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SECOND ORDER OSCILLATION OF MIXED NONLINEAR DYNAMIC EQUATIONS WITH SEVERAL POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract. New oscillation criteria are obtained for superlinear and sublinear forced dynamic equations having positive and negative coefficients by means of nonprincipal solutions.

1. **Introduction.** Let $\mathbb{T}$ be an arbitrary time scale which is unbounded from above. Let $[t_0, \infty)_{\mathbb{T}}$ denote a time scale interval, where $t_0 \in \mathbb{T}$ is fixed. As usual, $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

$x^\sigma(t) = x(\sigma(t))$, and $x^\Delta(t)$ denotes the time scale derivative of x. It is assumed that the reader is familiar with the time scale calculus.

In this paper we consider nonlinear dynamic equations of the form

$$(r(t)x^\Delta)^\Delta + \sum_{i=1}^{n} p_i(t)|x^\sigma|^{\beta_i - 1}x^\sigma = f(t), \qquad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1}$$

where

(i) $0 < \beta_1 < \ldots < \beta_m < 1 < \beta_{m+1} < \ldots < \beta_n$,
(ii) $p_i : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is a right-dense continuous function such that

$$p_i(t) \begin{cases} \leq 0, & i = 1, 2, \ldots, m \\ \geq 0, & i = m+1, m+2, \ldots, n \end{cases} \tag{2}$$

(iii) $r : [t_0, \infty)_{\mathbb{T}} \to (0, \infty)$ is a right-dense continuous function.

Definition 1.1. A nontrivial solution x of (1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, the solution is called oscillatory. Equation (1) is called oscillatory (nonoscillatory) if all of its solutions are oscillatory (nonoscillatory).

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The Emden-Fowler super-linear dynamic equation

$$(r(t)x^{\Delta})^{\Delta} + p(t)|x^{\sigma}|^{\alpha-1}x^{\sigma} = f(t), \qquad \alpha > 1 \tag{3}$$

and the Emden-Fowler sub-linear dynamic equation

$$(r(t)x^{\Delta})^{\Delta} - q(t)|x^{\sigma}|^{\gamma-1}x^{\sigma} = f(t), \qquad 0 < \gamma < 1 \tag{4}$$

are special cases of (1). Moreover, as $\beta_j \to 1^+$, $j > m$, and $\beta_k \to 1^-$, $k \le m$, Eq. (1) reduces to the linear equation

$$(r(t)x^{\Delta})^{\Delta} + p_0(t)x^{\sigma} = f(t) \tag{5}$$

where $p_0(t) = \sum_{i=1}^{n} p_i(t)$.

If $\mathbb{T} = \mathbb{R}$, then Eq. (1) reduces to a forced second-order differential equation

$$(r(t)x')' + \sum_{i=1}^{n} p_i(t)|x|^{\beta_i-1}x = f(t), \qquad t \ge t_0. \tag{6}$$

This equation in several special cases has been considered by many authors. Interval approach, Riccati technique, and Picone type formulas are commonly used to study these equations, see [15, 16, 17, 18, 19, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 34, 35, 38] and the references therein.

Note that the oscillation of the solutions of (3), (4) and (5) has also been studied by many authors, see for instance [1, 2, 3, 4, 5, 6, 8, 11, 12, 13, 14].

In this paper, we will use the notion of a nonprincipal solution of the linear dynamic equation

$$(r(t)x^{\Delta})^{\Delta} + \sum_{i=1}^{n} p_i(t)x^{\sigma} = 0 \tag{7}$$

associated with Eq. (1) as a tool to derive new oscillation criteria for Eq. (1).

Very recently, this technique has been used to establish oscillation criteria for a more general form of Eq. (6) when $n = 2$, see [25].

It is well-known that, see [7, Theorem 4.61], if (7) has a positive solution (the equation is then nonoscillatory by the Sturm comparison theorem), then there are two special linearly independent positive solutions u and z of (7) defined on $[a, \infty)_{\mathbb{T}}$, where $a \ge t_0$ is sufficiently large, such that

$$\int_a^{\infty} \frac{1}{r(s)u(s)u^{\sigma}(s)}\,\Delta s = \infty,$$

$$\lim_{t\to\infty} \frac{u(t)}{z(t)} = 0,$$

$$\int_a^{\infty} \frac{1}{r(s)z(s)z^{\sigma}(s)}\,\Delta s < \infty \tag{8}$$

and

$$\frac{r(t)z^{\Delta}(t)}{z(t)} > \frac{r(t)u^{\Delta}(t)}{u(t)}, \qquad t \in [a, \infty)_{\mathbb{T}}.$$

The solutions u and z are called principal and nonprincipal solutions, respectively.

We note that the concept of the principal solution was first introduced in 1936 by W. Leighton and M. Morse [20] for

$$(r(t)x')' + \nu(t)x = 0, \qquad t \ge t_0 \tag{9}$$

in studying positive solutions of certain differential equations. Since then the principal and nonprincipal solutions have been used successfully in connection with oscillation and asymptotic theory of several related equations, see for instance [1, 20, 33, 36] and the references cited therein.

To the best of our knowledge the first result on oscillation of dynamic equations making use of this concept was given by Zafer [37] for the second order forced dynamic equations of the form

$$(r(t)x^{\Delta})^{\Delta} + \nu(t)x^{\sigma} = f(t), \tag{10}$$

where r, ν, f are rd-continuous functions with $r(t) > 0$.

Theorem 1.1. *Suppose that*

$$(r(t)x^{\Delta})^{\Delta} + \nu(t)x^{\sigma} = 0$$

is nonoscillatory, and let $z(t)$ be a nonprincipal solution of it. If

$$\overline{\lim_{t\to\infty}}\, \mathcal{H}(t) = -\underline{\lim_{t\to\infty}}\, \mathcal{H}(t) = \infty, \tag{11}$$

where

$$\mathcal{H}(t) := \int_a^t \frac{1}{r(s)z(s)z^{\sigma}(s)} \left(\int_a^s f(\tau)z^{\sigma}(\tau)\Delta\tau \right) \Delta s, \qquad a \in [t_0, \infty)_{\mathbb{T}},$$

then Eq. (10) is oscillatory.

Theorem 1.1 is the time scale extension of a result obtained by Wong [36]. Our goal in this paper is to establish oscillation criteria similar to Theorem 1.1 for nonlinear dynamic equations of the form (1), and in particular, for (3) and (4).

If $\mathbb{T} = \mathbb{N}$, then $\sigma(t) = t + 1$ and $g^{\Delta}(t) = g(t + 1) - g(t)$. Setting $\Delta x_k = x^{\Delta}(k)$ as usual, the dynamic equation (1) reduces to a forced second-order difference equation of the form

$$\Delta(r_k \Delta x_k) + \sum_{i=1}^{n} p_{i,k} |x_{k+1}|^{\beta_i - 1} x_{k+1} = f_k, \qquad k \in \{k_0, k_0 + 1, \ldots\} \tag{12}$$

where $k_0 \in \mathbb{N}$, $\Delta z_k = z_{k+1} - z_k$, $r_k > 0$, and (2) takes the form

$$p_{i,k} \begin{cases} \leq 0, & i = 1, 2, \ldots, m \\ \geq 0, & i = m + 1, m + 2, \ldots, n \end{cases}$$

for $k \in \{k_0, k_0 + 1, \ldots\}$.

If $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$, where $q > 1$ is a fixed real number and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, then $\sigma(t) = qt$ and

$$z^{\Delta}(t) = \Delta_q z(t) := \frac{z(qt) - z(t)}{(q - 1)t}$$

which is known as "q-derivative"of z at t. In this case, the dynamic equation (1) reduces to a forced q-difference equation of the form

$$\Delta_q(r(t)\Delta_q x(t)) + \sum_{i=1}^{n} p_i(t)|x(qt)|^{\beta_i - 1} x(qt) = f(t), \qquad t \in \{q^{k_0}, q^{k_0+1}, \ldots\} \tag{13}$$

where $r(t) > 0$ and the functions $p_i(t)$ satisfy

$$p_i(t) \begin{cases} \leq 0, & i = 1, 2, \ldots, m \\ \geq 0, & i = m + 1, m + 2, \ldots, n \end{cases}$$

for $t \in \{q^{k_0}, q^{k_0+1}, \ldots\}$.

2. **Main Results.** Our main assumption is that Eq. (7) is nonoscillatory. Then we denote by $z(t)$ a nonprincipal solution of (7) defined on an interval $[a,\infty)_{\mathbb{T}}$.

Noting that

$$\int_a^\infty \frac{1}{r(s)z(s)z^\sigma(s)}\,\Delta s < \infty, \tag{14}$$

we define

$$\mathcal{T}(t) := \int_a^t \frac{1}{r(s)z(s)z^\sigma(s)}\left(\int_a^s f(\tau)z^\sigma(\tau)\Delta\tau\right)\Delta s,$$

and

$$\widetilde{\mathcal{T}}(t) := \int_a^t \frac{1}{r(s)z(s)z^\sigma(s)}\left(\int_a^s \left\{\sum_{i=1}^n \delta_i p_i(t)\right\} z^\sigma(\tau)\Delta\tau\right)\Delta s,$$

where $a \in [t_0,\infty)_{\mathbb{T}}$ is sufficiently large and

$$\delta_i = (\beta_i - 1)\beta_i^{\beta_i/(1-\beta_i)}.$$

The main result of this paper is the next theorem.

Theorem 2.1. *Suppose that Eq. (7) is nonoscillatory and let $z(t)$ be a positive solution of it satisfying (14), i.e. a nonprincipal solution. If*

$$\overline{\lim_{t\to\infty}}\left\{\mathcal{T}(t) - \widetilde{\mathcal{T}}(t)\right\} = -\underline{\lim_{t\to\infty}}\left\{\mathcal{T}(t) + \widetilde{\mathcal{T}}(t)\right\} = \infty, \tag{15}$$

then Eq. (1) is oscillatory.

Proof. Let $x(t)$ be a solution of (1) and let $z(t)$ be a positive solution of (7) satisfying (14), i.e., a nonprincipal solution. Then by employing the Lagrange identity [7, Theorem 4.30], we have

$$z^\sigma Lx - x^\sigma Lz = \left[rzz^\sigma\left(\frac{x}{z}\right)^\Delta\right]^\Delta$$

where

$$Lu = (r(t)u^\Delta)^\Delta + \sum_{i=1}^n p_i(t)u^\sigma.$$

In the view of $Lz = 0$,

$$Lx = f(t) + \sum_{i=1}^n p_i(t)F_{\beta_i}(x^\sigma),$$

where $F_*(s) = s - |s|^{*-1}s$, and using the substitution $x = z(t)w$, we get

$$(r(t)zz^\sigma\, w^\Delta)^\Delta = \left\{f(t) + \sum_{i=1}^n p_i(t)F_{\beta_i}(x^\sigma)\right\} z^\sigma, \qquad t \in [a,\infty)_{\mathbb{T}}. \tag{16}$$

Hence,

$$\begin{aligned} w(t) &= c_1 + c_2\int_a^t \frac{1}{r(s)z(s)z^\sigma(s)}\Delta s + \mathcal{T}(t) \\ &\quad + \int_a^t \frac{1}{r(s)z(s)z^\sigma(s)}\int_a^s \left\{\sum_{i=1}^n p_i(\tau)F_{\beta_i}(x^\sigma(\tau))\right\} z^\sigma(\tau)\Delta\tau\Delta s \end{aligned} \tag{17}$$

where $c_1 = w(a)$ and $c_2 = r(a)z(a)z^\sigma(a)w^\Delta(a)$ are constants.

It can be easily seen that

$$\max_{s\in[0,\infty)} F_{\beta_i}(s) = -\min_{s\in(-\infty,0]} F_{\beta_i}(s) = (\beta_i - 1)\beta_i^{\beta_i/(1-\beta_i)} = \delta_i, \qquad i > m \tag{18}$$

and

$$\max_{s\in(-\infty,0]} F_{\beta_i}(s) = -\min_{s\in[0,\infty)} F_{\beta_i}(s) = (1 - \beta_i)\beta_i^{\beta_i/(1-\beta_i)} = -\delta_i, \qquad i \le m \tag{19}$$

for $0 < \beta_1 < \ldots < \beta_m < 1 < \beta_{m+1} < \ldots < \beta_n$.

If $x(t) > 0$ on $[a,\infty)_{\mathbb{T}}$, then using (18) and (19) we obtain

$$w(t) \le c_1 + c_2 \int_a^t \frac{1}{r(s)z(s)z^\sigma(s)}\Delta s + \mathcal{T}(t) + \widetilde{\mathcal{T}}(t). \tag{20}$$

Similarly, if $x(t) < 0$ on $[a,\infty)_{\mathbb{T}}$, then again using (18) and (19) we obtain

$$w(t) \ge c_1 + c_2 \int_a^t \frac{1}{r(s)z(s)z^\sigma(s)}\Delta s + \mathcal{T}(t) - \widetilde{\mathcal{T}}(t). \tag{21}$$

Note that (14), (15), (20) and (21) imply that

$$\overline{\lim_{t\to\infty}}\, w(t) = -\underline{\lim_{t\to\infty}}\, w(t) = +\infty. \tag{22}$$

Because $z(t)$ is positive, (22) implies that $x(t)$ has no definite sign on $[a,\infty)_{\mathbb{T}}$, i.e. it is oscillatory. □

Remark 2.1. In view of

$$\lim_{\lambda\to 1^\pm} \lambda^{\lambda/(1-\lambda)} = 1/e,$$

we have

$$\lim_{\substack{\beta_i \to 1^+ \\ (i>m)}} \delta_i = \lim_{\substack{\beta_i \to 1^- \\ (i\le m)}} \delta_i = 0.$$

So, $\widetilde{\mathcal{T}}(t) \to 0$ as $\beta_i \to 1^+$ $(i > m)$ and $\beta_i \to 1^-$ $(i \le m)$. Then the condition (15) turns into the condition (11) and hence we recover Theorem 1.1 for Eq. (5).

Corollary 2.1. *Suppose that*

$$(r(t)x^\Delta)^\Delta + \sum_{i=m+1}^{n} p_i(t)x^\sigma = 0$$

is nonoscillatory and let $z(t)$ be a positive solution of it satisfying (14), i.e. a nonprincipal solution. If

$$\overline{\lim_{t\to\infty}} \left\{\mathcal{T}(t) - \widetilde{\mathcal{T}}_1(t)\right\} = -\underline{\lim_{t\to\infty}} \left\{\mathcal{T}(t) + \widetilde{\mathcal{T}}_1(t)\right\} = \infty,$$

where

$$\widetilde{\mathcal{T}}_1(t) := \int_a^t \frac{1}{r(s)z(s)z^\sigma(s)} \left(\int_a^s \left\{\sum_{i=m+1}^{n} p_i(\tau)\right\} z^\sigma(\tau)\Delta\tau\right)\Delta s, \qquad a \in [t_0,\infty)_{\mathbb{T}},$$

then equation

$$(r(t)x^\Delta)^\Delta + \sum_{i=m+1}^{n} p_i(t)|x^\sigma|^{\beta_i - 1}x^\sigma = f(t) \tag{23}$$

is oscillatory.

Remark 2.2. If we pick $p_j(t) = 0$, for all $j = m+2, m+3, \ldots, n$, then Eq. (23) reduces to Eq. (3) with $p(t) = p_{m+1}(t)$ and $\alpha = \beta_{m+1}$.

Corollary 2.2. *Let $z(t)$ be a positive solution of*

$$(r(t)x^{\Delta})^{\Delta} + \sum_{i=1}^{m} p_i(t)x^{\sigma} = 0$$

satisfying (14), i.e. a nonprincipal solution. If

$$\overline{\lim_{t\to\infty}} \left\{ \mathcal{T}(t) - \widetilde{\mathcal{T}}_2(t) \right\} = -\underline{\lim_{t\to\infty}} \left\{ \mathcal{T}(t) + \widetilde{\mathcal{T}}_2(t) \right\} = \infty$$

where

$$\widetilde{\mathcal{T}}_2(t) := \int_a^t \frac{1}{r(s)z(s)z^{\sigma}(s)} \left(\int_a^s \left\{ \sum_{i=1}^{m} p_i(\tau) \right\} z^{\sigma}(\tau)\Delta\tau \right) \Delta s, \qquad a \in [t_0, \infty)_{\mathbb{T}},$$

then equation

$$(r(t)x^{\Delta})^{\Delta} + \sum_{i=1}^{m} p_i(t)|x^{\sigma}|^{\beta_i - 1}x^{\sigma} = f(t) \tag{24}$$

is oscillatory.

Remark 2.3. If we pick $p_j(t) = 0$, for all $j = 2, 3, \ldots, m$, then Eq. (24) reduces to Eq. (4) with $q(t) = -p_1(t)$ and $\gamma = \beta_1$.

3. **Applications.** We restate Theorem 2.1 for the special time scales $\mathbb{R}, \mathbb{N}$ and $q^{\mathbb{N}_0}$. One can easily write down the analogs of Corollary 2.1 and Corollary 2.1 in a similar manner.

Theorem 3.1. *Suppose that*

$$(r(t)x')' + \sum_{i=1}^{n} p_i(t)x = 0$$

is nonoscillatory, and let $z(t)$ be a positive solution of it satisfying

$$\int_a^{\infty} \frac{1}{r(s)z^2(s)} \, \mathrm{d}s < \infty,$$

i.e. a nonprincipal solution. If

$$\overline{\lim_{t\to\infty}} \left\{ \mathcal{R}(t) - \widetilde{\mathcal{R}}(t) \right\} = -\underline{\lim_{t\to\infty}} \left\{ \mathcal{R}(t) + \widetilde{\mathcal{R}}(t) \right\} = \infty,$$

then Eq. (6) is oscillatory, where

$$\mathcal{R}(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s f(\tau)z(\tau)\mathrm{d}\tau \right) \mathrm{d}s$$

and

$$\widetilde{\mathcal{R}}(t) := \int_a^t \frac{1}{r(s)z^2(s)} \left(\int_a^s \left\{ \sum_{i=1}^{n} \delta_i p_i(t) \right\} z(\tau)\mathrm{d}\tau \right) \mathrm{d}s$$

for $a \geq t_0$.

Theorem 3.2. *Suppose that*

$$\Delta(r_k \Delta x_k) + \sum_{i=1}^{n} p_{i,k} x_{k+1} = 0, \qquad k \in \{k_0, k_0 + 1, \ldots\}$$

is nonoscillatory, and let $\{z_k\}$ *be a positive solution of it satisfying*

$$\sum_{j=1}^{\infty} \frac{1}{r_j z_j z_{j+1}} < \infty,$$

i.e. a nonprincipal solution. If

$$\overline{\lim_{k\to\infty}} \left\{ \mathcal{N}_k - \widetilde{\mathcal{N}}_k \right\} = - \underline{\lim_{k\to\infty}} \left\{ \mathcal{N}_k + \widetilde{\mathcal{N}}_k \right\} = \infty,$$

then Eq. (12) *is oscillatory, where*

$$\mathcal{N}_k := \sum_{j=1}^{k-1} \frac{1}{r_j z_j z_{j+1}} \sum_{s=1}^{j-1} f_s z_{s+1}$$

and

$$\widetilde{\mathcal{N}}_k := \sum_{j=1}^{k-1} \frac{1}{r_j z_j z_{j+1}} \sum_{s=1}^{j-1} \left\{ \sum_{i=1}^{n} \delta_i p_{i,s} \right\} z_{s+1}$$

for $k \in \{k_0, k_0+1, \ldots\}$.

Theorem 3.3. *Suppose that*

$$\Delta_q(r(t)\Delta_q x(t)) + \sum_{i=1}^{n} p_i(t) x(qt) = 0, \qquad t \in \{q^{k_0}, q^{k_0+1}, \ldots\}$$

is nonoscillatory, and let $z(t)$ *be a positive solution of it satisfying*

$$\sum_{k=0}^{\infty} \frac{q^k}{r(q^k) z(q^k) z(q^{k+1})} < \infty,$$

i.e. a nonprincipal solution. If

$$\overline{\lim_{k\to\infty}} \left\{ Q(k) - \widetilde{Q}(k) \right\} = - \underline{\lim_{k\to\infty}} \left\{ Q(k) + \widetilde{Q}(k) \right\} = \infty,$$

then Eq. (13) *is oscillatory, where*

$$Q(k) := \sum_{j=0}^{k-1} \frac{q^j}{r(q^j) z(q^j) z(q^{j+1})} \sum_{s=0}^{j-1} f(q^s) z(q^{s+1})$$

and

$$\widetilde{Q}(k) := \sum_{j=0}^{k-1} \frac{q^j}{r(q^j) z(q^j) z(q^{j+1})} \sum_{s=0}^{j-1} \left\{ \sum_{i=1}^{n} \delta_i p_i(q^s) \right\} z(q^{s+1})$$

for $k \in \{k_0, k_0+1, \ldots\}$.

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