. *Under the same assumption of Lemma 3.3, we have*

$$\|v_h\|_{1,\infty} \le C\|A_h v_h\|_{h,p} \qquad (v_h \in X_h). \tag{46}$$

Proof. Let $v_h \in X_h$. According to (32), (27), (29), (34), and (42), we deduce

$$\begin{aligned} \|L_h v_h\|_p &\le \|K_h A_h v_h\|_p + \|(K_h - I)v_h\|_p \\ &\le C\|A_h v_h\|_p + Ch\|\nabla v_h\|_p \\ &\le C\|A_h v_h\|_{h,p} + Ch\|A_h v_h\|_{h,p}. \end{aligned}$$

On the other hand, we know (cf. [25, Lemma 4.5])

$$\|v_h\|_{1,\infty} \le C\|L_h v_h\|_p.$$

Combining these inequalities, we obtain (46). □

Lemma 3.7 ([25, Lemma 4.7])**.** *Taking positive constants $\tau_1, \ldots, \tau_l$ ($l \in \mathbb{N}$), putting $t_n = \tau_1 + \cdots + \tau_n$ for $1 \le n \le l$ and $t_0 = 0$, suppose that a sequence $\{z_n\}_{n=0}^l \subset \mathbb{R}$ satisfies*

$$0 < z_n \le c_1 + c_2 \sum_{j=1}^{n} \frac{\tau_j}{(t_n - t_{j-1})^r}(z_{j-1} + z_j) \qquad (1 \le n \le l),$$

where c_1, c_2 and $r \in (0,1)$ are positive constants. Then we have

$$z_n \le c_1 c_3 \exp\left(c_4 c_2^{\frac{1}{1-r}} t_n\right) \qquad (0 \le n \le l),$$

where c_3 and c_4 are positive constants depending only on r.

3.3. Lemmas concerning b and b_h.

Lemma 3.8. *Let $p, q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$|b(v, u, \chi)| \leq \|u\|_\infty \|\nabla v\|_p \|\nabla \chi\|_q \quad (v \in W^{1,p}, u \in L^\infty, \chi \in W^{1,q}). \tag{47}$$

Furthermore, if $p > d$,

$$|b(v, u, \chi)| \leq C\|u\|_{1,p} \|\nabla v\|_{1,p} \|\chi\|_q \quad (v \in \mathcal{W}_p, u \in W^{1,p}, \chi \in W^{1,q}). \tag{48}$$

Proof. Inequality (47) is obvious in view of Schwarz's inequality. On the other hand, by integration by parts,

$$b(v, u, \chi) = \int_\Omega \nabla (u \nabla v) \ \chi \ dx$$

for $v \in \mathcal{W}_p$, $u \in W^{1,p}$ and $\chi \in W^{1,q}$. Since $p > d$, we can perform an estimation:

$$\|\nabla \cdot (u \nabla v)\|_p \leq C\|u\|_{1,p} \|\nabla v\|_{1,p}. \tag{49}$$

Combining these, we obtain (48). □

Remark 3.5. In virtue of Lemma 3.8, the trilinear form b primarily defined on $W^{1,p} \times L^\infty \times W^{1,q}$ can be re-defined as that on $\mathcal{W}_p \times W^{1,p} \times L^q$.

Lemma 3.9. *Suppose that* (H2) *is satisfied. Let $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\begin{aligned} &|b_h(v_h, u_h, \chi_h) - b_h(w_h, u_h, \chi_h)| \\ &\quad \leq Ch \left(\|\nabla v_h\|_\infty + \|\nabla w_h\|_\infty\right) \|u_h\|_{1,p} \|\nabla \chi_h\|_q \\ &\quad\quad + C\|\nabla (v_h - w_h)\|_\infty \|u_h\|_{1,p} \|\nabla \chi_h\|_q \quad (v_h, w_h, u_h, \chi_h \in X_h). \end{aligned} \tag{50}$$

Proof. Let $v_h, w_h, u_h, \chi_h \in X_h$. In general, for $v_h \in X_h$, we write as $v = v_h$ and $v_i = v(P_i) = v_h(P_i)$ for the sake of simplicity. Defining

$$\Gamma_h = \{\Gamma_{ij} = \partial D_i \cap \partial D_j \mid 1 \leq i, j \leq N\}, \tag{51}$$

we observe that

$$\begin{aligned} b_h(w_h, u_h, \chi_h) &= \sum_{\Gamma_{ij} \in \Gamma_h} (\chi_i - \chi_j) \left[\beta_{ij}^+(w) u_i - \beta_{ij}^-(w) u_j\right] \\ &= \sum_{\Gamma_{ij} \in \Gamma_h} (\chi_i - \chi_j) \left[\sigma_{ij}^+(w) u_i - \sigma_{ij}^-(w) u_j\right] \beta_{ij}(w), \end{aligned} \tag{52}$$

where

$$\sigma_{ij}^+(w) = \operatorname{sgn} \beta_{ij}^+(w), \qquad \sigma_{ij}^-(w) = 1 - \sigma_{ij}^+(w).$$

Using this expression, we can decompose as

$$\begin{aligned} b_h(v_h, u_h, \chi_h) - b_h(w_h, u_h, \chi_h) &= \sum_{\Gamma_{ij} \in \Gamma_h} (\chi_i - \chi_j) \, \sigma_{ij}^+(v) \, (u_i - u_j) \, \beta_{ij}(v) \\ &\quad - \sum_{\Gamma_{ij} \in \Gamma_h} (\chi_i - \chi_j) \, \sigma_{ij}^+(w) \, (u_i - u_j) \, \beta_{ij}(w) \\ &\quad + \sum_{\Gamma_{ij} \in \Gamma_h} (\chi_i - \chi_j) \, u_j \left[\beta_{ij}(v) - \beta_{ij}(w)\right] \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

Below, we use Sobolev's inequality

$$\|v\|_\infty \le C\|v\|_{1,p} \qquad (p \in (d,\infty],\ v \in W^{1,p}), \tag{53}$$

and

$$\max_{x,y\in T} |\chi_h(x) - \chi_h(y)| \le C h_T^{1-d/p} \|\nabla \chi_h\|_{L^p(T)} \quad (p \in [1,\infty],\ T \in \mathcal{T}_h). \tag{54}$$

Let $\Delta_{ij} = \{T \in \mathcal{T}_h \mid P_i \in T \text{ and } P_j \in T\}$ and set

$$h_{ij} = \max_{T\in\Delta_{ij}} h_T.$$

Further, we write $T_{ij} = T$ if $h_{ij} = h_T$ for $T \in \Delta_{ij}$.

Now, we have by (54)

$$\begin{aligned} |I_1| &\le C\|\nabla v\|_\infty \sum_{\Gamma_{ij}\in\Gamma_h} h_{ij}^{1-d/q}\|\nabla\chi\|_{L^q(T_{ij})} h_{ij}^{1-d/p}\|\nabla u\|_{L^p(T_{ij})} \mathrm{meas}_{\,d-1}(\Gamma_{ij}) \\ &\le C\|\nabla v\|_\infty \sum_{\Gamma_{ij}\in\Gamma_h} h_{ij}^{2-d/q-d/p+(d-1)}\|\nabla\chi\|_{L^q(T_{ij})}\|\nabla u\|_{L^p(T_{ij})} \\ &\le Ch\|\nabla v\|_\infty\|\nabla\chi\|_q\|\nabla u\|_p, \end{aligned}$$

and

$$|I_2| \le Ch\|\nabla w\|_\infty\|\nabla\chi\|_q\|\nabla u\|_p.$$

Moreover, in view of (H2), we have by (53)

$$\begin{aligned} |I_3| &\le C \sum_{\Gamma_{ij}\in\Gamma_h} h_{ij}^{1-d/q}\|\nabla\chi\|_{L^q(T_{ij})}\|u\|_\infty \cdot h_{ij}^{d-1}\|\nabla(v-w)\|_\infty \\ &\le C\|\nabla(v-w)\|_\infty\|u\|_\infty \sum_{\Gamma_{ij}\in\Gamma_h} h_{ij}^{d/p}\|\nabla\chi\|_{L^q(T_{ij})} \\ &\le C\|\nabla(v-w)\|_\infty\|u\|_{1,p}\left(\sum_{\Gamma_{ij}\in\Gamma_h} h_{ij}^d\right)^{1/p}\left(\sum_{\Gamma_{ij}\in\Gamma_h}\|\nabla\chi\|_{L^q(T_{ij})}^q\right)^{1/q} \\ &\le C\|\nabla(v-w)\|_\infty\|u\|_{1,p}\|\nabla\chi\|_q. \end{aligned}$$

Here, we used

$$\begin{aligned} \sum_{\Gamma_{ij}\in\Gamma_h} h_{ij}^d \le \sum_{T\in\mathcal{T}_h} h_T^d &\le \frac{(2\gamma_1)^d}{c_d}\sum_{T\in\mathcal{T}_h} c_d\left(\frac{\rho_T}{2}\right)^d \\ &\le \frac{(2\gamma_1)^d}{c_d}\mathrm{meas}_d(\Omega), \end{aligned} \tag{55}$$

where $c_2 = \pi$ and $c_3 = 4\pi/3$.

Summing these estimates, we obtain (50). □

Lemma 3.10 ([25, Lemma 5.2])**.** *Let $p, q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$. Then,*

$$|b_h(v_h,u_h,\chi_h)| \le \begin{cases} C\|\nabla v_h\|_p\|u_h\|_\infty\|\nabla\chi_h\|_q, \\ C\|\nabla v_h\|_\infty\|u_h\|_p\|\nabla\chi_h\|_q \end{cases} \qquad (v_h,u_h,\chi_h \in X_h). \tag{56}$$

Lemma 3.11. *Suppose that* (H2) *is satisfied. Let $p \in (d,\infty)$ and $\frac{1}{p}+\frac{1}{q}=1$. Then,*

$$|b(v,u,\chi_h) - b_h(P_h v, \pi_h u, \chi_h)| \le C(h^{1-d/p}+h)\|v\|_{2,p}\|u\|_{1,p}\|\nabla\chi_h\|_q \qquad (v \in \mathcal{W}_p, u \in W^{1,p}, \chi_h \in X_h). \tag{57}$$

Proof. We simply write as $u_h = \pi_h u$, $v_h = P_h v$, $\overline{\chi} = M_h \chi_h$, $u_i = u(P_i) = u_h(P_i)$, $\chi_i = \chi_h(P_i)$, $\sigma_{ij}^{\pm} = \sigma_{ij}^{\pm}(v_h)$, and $\beta_{ij} = \beta_{ij}(v_h)$. Further, we set

$$\hat{\beta}_{ij} = \int_{\Gamma_{ij}} u \left(\nabla v \cdot \nu_{ij}\right) \, dS, \quad \hat{\beta}'_{ij} = \int_{\Gamma_{ij}} \nabla v \cdot \nu_{ij} \, dS.$$

In order to prove (57), we divide it as

$$\begin{aligned} & b(v,u,\chi_h) - b_h(v_h,u_h,\chi_h) \\ = \; & -\int_\Omega u \nabla v \cdot \nabla \chi \, dx - \int_\Omega \overline{\chi} \, \nabla(u \nabla v) \, dx \\ & + \int_\Omega \overline{\chi} \, \nabla(u\nabla v) \, dx - \sum_{\Gamma_{ij}\in\Gamma_h} (\chi_i - \chi_j) \left(\sigma_{ij}^+ u_i + \sigma_{ij}^- u_j\right) \hat{\beta}'_{ij} \\ & + \sum_{\Gamma_{ij}\in\Gamma_h} (\chi_i - \chi_j) \left(\sigma_{ij}^+ u_i + \sigma_{ij}^- u_j\right) \hat{\beta}'_{ij} - b_h(v_h,u_h,\chi_h) \\ \equiv \; & I_1 + I_2 + I_3, \end{aligned}$$

where Γ_h is defined as (51). First, by the integration by parts, we have

$$I_1 = \int_\Omega \chi_h \, \nabla(u\nabla v) \, dx - \int_\Omega \overline{\chi} \, \nabla(u\nabla v) \, dx.$$

Hence, by (49) and (28),

$$\begin{aligned} |I_1| & \le C\|u\|_{1,p}\|\nabla v\|_{1,p}\|\chi_h - M_h\chi_h\|_q \\ & \le Ch\|u\|_{1,p}\|v\|_{2,p}\|\nabla\chi_h\|_q. \end{aligned}$$

Next, because of

$$\int_\Omega \overline{\chi} \, \nabla(u\nabla v) \, dx = \sum_{\Gamma_{ij}\in\Gamma_h} (\chi_i - \chi_j)\hat{\beta}_{ij},$$

we can express I_2 as

$$\begin{aligned} I_2 & = \sum_{\Gamma_{ij}\in\Gamma_h} (\chi_i - \chi_j) \left[\hat{\beta}_{ij} - (\sigma_{ij}^+ u_i + \sigma_{ij}^- u_j)\hat{\beta}'_{ij}\right] \\ & = \sum_{\Gamma_{ij}\in\Gamma_h} (\chi_i - \chi_j) \int_{\Gamma_{ij}} \left[\sigma_{ij}^+(u(x) - u_i) + \sigma_{ij}^-(u(x) - u_j)\right] (\nabla v \cdot \nu_{ij}) \, dS. \end{aligned}$$

Therefore, in view of (53) and (54), we deduce

$$\begin{aligned} |I_2| & \le C \sum_{\Gamma_{ij}\in\Gamma_h} h_{ij}^{1-d/q} \|\nabla\chi_h\|_{L^q(T_{ij})} \|\nabla v\|_\infty \cdot \\ & \qquad \cdot \int_{\Gamma_{ij}} \left(|u(x) - u_i| + |u(x) - u_j|\right) \, dS \\ & \le C\|\nabla v\|_\infty \sum_{\Gamma_{ij}\in\Gamma_h} h_{ij}^{1-d/q} \|\nabla\chi_h\|_{L^q(T_{ij})} h_{ij}^{d-d/p} \|u\|_{W^{1,p}(T_{ij})} \\ & \le Ch\|u\|_{1,p}\|\nabla v\|_\infty\|\nabla\chi_h\|_q, \end{aligned}$$

where h_{ij} and T_{ij} are those defined in the proof of Lemma 3.9. Finally, by virtue of (52), (21) and (55), we have

$$\begin{aligned}
|I_3| &\le \sum_{\Gamma_{ij}\in\Gamma_h} |\chi_i-\chi_j|\cdot\left|\sigma_{ij}^{+}u_i+\sigma_{ij}^{-}u_j\right|\cdot\left|\hat\beta'_{ij}-\beta_{ij}\right| \\
&\le C\sum_{\Gamma_{ij}\in\Gamma_h} h_{ij}^{1-d/q}\|\nabla\chi_h\|_{L^q(T_{ij})}\|u\|_\infty\int_{\Gamma_{ij}}\|\nabla(v-P_hv)\|_\infty|\nu_{ij}|\,dS \\
&\le Ch^{1-d/p}\|v\|_{2,p}\|u\|_\infty\sum_{\Gamma_{ij}\in\Gamma_h} h_{ij}^{1-d/q+(d-1)}\|\nabla\chi_h\|_{L^q(T_{ij})} \\
&\le Ch^{1-d/p}\|v\|_{2,p}\|u\|_{1,p}\|\nabla\chi_h\|_q.
\end{aligned}$$

□

4. **Proof of Theorem 2.4.**

4.1. **Expression of the error.** We shall give the proof of Theorem 2.4. Throughout this section, we suppose that (H1) and (H2) are satisfied. We set $\delta=1/8$ and $\theta=7/8$; then $\theta+\delta=1$ and $\theta-\delta=3/4>1/2$. Moreover, we suppose that (A1) and (A2) are satisfied, respectively, with some $\mu\in(d,\infty)$ and for some $p\in(d,\mu)$. Further, we take $k=k_1=k_2=D_u=D_v=\lambda=1$ without loss of generality, since the contributions of those values are not essential in the following discussion. Recall that the solution (u,v) of (3) satisfies the regularity condition (7) for some $J\in(0,\infty)$ and $\sigma\in(0,1]$. Then we note that the system (3) can be expressed as

$$\frac{du(t)}{dt}+Au(t)+B(v(t))u(t)=u(t),\qquad 0<t<J, \tag{58a}$$

$$\frac{dv(t)}{dt}+Av(t)=u(t),\qquad 0<t<J, \tag{58b}$$

$$u(0)=u_0,\qquad v(0)=v_0, \tag{58c}$$

where $A=A_p:\mathcal{D}(A)\subset L^p\to L^p$ is the operator defined as (6), and, for every $v\in W^{2,p}$, $B(v):W^{1,p}\to L^p$ is defined by $B(v)u=\nabla(u\nabla v)$. Moreover, we set

$$\alpha_1=\sup_{t\in[0,J]}\|u(t)\|_{2,p},\quad \hat\alpha_1=\sup_{t\in[0,J]}\|v(t)\|_{2,p},$$

$$\alpha_2=\sup_{t\in[0,J]}\|u'(t)\|_{2,p},\quad \hat\alpha_2=\sup_{t\in[0,J]}\|v'(t)\|_{2,p},$$

$$\alpha_3=\sup_{t,s\in[0,J]}\frac{\|u'(t)-u'(s)\|_p}{|t-s|^\sigma},\quad \hat\alpha_3=\sup_{t,s\in[0,J]}\frac{\|v'(t)-v'(s)\|_{1,p}}{|t-s|^\sigma},$$

where $u'=u_t$ and $v'=v_t$.

Let (u_h^n,v_h^n) be the solution of (4). The errors are decomposed as

$$\begin{aligned}
u(t_n)-u_h^n &= \eta_h^n+w_h^n,\\
v(t_n)-v_h^n &= \hat\eta_h^n+\hat w_h^n,
\end{aligned}$$

where $\eta_h^n=u(t)-U_h(t_n)$, $\hat\eta_h^n=v(t)-V_h(t_n)$, $w_h^n=U_h(t_n)-u_h^n$, $\hat w_h^n=V_h(t_n)-v_h^n$, $U_h(t)=R_hu(t)$ and $V_h(t)=R_hv(t)$. We have by (24) and (25)

$$\|\eta_h^n\|_p\le Ch^2\alpha_1,\quad \|\hat\eta_h^n\|_{1,\infty}\le Ch^{1-d/p}\hat\alpha_1. \tag{59}$$

Hence, it suffices to consider the estimates for w_h^n and $\hat w_h^n$. To this end, we first characterize w_h^n and $\hat w_h^n$ as solutions of discrete parabolic equations and then apply the discrete Duhamel's principle to obtain estimations for them. Now, we introduce, for any $v_h \in X_h$, the operator $B_h(v_h) : X_h \to X_h$ defined by

$$(B_h(v_h)u_h, \chi_h) = b_h(v_h, u_h, \chi_h) \qquad (u_h, \chi_h \in X_h),$$

and recall that $A_h : X_h \to X_h$ is defined by (31). We set $U_h^n = U_h(t_n)$ and $V_h^n = V_h(t_n)$. Moreover, we introduce the backward Euler difference operator

$$\partial_{\Delta t} w_h^n = \frac{w_h^n - w_h^{n-1}}{\Delta t}$$

for $\Delta t > 0$. Then, using (4a), we observe that

$$\begin{aligned}(\partial_{\tau_n} w_h^n, \chi_h)_h &+ (\nabla w_h^n, \nabla \chi_h) + (w_h^n, \chi_h)_h \\ &= (\partial_{\tau_n} U_h^n, \chi_h)_h + (\nabla U_h^n, \nabla \chi_h) + (U_h^n, \chi_h)_h + b_h(v_h^{n-1}, u_h^n, \chi_h) - (u_h^n, \chi_h)_h\end{aligned}$$

for any $\chi_h \in X_h$; equivalently,

$$\begin{aligned}\partial_{\tau_n} w_h^n + A_h w_h^n &= \partial_{\tau_n} U_h^n + A_h U_h^n + K_h^{-1} B_h(v_h^{n-1}) u_h^n - u_h^n \\ &\equiv F_h^n.\end{aligned}$$

Thus, by the discrete Duhamel's principle, we obtain the following identity:

$$w_h^n = E_{n,1} w_h^0 + \sum_{j=1}^{n} \tau_j E_{n,j} F_h^j, \tag{60}$$

where

$$\begin{aligned}r(\tau_j A_h) &= (I + \tau_j A_h)^{-1}, \quad \left(r(s) = (1+s)^{-1}\right), \\ E_{n,j} &= r(\tau_n A_h) r(\tau_{n-1} A_h) \cdots r(\tau_j A_h)\end{aligned}$$

for $1 \le j \le n$. By virtue of (33) and (32), we have

$$\begin{aligned}&K_h^{-1} P_h u'(t_j) + A_h U_h^j \\ &\quad = K_h^{-1} P_h u'(t_j) + K_h^{-1} L_h \cdot L_h^{-1} P_h A u(t_j) - K_h^{-1} L_h U_h^j + A_h U_h^j \\ &\quad = K_h^{-1} P_h \left[u(t_j) - B(u(t_j)) u(t_j)\right] + (I - K_h^{-1}) U_h^j.\end{aligned}$$

So F^j is written as

$$\begin{aligned}F^j &= P_h \partial_{\tau_j} (U_h^j - u(t_j)) + P_h(\partial_{\tau_j} u(t_j) - u'(t_j)) + (I - K_h^{-1}) P_h u'(t_j) \\ &\quad + K_h^{-1} P_h \left[B_h(v_h^{j-1}) u_h^j - B(v(t_j)) u(t_j)\right] + K_h^{-1} P_h u(t_j) - u_h^j + (I - K_h^{-1}) U_h^j.\end{aligned}$$

Therefore, we have

$$w_h^n = I_0 + I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_0 &= E_{n,1}w_h^0 - \sum_{j=1}^{n} \tau_j E_{n,j} P_h \partial_{\tau_j} \eta_h^j, \\ I_1 &= \sum_{j=1}^{n} \tau_j E_{n,j} (K_h^{-1} P_h u(t_j) - u_h^j), \\ I_2 &= \sum_{j=1}^{n} \tau_j E_{n,j} P_h (\partial_{\tau_j} u(t_j) - u'(t_j)), \\ I_3 &= \sum_{j=1}^{n} \tau_j E_{n,j} (I - K_h^{-1}) \left[P_h u'(t_j) + U_h^j \right], \\ I_4 &= \sum_{j=1}^{n} \tau_j E_{n,j} K_h^{-1} \left[B_h(v_h^{j-1}) u_h^j - P_h B(v(t_j)) u(t_j) \right]. \end{aligned}$$

In the similar way, we obtain

$$\hat{w}_h^n = \hat{I}_0 + I_1 + \hat{I}_2 + \hat{I}_3,$$

where

$$\begin{aligned} \hat{I}_0 &= E_{n,1}\hat{w}_h^0 - \sum_{j=1}^{n} \tau_j E_{n,j} P_h \partial_{\tau_j} \hat{\eta}_h^j, \\ \hat{I}_2 &= \sum_{j=1}^{n} \tau_j E_{n,j} P_h (\partial_{\tau_j} v(t_j) - v'(t_j)), \\ \hat{I}_3 &= \sum_{j=1}^{n} \tau_j E_{n,j} (I - K_h^{-1}) \left[P_h v'(t_j) + V_h^j \right]. \end{aligned}$$

4.2. **Some estimates.** In the following lemmas, we always assume all assumptions described in the beginning of the previous subsection.

Lemma 4.1.

$$\|A_h^\delta I_0\|_{h,p} \le Ch(\alpha_1 + \alpha_0) + ChJ\alpha_2, \tag{61}$$

$$\|A_h \hat{I}_0\|_{h,p} \le Ct_n^{-1} h^2 (\hat{\alpha}_1 + \alpha_0) + ChJ^\delta \hat{\alpha}_2. \tag{62}$$

Proof. Since

$$\eta_h^j - \eta_h^{j-1} = \int_{t_{j-1}}^{t_j} [u'(s) - R_h u'(s)]\, ds,$$

we have by (41) and (23)

$$\begin{aligned} \|A_h^\delta P_h(\eta_h^j - \eta_h^{j-1})\|_{h,p} &\le C\|P_h(\eta_h^j - \eta_h^{j-1})\|_{1,p} \\ &\le C\|\eta_h^j - \eta_h^{j-1}\|_{1,p} \\ &\le C\int_{t_{j-1}}^{t_j} \|u'(s) - R_h u'(s)\|_{1,p}\, ds \\ &\le C\tau_j \cdot Ch\alpha_2. \end{aligned}$$

On the other hand, by (38) and (39), we obtain

$$\begin{aligned}\|A_h^\delta E_{n,1} w_h^0\|_{h,p} &\leq \|E_{n,1}\|_{h,p} \|A_h^\delta (U_h^0 - P_h u_0 + P_h u_0 - u_{0h})\|_{h,p} \\ &\leq C\left(\|P_h(R_h - I)u_0\|_{1,p} + \|P_h(u_0 - u_{0h})\|_{1,p}\right) \\ &\leq C\left(\|u_0 - R_h u_0\|_{1,p} + \|u_0 - u_{0h}\|_{1,p}\right) \\ &\leq Ch(\alpha_1 + \alpha_0).\end{aligned}$$

Hence, we can estimate as

$$\begin{aligned}\|A_h^\delta I_0\|_{h,p} &\leq \|A_h^\delta E_{n,1} w_h^0\|_{h,p} + \sum_{j=1}^n \tau_j \|E_{n,1}\|_{h,p} \|A_h^\delta P_h \partial_{\tau_j} \eta_h^j\|_{h,p} \\ &\leq Ch(\alpha_1 + \alpha_0) + Ch\alpha_2 t_n\end{aligned}$$

Similarly, we have

$$\|A_h^\delta P_h(\hat\eta_h^j - \hat\eta_h^{j-1})\|_{h,p} \leq C\tau_j h \hat\alpha_2,$$

and

$$\begin{aligned}\|A_h E_{n,1} \hat w_h^0\|_{h,p} &\leq \|A_h E_{n,1}\|_{h,p} \cdot C\left(\|(R_h - I)v_0\|_p + \|P_h(v_0 - v_{0h})\|_p\right) \\ &\leq C t_n^{-1} h^2 \left(\hat\alpha_1 + \alpha_0\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\|A_h \hat I_0\|_{h,p} &\leq \|A_h E_{n,1} \hat w_h^0\|_{h,p} + \sum_{j=1}^n \tau_j \|A_h^{1-\delta} E_{n,1}\|_{h,p} \|A_h^\delta P_h \partial_{\tau_j} \hat\eta_h^j\|_{h,p} \\ &\leq C t_n^{-1} h^2 \left(\hat\alpha_1 + \alpha_0\right) + Ch\hat\alpha_2 \sum_{j=1}^n \frac{\tau_j}{(t_n - t_{j-1})^{1-\delta}}.\end{aligned}$$

This, together with an elementary inequality

$$\sum_{j=1}^n \frac{\tau_j}{(t_n - t_{j-1})^\xi} \leq \int_0^{t_n} \frac{ds}{(t_n - s)^\xi} = t_n^{1-\xi} \leq J^{1-\xi} \qquad (0 \leq \xi \leq 1),$$

implies (62). □

Lemma 4.2.

$$\|A_h^\delta I_1\|_{h,p} \leq Ch^2 J^{1-\theta} \alpha_1 + C \sum_{j=1}^n \frac{\tau_j}{(t_n - t_{j-1})^\theta} \|A_h^\delta w_h^j\|_{h,p}, \tag{63}$$

$$\begin{aligned}\|A_h I_1\|_{h,p} &\leq Ch(\alpha_1 + \alpha_1 J^\delta + \alpha_2 J) \\ &\quad + C \sum_{j=1}^n \frac{\tau_j}{(t_n - t_{j-1})^{1-\delta}} \|A_h^\delta w_h^j\|_{h,p}.\end{aligned} \tag{64}$$

Proof. First we have by (44) and (40)

$$\begin{aligned}&\|A_h^{-\theta+\delta}(K_h^{-1} P_h u(t_j) - u_h^j)\|_{h,p} \\ &\leq \|A_h^{-\theta+\delta}(K_h^{-1} - I) P_h u(t_j)\|_{h,p} + \|A_h^{-\theta+\delta}(P_h u(t_j) - u_h^j)\|_{h,p} \\ &\leq Ch^2 \|\nabla P_h u(t_j)\|_{h,p} + C\|P_h u(t_j) - u_h^j\|_{h,p} \\ &\leq Ch^2 \alpha_1 + C(\|P_h u(t_j) - U_h(t_j)\|_{h,p} + \|U_h(t_j) - u_h^j\|_{h,p}) \\ &\leq Ch^2 \alpha_1 + C\|w_h^j\|_{h,p} \\ &\leq Ch^2 \alpha_1 + C\|A_h^\delta w_h^j\|_{h,p}.\end{aligned}$$

Hence,

$$\begin{aligned}\|A_h^{\delta} I_1\|_{h,p} &\leq \sum_{j=1}^{n} \tau_j \|E_{n,j} A_h^{\theta}\|_{h,p} \|A_h^{-\theta+\delta}(K_h^{-1} P_h u(t_j) - u_h^j)\|_{h,p} \\ &\leq C \sum_{j=1}^{n} \frac{\tau_j}{(t_n - t_{j-1})^{\theta}} \left(h^2 \alpha_1 + \|A_h^{\delta} w_h^j\|_{h,p} \right),\end{aligned}$$

which implies (63).

In order to derive (64), we write it as

$$\begin{aligned}I_1 &= \sum_{j=1}^{n} \tau_j E_{n,j} (K_h^{-1} - I) P_h u(t_j) \\ &\quad + \sum_{j=1}^{n} \tau_j E_{n,j} P_h (I - R_h) u(t_j) + \sum_{j=1}^{n} \tau_j E_{n,j} w_h^j \\ &\equiv I_{11} + I_{12} + I_{13}.\end{aligned}$$

Then, by (41), (38), (19) and (23),

$$\begin{aligned}\|A_h I_{12}\|_{h,p} &\leq \sum_{j=1}^{n} \tau_j \|E_{n,j} A_h^{1-\delta}\|_{h,p} \|A_h^{\delta} P_h (I - R_h) u(t_j)\|_{h,p} \\ &\leq C \sum_{j=1}^{n} \frac{\tau_j}{(t_n - t_{j-1})^{1-\delta}} \|P_h (I - R_h) u(t_j)\|_{1,p} \\ &\leq C \sum_{j=1}^{n} \frac{\tau_j}{(t_n - t_{j-1})^{1-\delta}} \|(I - R_h) u(t_j)\|_{1,p} \\ &\leq C \sum_{j=1}^{n} \frac{\tau_j}{(t_n - t_{j-1})^{1-\delta}} h \alpha_1 \\ &\leq C J^{\delta} h \alpha_1.\end{aligned}$$

Moreover,

$$\|A_h I_{13}\|_{h,p} \leq C \sum_{j=1}^{n} \frac{\tau_j}{(t_n - t_{j-1})^{1-\delta}} \|A_h^{\delta} w_h^j\|_{1,p}.$$

Therefore, it suffices to verify

$$\|A_h I_{11}\|_{h,p} \leq C h (\alpha_1 + J \alpha_2) \tag{65}$$

in order to prove (64).

To this end, we set $\chi_h^j = (K_h^{-1} - I) P_h u(t_j)$. Then, we have by (45) and (19)

$$\begin{aligned}\|\chi_h^n\|_{h,p} &\leq C h \|\nabla P_h u(t_n)\|_p \\ &\leq C h \alpha_1\end{aligned}$$

and

$$\begin{aligned}\|\chi_h^j - \chi_h^n\|_{h,p} &\leq C h \|\nabla (u(t_j) - u(t_n))\|_p \\ &\leq C h \int_{t_j}^{t_n} \|u'(s)\|_{1,p} \, ds \\ &\leq C h \alpha_2 (t_n - t_{j-1}).\end{aligned}$$

Moreover, we observe

$$\begin{aligned}\sum_{j=1}^{n}\tau_j A_h E_{n,j}\chi_h^n &= \sum_{j=1}^{n-1}(E_{n,j+1}-E_{n,j})\chi_h^n+\tau_n A_h E_{n,n}\chi_h^n\\ &= (E_{n,n}-E_{n,1})\chi_h^n+\tau_n A_h r(\tau_n A_h)\chi_h^n,\end{aligned}$$

and, hence,

$$\begin{aligned}\left\|\sum_{j=1}^{n}\tau_j A_h E_{n,j}\chi_h^n\right\|_{h,p} &\le (1+1)\|\chi_h^n\|_{h,p}+\tau_n\cdot C\tau_n^{-1}\|\chi_h^n\|_{h,p}\\ &\le C\|\chi_h^n\|_{h,p}. \end{aligned}\tag{66}$$

We combine these inequalities in the following way. Thus, we write as

$$A_h I_{11}=\sum_{j=1}^{n}\tau_j A_h E_{n,j}(\chi_h^j-\chi_h^n)-\sum_{j=1}^{n}\tau_j A_h E_{n,j}\chi_h^n$$

and estimate as

$$\begin{aligned}\|A_h I_{11}\|_{h,p} &\le \sum_{j=1}^{n}\tau_j\|A_h E_{n,j}(\chi_h^j-\chi_h^n)\|_{h,p}+\left\|\sum_{j=1}^{n}\tau_j A_h E_{n,j}\chi_h^n\right\|_{h,p}\\ &\le C\sum_{j=1}^{n}\frac{\tau_j}{t_n-t_{j-1}}\|\chi_h^j-\chi_h^n\|_{h,p}+C\|\chi_h^n\|_{h,p}\\ &\le Ch\alpha_2\sum_{j=1}^{n}\tau_j+Ch\alpha_1\\ &\le ChJ\alpha_2+Ch\alpha_1\end{aligned}$$

which implies (65). □

Lemma 4.3.

$$\|A_h^\delta I_2\|_{h,p} \le C\tau^\sigma J^{1-\delta}\alpha_3, \tag{67}$$

$$\|A_h\hat{I}_2\|_{h,p} \le C\tau^\sigma J^{1-\delta}\hat{\alpha}_3. \tag{68}$$

Proof. Since

$$I_2=\sum_{j=1}^{n}E_{n,j}P_h\int_{t_{j-1}}^{t_j}[u'(s)-u'(t_j)]\ ds,$$

we have

$$\begin{aligned}\|A_h^\delta I_2\|_{h,p} &\le C\sum_{j=1}^{n}\|A_h^\delta E_{n,j}\|_{h,p}\int_{t_{j-1}}^{t_j}\|u'(s)-u'(t_{j-1})\|_p\ ds\\ &\le C\sum_{j=1}^{n}\frac{1}{(t_n-t_{j-1})^\delta}\cdot\alpha_3\int_{t_{j-1}}^{t_j}(s-t_{j-1})^\sigma\ ds\\ &\le C\tau^\sigma J^{1-\delta}\alpha_3.\end{aligned}$$

Inequality (68) is obtained similarly. □

Lemma 4.4.

$$\|A_h^\delta I_3\|_{h,p} \leq Ch^2 J^{1-\theta}(\alpha_1+\alpha_2), \tag{69}$$

$$\|A_h \hat{I}_3\|_{h,p} \leq ChJ^\sigma \hat{\alpha}_3 + Ch(\hat{\alpha}_1+\hat{\alpha}_2). \tag{70}$$

Proof. Using (44), (19) and (22), we deduce

$$\begin{aligned}
\|A_h^\delta I_3\|_{h,p} &\leq \sum_{j=1}^n \|A_h^\theta E_{n,j}\|_{h,p}\|A_h^{-\theta+\delta}(I-K_h^{-1})P_h u'(t_j)\|_{h,p} \\
&\quad + \sum_{j=1}^n \|A_h^\theta E_{n,j}\|_{h,p}\|A_h^{-\theta+\delta}(I-K_h^{-1})U_h^j\|_{h,p} \\
&\leq \sum_{j=1}^n \frac{C\tau_j}{(t_n-t_{j-1})^\theta}\cdot Ch^2\|\nabla P_h u'(t_j)\|_p \\
&\quad + \sum_{j=1}^n \frac{C\tau_j}{(t_n-t_{j-1})^\theta}\cdot Ch^2\|\nabla R_h u(t_j)\|_p \\
&\leq Ch^2 J^{1-\theta}(\alpha_1+\alpha_2).
\end{aligned}$$

On the other hand, Inequality (70) is obtained in the same way as the proof of (65). Letting $\chi_h^j = (I-K_h^{-1})\left(P_h v'(t_j)+V_h^j\right)$, we have by (44) and (19),

$$\begin{aligned}
\|\chi_h^n\|_{h,p} &\leq Ch\left(\|\nabla P_h v'(t_n)\|_p + \|\nabla R_h v(t_n)\|_p\right) \\
&\leq Ch\left(\|v'(t_n)\|_{1,p} + \|v(t_n)\|_{1,p}\right) \\
&\leq Ch\left(\hat{\alpha}_1+\hat{\alpha}_2\right),
\end{aligned}$$

and

$$\begin{aligned}
\|\chi_h^j - \chi_h^n\|_{h,p} &\leq \|(I-K_h^{-1})P_h\left[v'(t_j)-v'(t_n)\right]\|_{h,p} \\
&\quad + \|(I-K_h^{-1})R_h\left[v'(t_j)-v'(t_n)\right]\|_{h,p} \\
&\leq Ch\|v'(t_j)-v'(t_n)\|_{1,p} \\
&\leq Ch\hat{\alpha}_3(t_n-t_j)^\sigma \\
&\leq Ch\hat{\alpha}_3(t_n-t_{j-1})^\sigma.
\end{aligned}$$

Then, writing as

$$A_h\hat{I}_3 = \sum_{j=1}^n \tau_j A_h E_{n,j}(\chi_h^j-\chi_h^n) - \sum_{j=1}^n \tau_j A_h E_{n,j}\chi_h^n,$$

we estimate by (66)

$$\begin{aligned}
\|A_h\hat{I}_3\|_{h,p} &\leq C\sum_{j=1}^n \frac{\tau_j}{t_n-t_{j-1}}\|\chi_h^j-\chi_h^n\|_{h,p} + C\|\chi_h^n\|_{h,p} \\
&\leq Ch\hat{\alpha}_3\sum_{j=1}^n \frac{\tau_j}{(t_n-t_{j-1})^{1-\sigma}} + Ch\left(\hat{\alpha}_1+\hat{\alpha}_2\right) \\
&\leq ChJ^\sigma\hat{\alpha}_3 + Ch\left(\hat{\alpha}_1+\hat{\alpha}_2\right)
\end{aligned}$$

which implies (70). □

Lemma 4.5.

$$
\begin{aligned}
\|A_h^\delta I_4\|_{h,p} \quad \le \quad & CJ^{1-\theta}(\tau\alpha_1\hat{\alpha}_2 + h^2\alpha_1\hat{\alpha}_1 + h^{1-d/p}\alpha_1\hat{\alpha}_1) \\
& + C\alpha_1(1+h+h^2)\sum_{j=1}^{n}\frac{\tau_j}{(t_n - t_{j-1})^\theta}\|\nabla\hat{w}_h^{j-1}\|_\infty \\
& + C\sum_{j=1}^{n}\frac{\tau_j}{(t_n-t_{j-1})^\theta}\left(\hat{\alpha}_1 + \|\nabla\hat{w}_h^{j-1}\|_\infty\right)\|A_h^\delta w_h^j\|_{h,p}. \qquad (71)
\end{aligned}
$$

Proof. If it can be shown that

$$
\begin{aligned}
&\left\|A_h^{-\theta}K_h^{-1}\left[B_h(v_h^{j-1})u_h^j - P_hB(v^j)u^j\right]\right\|_{h,p} \\
&\quad \le C\Big(\tau\alpha_1\hat{\alpha}_2 + h^2\alpha_1\hat{\alpha}_1 + h^{1-d/p}\alpha_1\hat{\alpha}_1 + \hat{\alpha}_1\|A_h^\delta w_h^j\|_{h,p} \\
&\qquad + \alpha_1(1+h+h^2)\|\nabla\hat{w}_h^{j-1}\|_\infty + \|A_h^\delta w_h^j\|_{h,p}\|\nabla w_h^{j-1}\|_\infty\Big), \qquad (72)
\end{aligned}
$$

we have (71) in the same manner as for the estimation of I_1. In order to get (72), it suffices to verify that

$$
\begin{aligned}
|(B_h(v_h^{j-1})u_h^j - B(v^j)u^j, \chi_h)| \le C\Big[&\tau\alpha_1\hat{\alpha}_2 + h^2\alpha_1\hat{\alpha}_1 + h^{1-d/p}\alpha_1\hat{\alpha}_1 + \hat{\alpha}_1\|A_h^\delta w_h^j\|_{h,p} \\
&+ \alpha_1(1+h+h^2)\|\nabla\hat{w}_h^{j-1}\|_\infty + \|A_h^\delta w_h^j\|_{h,p}\|\nabla w_h^{j-1}\|_\infty\Big]\|\nabla\chi_h\|_q \qquad (73)
\end{aligned}
$$

for $\chi_h \in X_h$. Indeed, by virtue of (36) and (34),

$$
\begin{aligned}
&\left\|A_h^{-\theta}K_h^{-1}[B_h(v_h^{j-1})u_h^j - P_hB(v^j)u^j]\right\|_{h,p} \\
&\quad \le \quad C\sup_{\chi_h\in X_h}\frac{\left(A_h^{-\theta}K_h^{-1}[B_h(v_h^{j-1})u_h^j - P_hB(v^j)u^j], \chi_h\right)_h}{\|\chi_h\|_{h,q}} \\
&\quad \le \quad C\sup_{\chi_h\in X_h}\frac{\left(B_h(v_h^{j-1})u_h^j - B(v^j)u^j, A_h^{-\theta}\chi_h\right)}{\|\chi_h\|_q},
\end{aligned}
$$

which, together with (43) and (73), implies (72). Consequently, (72) is reduced to (73). To prove that, letting $\chi_h \in X_h$, we write it as

$$
\begin{aligned}
&b(v^j, u^j, \chi_h) - b_h(v_h^{j-1}, u_h^j, \chi_h) \\
&\quad = \quad b(v^j, u^j, \chi_h) - b(v^{j-1}, u^j, \chi_h) \\
&\qquad\quad + b(v^{j-1}, u^j, \chi_h) - b_h(P_hv^{j-1}, \pi_hu^j, \chi_h) \\
&\qquad\quad + b_h(P_hv^{j-1}, \pi_hu^j, \chi_h) - b_h(P_hv^{j-1}, P_hu^j, \chi_h) \\
&\qquad\quad + b_h(P_hv^{j-1}, P_hu^j, \chi_h) - b_h(v_h^{j-1}, P_hu^j, \chi_h) \\
&\qquad\quad + b_h(v_h^{j-1}, P_hu^j, \chi_h) - b_h(v_h^{j-1}, u_h^j, \chi_h) \\
&\quad \equiv \quad I_{41} + I_{42} + I_{43} + I_{44} + I_{45}.
\end{aligned}
$$

First, in view of Lemma 3.8,

$$
\begin{aligned}
|I_{41}| \quad & \le \quad \|u^j\|_\infty\|\nabla v^j - \nabla v^{j-1}\|_p\|\nabla\chi_h\|_q \\
& \le \quad \|u^j\|_\infty\|\nabla\chi_h\|_q\int_{t_{j-1}}^{t_j}\|\nabla v'(s)\|_p\,ds \\
& \le \quad \tau\alpha_1\hat{\alpha}_2\|\nabla\chi_h\|_q.
\end{aligned}
$$

We apply Lemma 3.11 to obtain

$$|I_{42}| \leq Ch^{1-d/p}\|v^{j-1}\|_{2,p}\|u^j\|_{1,p}\|\nabla\chi_h\|_q \leq Ch^{1-d/p}\alpha_1\hat{\alpha}_1\|\nabla\chi_h\|_q.$$

Since b_h is linear with respect to the second argument, we calculate as

$$\begin{aligned} |I_{43}| &\leq \|\nabla P_h v^{j-1}\|_\infty \|\pi_h u^j - P_h u^j\|_p \|\nabla\chi_h\|_q \\ &\leq C\|\nabla v^{j-1}\|_\infty \left(\|\pi_h u^j - u^j\|_p + \|u^j - P_h u^j\|_p\right)\|\nabla\chi_h\|_q \\ &\leq C\hat{\alpha}_1 \cdot Ch^2\|u^j\|_{2,p}\|\nabla\chi_h\|_q \\ &\leq Ch^2\hat{\alpha}_1\alpha_1\|\nabla\chi_h\|_q \end{aligned}$$

by (48) and (19). By virtue of Lemma 3.10, we deduce

$$\begin{aligned} |I_{45}| &\leq C\|\nabla v_h^{j-1}\|_\infty \|P_h u^j - u_h^j\|_p \|\nabla\chi_h\|_q \\ &\leq C\left(\|\nabla R_h v^{j-1}\|_\infty + \|\nabla \hat{w}_h^{j-1}\|_\infty\right) \\ &\qquad \cdot \left(\|P_h u^j - R_h u^j\|_p + \|w_h^j\|_p\right)\|\nabla\chi_h\|_q \\ &\leq C\left(\hat{\alpha}_1 + \|\nabla\hat{w}_h^{j-1}\|_\infty\right)\left(Ch^2\alpha_1 + \|A_h^\delta w_h^j\|_p\right)\|\nabla\chi_h\|_q. \end{aligned}$$

We apply Lemma 3.9 to obtain

$$\begin{aligned} |I_{44}| \leq\ & Ch\left(\|\nabla P_h v^{j-1}\|_\infty + \|\nabla v_h^{j-1}\|_\infty\right)\|P_h u^j\|_{1,p}\|\nabla\chi_h\|_q \\ &+ C\|\nabla(P_h v^{j-1} - v_h^{j-1})\|_\infty \|P_h u^j\|_{1,p}\|\nabla\chi_h\|_q. \end{aligned}$$

Now, we note, by (19) and (22),

$$\begin{aligned} &\|\nabla P_h v^{j-1}\|_\infty + \|\nabla v_h^{j-1}\|_\infty \\ &\quad\leq \|\nabla P_h v^{j-1}\|_\infty + \|R_h v^{j-1}\|_\infty + \|\nabla\hat{w}_h^{j-1}\|_\infty \\ &\quad\leq C\|v^{j-1}\|_{1,\infty} + \|\nabla\hat{w}_h^{j-1}\|_\infty \\ &\quad\leq C\hat{\alpha}_1 + \|\nabla\hat{w}_h^{j-1}\|_\infty, \end{aligned}$$

where we have used Sobolev's inequality

$$\|w\|_{1,\infty} \leq C\|w\|_{2,p} \qquad (p \in (d,\infty), w \in W^{2,p}).$$

Furthermore, by (19), (23) and (25),

$$\begin{aligned} \|\nabla(P_h v^{j-1} - v_h^{j-1})\|_\infty &\leq \|\nabla P_h(I - R_h)v^{j-1}\|_\infty + \|\nabla\hat{w}_h^{j-1}\|_\infty \\ &\leq Ch^{1-d/p}\hat{\alpha}_1 + \|\nabla\hat{w}_h^{j-1}\|_\infty. \end{aligned}$$

Hence,

$$|I_{44}| \leq C\left[h^{1-d/p}\alpha_1\hat{\alpha}_1 + (1+h)\alpha_1\|\nabla\hat{w}_h^{j-1}\|_\infty\right]\|\nabla\chi_h\|_q.$$

Combining these inequalities, we obtain (73); thus we finish the proof of (71). □

4.3. **Completion of the proof.** We can complete the proof of Theorem 2.4 in the following way. First, we recall that τ is defined as (9). Then, we have

$$\frac{h^2}{t_n} \leq Ch\frac{2d}{\varepsilon}\left(\hat{\alpha}_1 + \alpha_0 h\right).$$

Summing the estimates for $I_0, \ldots, \hat{I}_3$ and using (46), we obtain, for $h \in (0, h_0)$ with some $0 < h_0 < 1$,

$$\|A_h^\delta w_h^n\|_{h,p} + \|\hat{w}_h^n\|_{1,\infty} \leq C_2\left(\tau^\sigma + h^{1-d/p}\right) + C\alpha_1 \sum_{j=1}^{n} \frac{\tau_j}{(t_n - t_{j-1})^{7/8}} \|\hat{w}_h^{j-1}\|_{1,\infty} + C\sum_{j=1}^{n} \frac{\tau_j}{(t_n - t_{j-1})^{7/8}} \left(1 + \hat{\alpha}_1 + \alpha_1 J + \|\hat{w}_h^{j-1}\|_{1,\infty}\right) \|A_h^\delta w_h^j\|_{h,p}, \quad (74)$$

where $C_2 = C(1 + J + \varepsilon^{-1}) \left[\alpha_0 + \sum_{i=1}^{3}(\alpha_i + \hat{\alpha}_i) + \alpha_1\hat{\alpha}_1 + \alpha_1\hat{\alpha}_2\right]$.

We define $z_j = \|A_h^\delta w_h^j\|_{h,p} + \|\hat{w}_h^j\|_{1,\infty}$ and $Z_n = \sup_{0\leq j\leq n} z_j$. First we assume that

$$Z_{n-1} \leq 1.$$

Thereby, (74) implies that

$$z_n \leq C_2\left(\tau^\sigma + h^{1-d/p}\right) + C_3 \sum_{j=1}^{n} \frac{\tau_j}{(t_n - t_{j-1})^{7/8}} (z_{j-1} + z_j),$$

with $C_3 = C(1 + J)(1 + \alpha_1 + \hat{\alpha}_1)$. Hence, according to Lemma 3.7,

$$z_n \leq C_4\left(\tau^\sigma + h^{1-d/p}\right) \exp(C_5 J) \equiv \hat{z},$$

where $C_4 = \tilde{C}C_2$, $C_5 = \tilde{C}C_3^8$ and $\tilde{C}$ is the absolute positive constant. Since C_4 and C_5 are independent of h and n, we can choose sufficiently small $h_2 > 0$ such that $\hat{z} \leq 1$ for $h \in (0, h_2)$. On the other hand, since u_{0h} and v_{0h} are chosen so that (8) holds, we have

$$\begin{aligned} z_0 \quad &\leq \quad C\left(\|R_h u_0 - u_0\|_{1,p} + \|u_0 - u_{0h}\|_{1,p}\right) \\ &\qquad + \quad \|R_h v_0 - v_0\|_{1,\infty} + \|v_0 - v_{0h}\|_{1,\infty} \\ &\leq \quad Ch^{1-d/p}\left(\|u_0\|_{2,p} + \|v_0\|_{2,p} + \alpha_0\right). \end{aligned}$$

Hence, we can take $h_3 > 0$ such that $z_0 \leq 1$ for $h \in (0, h_3)$.

At this stage, we set $h_1 = \min(h_0, h_2, h_3)$. Then, since $h \in (0, h_1)$, we have $Z_n \leq 1$ for all $n \geq 0$ such that $t_n < J$ by induction. In conclusion, we have by (40)

$$\begin{aligned} \|w_h^n\|_p + \|\hat{w}_h^n\|_{1,\infty} \quad &\leq \quad C\left(\|A_h^\delta w_h^n\|_{h,p} + \|\hat{w}_h^n\|_{1,\infty}\right) \\ &\leq \quad C\left[C_4 \exp(C_5 J)\right]\left(\tau^\sigma + h^{1-d/p}\right). \end{aligned}$$

This, together with (59), implies the desired error estimate (10). Thus we complete the proof of Theorem 2.4.

5. Concluding remarks.

5.1. The case of smooth domains. Now we suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega$. Consequently, if in addition u_0 and v_0 are smooth and satisfy the compatibly condition for (1c), the system (1) admits a unique classical solution satisfying

$$u,\ \frac{\partial u}{\partial x_i},\ \frac{\partial^2 u}{\partial x_i x_j},\ u_t,\ v,\ \frac{\partial v}{\partial x_i},\ \frac{\partial^2 v}{\partial x_i x_j},\ v_t \in C(\overline{\Omega} \times [0, J])$$

for a sufficiently small J. See, for the proof of this fact, [27] and [28].

In this case, we take a regular family of (curved) triangulations $\{\mathcal{T}_h\}$, which *exactly fit the boundary*:

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T.$$

The definitions of X_h, $\hat{\phi}_i$, D_i, etc. are similar to those before with obvious modifications (see, for example, [3]).

Then, under Assumptions (H1) and (H2), we can derive the following error estimates:

$$\sup_{0 \le t_n \le J} (\|u(t_n) - u_h^n\|_p + \|v(t_n) - v_h^n\|_{1,\infty}) \le C_1'(h + \tau)$$

for $h \in (0, h_1')$; this can be achieved similarly to the proof of Theorem 2.4. See [25, §7] for further detail.

5.2. **General sensitive function.** We have considered the linear sensitive function λv; we now deal with a general sensitive function $\phi(v)$ instead of λv (cf. [17]). Thus, we are concerned with

$$u_t - \nabla \cdot (D_u \nabla u - u \nabla \phi(v)) = 0 \qquad \text{in } \Omega \times (0, J), \tag{75a}$$

$$k v_t - D_v \Delta v + k_1 v - k_2 u = 0 \qquad \text{in } \Omega \times (0, J), \tag{75b}$$

$$\partial u / \partial \nu = 0, \ \partial v / \partial \nu = 0 \qquad \text{on } \partial\Omega \times (0, J), \tag{75c}$$

$$u|_{t=0} = u_0, \quad v|_{t=0} = v_0 \qquad \text{on } \Omega. \tag{75d}$$

Here, $\phi : [0, \infty) \to \mathbb{R}$ denotes a smooth non-decreasing function. For example, we may take $\phi(v) = \lambda \log v$, $\phi(v) = \lambda v^2/(1 + v^2)$, and so on. For the system (75), we consider the finite-element scheme (4) where $\beta_{ij}^{\pm}(v_h)$ should be replaced by

$$\beta_{ij}^{\pm}(v_h) = \int_{\Gamma_{ij}} [\nabla \pi_h \phi(v_h) \cdot \nu_{ij}]_{\pm} \, dS.$$

Then, Theorem 2.1 remains true. On the other hand, Theorems 2.2 and 2.3 also remain true, if the time increment control (5) is replaced by

$$\tau_n = \min \left\{ \tau, \quad \frac{\varepsilon \kappa_h}{2d \|\nabla \pi_h \phi(v_h^{n-1})\|_\infty} \right\}.$$

Error estimate, however, is open at present. Several numerical examples which validate this scheme are presented in [26].

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