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IS THE TRUDINGER-MOSER NONLINEARITY A TRUE CRITICAL NONLINEARITY?

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Abstract. While the critical nonlinearity $\int |u|^{2^*}$ for the Sobolev space H^1 in dimension $N > 2$ lacks weak continuity at any point, Trudinger-Moser nonlinearity $\int e^{4\pi u^2}$ in dimension $N = 2$ is weakly continuous at any point except zero. In the former case the lack of weak continuity can be attributed to invariance with respect to actions of translations and dilations. The Sobolev space H_0^1 of the unit disk $\mathbb{D} \subset \mathbb{R}^2$ possesses transformations analogous to translations (Möbius transformations) and nonlinear dilations $r \mapsto r^s$. We present improvements of the Trudinger-Moser inequality with nonlinearities sharper than $\int e^{4\pi u^2}$, that lack weak continuity at any point and possess (separately), translation and dilation invariance. We show, however, that no nonlinearity of the form $\int F(|x|, u(x))\mathrm{d}x$ is both dilation- and Möbius shift-invariant. The paper also gives a new, very short proof of the conformal-invariant Trudinger-Moser inequality obtained recently by Mancini and Sandeep [10] and of a sharper version of Onofri-type inequality of Beckner [4].

1. **Introduction.** The classical (Pohozhaev)-Trudinger-Moser inequality ([13, 15, 11]) on a bounded domain $\Omega \subset \mathbb{R}^2$,

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2 \le 1} \int_\Omega e^{4\pi u^2} dx < \infty, \tag{1}$$

is usually regarded as a natural analog of the Sobolev inequality

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2 \le 1} \int_\Omega |u|^{2^*} \mathrm{d}x < \infty, \tag{2}$$

where $\Omega \subset \mathbb{R}^N$, $N > 2$, and $2^* = \frac{2N}{N-2}$. Indeed, both inequalities correspond to the end points of respective parameter scales: replacing the number 4π in (1) by any $p > 4\pi$, or the number 2^* in (2) by any $q > 2^*$, results in the respective supremum taking the value $+\infty$. When $p < 4\pi$, or $q > 2^*$, both nonlinearities become weakly continuous.

There is a significant difference, however, between the weak continuity properties of the two functionals at the endpoint value 4π resp. 2^*. The Trudinger-Moser nonlinearity $\int_\Omega e^{4\pi u^2}$ is weakly continuous at any non-zero point of the ball $\{u \in H_0^2(\Omega), \|\nabla u\|_2 \le 1\}$ (see [9]), while the functional $\int_\Omega |u|^{2^*}$ lacks weak continuity at

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any point. Indeed, assuming, for the sake of simplicity, that Ω is a unit ball, and taking a $w \in H_0^1(\Omega)\setminus\{0\}$, extended by zero to the whole $\mathbb{R}^N$), we have the sequence $w_k(x) = 2^{\frac{N-2}{2}k} w(2^k x)$ that weakly converges to zero, while $\|\nabla w_k\|_2 = \|\nabla w\|_2$ and $\|w_k\|_{2^*} = \|w\|_{2^*}$. Let now $u_k = u + w_k$. From Brezis-Lieb lemma it follows that

$$\lim \int_\Omega |u_k|^{2^*} = \int_\Omega |u|^{2^*} + \int_\Omega |w|^{2^*} \neq \int_\Omega |u|^{2^*}.$$

Since $u_k \rightharpoonup u$, this verifies that the functional $\int_\Omega |u|^{2^*}$ is not weakly continuous at u.

On the other hand, the Trudinger-Moser nonlinearity is not invariant with respect to any non-compact semigroup of transformations that we know, that preserves the gradient norm.

The subject of this paper is to show that Trudinger-Moser nonlinearity is not a true critical nonlinearity, in the sense that it is dominated by invariant nonlinearities that lack weak semicontinuity at any point. In Section 2 we present such nonlinearity in the radial subspace of $H_0^1(\mathbb{D})$, invariant with respect to nonlinear dilations. By $\mathbb{D}$ we denote the open unit disk in $\mathbb{R}^2$. In Section 3 we consider another, Möbius shift-invariant functional on $H_0^1(\mathbb{D})$, that yields an improved Trudinger-Moser inequality, and give a new, greatly simplified, proof of the latter. We also state and prove a related version of Onofri inequality on $\mathbb{D}$. In Section 4 we show that there is no functional of the form $\int F(|x|, u)$ that is both dilation- and Möbius shift-invariant.

2. **Dilation-invariant nonlinearity.** Let $H_{0,r}^1(\mathbb{D})$ denote the subspace of radial functions of $H_0^1(\mathbb{D})$. The transformations

$$h_s u(r) \stackrel{\text{def}}{=} s^{-\frac{1}{2}} u(r^s),\ u \in H_{0,r}^1(\mathbb{D})\ s > 0, \tag{3}$$

preserve the norm $\|\nabla u\|_2$ of $H_{0,r}^1(\mathbb{D})$, as well as the functional that appears in the 2-dimensional analog of Hardy inequality due to Leray ([8]) $\int_{\mathbb{D}} \frac{u^2}{|x|^2(\log 1/|x|)^2} dx$ (for the inequality of Leray type in higher dimensions see Adimurthi and Sandeep [1] and Adimurthi and Sekar [2]). Furthermore, these transformations preserve the norms of a family of weighted L^p-spaces, $p = [2, \infty]$, analogous to the weighted-L^p scale with $p \in [2, 2^*]$ produced by Hölder inequality in the case $N > 2$, interpolating between the Hardy term $\int \frac{u^2}{|x|^2} dx$ and the critical nonlinearity $\int |u|^{2^*} dx$. In the case $N = 2$, the critical exponent is formally $2^* = +\infty$ and the dilation-invariant L^{2^*}-norm is

$$\|u\|_{2^*} = \sup_{r\in(0,1)} \frac{|u(r)|}{(2\pi \log \frac{1}{r})^{1/2}}. \tag{4}$$

The following statement asserts that the Trudinger-Moser functional is dominated by the 2^*-norm.

Proposition 1. *The functional* $\int_{\mathbb{D}} e^{4\pi u^2}$ *on the set* $\{u \in H_{0,r}^1(\mathbb{D}), \|u\|_{2^*} < 1\}$ *is continuous in the norm* (4).

Proof. From the definition of the 2^*-norm (4) it follows that $e^{4\pi u^2} \leq r^{-a}$, where $a = 2\|u\|_{2^*}^2 < 2$. Continuity of $\int_{\mathbb{D}} e^{4\pi u^2}$ is now a consequence of Lebesgue convergence theorem. □

Moreover, it is known ([5]) that the Orlicz norm associated with the functional $\int_{\mathbb{D}} (e^{4\pi u^2} - 1)$ is equivalent to the norm (4), and in the general, non-radial, case, to

the quasinorm $\|u^*\|_{2^*}$, where u^* denotes decreasing rearrangement. Note that it is well known that the unit ball in the 2^*-norm contains the unit ball in the gradient norm. Since the proof is elementary, we provide this as the following

Lemma 2.1. *For every $u \in H^1_{0,r}(\mathbb{D})$,*

$$2\pi|u(r)|^2 \leq \|\nabla u\|_2^2 \log\frac{1}{r},\ r \in [0,1]. \tag{5}$$

Proof. Use the Newton-Leibniz formula:

$$|u(\rho)|^2 = \left|\int_1^\rho u'(r)dr\right|^2 \leq \left|\int_1^\rho u'(r)r^{-1}rdr\right|^2,$$

and apply Cauchy inequality (with respect to the measure rdr) to the product $u'(r)r^{-1}$ in the right hand side:

$$|u(\rho)|^2 \leq \left|\int_0^1 |u'(r)|^2 rdr\right| \left|\int_\rho^1 r^{-2}rdr\right| \leq \frac{1}{2\pi}\|\nabla u\|_2^2 \log\frac{1}{\rho}.$$

□

Proposition 2. *The norm (4) lacks weak continuity at any $u \in H^1_{0,r}(\mathbb{D})$.*

Proof. Observe that that for every $u, v \in C^\infty_{0,r}(\mathbb{D}\setminus\{0\})$,

$$\|u + k^{1/2}v(r^{-k})\|_{2^*} \to \max\{\|u\|_{2^*}, \|v\|_{2^*}\}. \tag{6}$$

Indeed, for all k sufficiently large, the functions u and $k^{1/2}v(r^{-k})$ have disjoint support. By density of $C^\infty_{0,r}(\mathbb{D}\setminus\{0\})$ in $H^1_{0,r}(\mathbb{D})$, and by Lemma 2.1, we may extend (6) to all $u, v \in H^1_{0,r}(\mathbb{D})$. If, moreover, $\|v\|_{2^*} > \|u\|_{2^*}$ and $u_k(r) = u(r) + k^{1/2}v(r^{-k})$, then $u_k \rightharpoonup u$, but $\|u_k\|_{2^*} = \|v\|_{2^*} > \|u\|_{2^*}$. Consequently the map $u \mapsto \|u\|_{2^*}$ lacks weak continuity at any point. □

3. **Translation-invariant nonlinearity.** Adopting, for the sake of convenience, the complex numbers notation $z = x_1 + ix_2$ for points (x_1, x_2) on $\mathbb{D}$, we consider the following set of automorphisms of $\mathbb{D}$, known as Möbius transformations.

$$\eta_\zeta(z) = \frac{z-\zeta}{1-\bar{\zeta}z}, \zeta \in \mathbb{D}. \tag{7}$$

Since the maps (7) are conformal automorphisms of $\mathbb{D}$, one has $|\nabla u \circ \eta_\zeta|_2 = |\nabla u|_2$, which implies that the Möbius shifts $u \mapsto u \circ \eta_\zeta$, $\zeta \in \mathbb{D}$, preserve the gradient norm $\|\nabla u\|_2$. Moreover, they preserve the measure $\frac{4dx}{(1-|x|^2)^2}$. In fact, the gradient norm can be interpreted, under the Poincaré disk model of the hyperbolic space $\mathbb{H}^2$, as the norm associated with the Laplace-Beltrami operator on the hyperbolic space $\dot{H}^1(\mathbb{H}^2)$, defined by completion of $C^\infty_0(\mathbb{H}^2)$, and the measure $\frac{4dx}{(1-|x|^2)^2}$ is the Riemann measure on $\mathbb{H}^2$. Moreover, transformations (7) form a non-compact group of isometries of $\mathbb{H}^2$.

The following inequality (originally expressed in terms of $\mathbb{H}^2$) has been shown by Mancini and Sandeep [10].

Theorem 3.1. *Let $\mathbb{D}$ be the open unit disk. The following relation holds true:*

$$\sup_{u \in H^1_0(\mathbb{D}), \|\nabla u\|_2 \leq 1} \int_{\mathbb{D}} \frac{e^{4\pi u^2} - 1}{(1-|x|^2)^2} dx < \infty. \tag{8}$$

Note that the nonlinearity in (8) dominates the Trudinger-Moser nonlinearity $e^{4\pi u^2}-1$. Furthermore,

Proposition 3. *The functional*

$$J(u)=\int_{\mathbb{D}}\frac{e^{4\pi u^2}-1}{(1-|x|^2)^2}dx$$

lacks weak continuity at any point in $H_0^1(\mathbb{D})$.

Proof. We give the proof for $u\in C_0^\infty(\mathbb{D})$. Extension of the proof to general $u\in H_0^1(\mathbb{D})$, based on the continuity of $J(u)$ and the density of $C_0^\infty(\mathbb{D})$ in $H_0^1(\mathbb{D})$, is left for the reader. let $w\in C_0^\infty(\mathbb{D})$, $w\neq 0$, let $\zeta_k=1-1/k$ and define $w_k=w\circ\eta_{\zeta_k}$, $u_k=u+w_k$. Then $u_k\rightharpoonup u$ and, for k sufficiently large, u and w have disjoint supports. Therefore, for k large, taking into account Möbius shift-invariance of the functional j, we have

$$J(u_k)=J(u)+J(w_k)=J(u)+J(w)\neq J(u),$$

and thus J is not weakly continuous at u. □

We give now a new proof of Theorem 3.1.

Proof. Note that the standard rearrangement argument applies on $\mathbb{H}^2$ in an analogous way to that in the Euclidean case, with the Riemannian measure on $\mathbb{H}^2$ replacing the Lebesgue measure (see [6]). Consequently, it suffices to consider the inequality only for radial functions on the Poincaré disk.

Let $u\in H_0^1(\mathbb{D})$ be an arbitrary function satisfying $\|\nabla u\|_2\le 1$. We evaluate the integral for $r\le\frac12$ by the standard Trudinger-Moser inequality. For $\frac12\le r\le 1$ we estimate the weight in the integral by the distance to the boundary: $\frac{1}{(1-r^2)^2}\le\frac49\frac{1}{(1-r)^2}$. Then

$$\begin{aligned}\int_{\mathbb{D}}\frac{e^{4\pi u^2}-1}{(1-r^2)^2}dx\le&\frac{16}{9}\int_{\mathbb{D}}e^{4\pi u^2}dx+2\pi\frac49\int_{\frac12}^1\frac{e^{4\pi u^2}-1}{(1-r)^2}rdr\\ \le&\frac{16}{9}\int_{\mathbb{D}}e^{4\pi u^2}dx+2\pi\frac49\int_{\frac12}^1\frac{e^{4\pi u^2}-1}{(1-r)^2}rdr.\end{aligned}\tag{9}$$

Let us apply now to the right hand side Lemma 2.1 (which gives $e^{4\pi u^2}\le\frac{1}{r^2}\le 4$ for $r\in[\frac12,1]$), and use the elementary inequality $e^t-1\le te^t$ that holds for $t>0$:

$$\begin{aligned}2\pi\frac49\int_{\frac12}^1\frac{e^{4\pi u^2}-1}{(1-r)^2}rdr\ &\le 2\pi\frac49\int_{\frac12}^1\frac{u^2e^{4\pi u^2}}{(1-r)^2}rdr\\ &\le\frac{16}{9}\int_{\mathbb{D}}\frac{u^2}{(1-r)^2}dx\le\frac{64}{9}\ .\end{aligned}\tag{10}$$

The bound in the right hand side is due the Hardy inequality (with the distance from the boundary). Thus (8) follows from substitution of (10) into (9) and the standard Trudinger-Moser inequality. □

The argument above is not surprising in the sense that in the higher dimensions one can derive the Sobolev inequality on $\mathbb{R}^N$ (although not with the optimal constant) from the Hardy inequality using the pointwise estimate for the radial functions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $\sup|u(r)|r^{\frac{N-2}{2}}\le C\|\nabla u\|_2^2$. This argument also leads to

improvements (without an optimal constant) in Onofri-type inequalities. Here we consider an Onofri-type inequality on the unit disk due to Beckner [4],

$$\log\left(\frac{1}{\pi}\int_{\mathbb{D}} e^u\right) + \left(\frac{1}{\pi}\int_{\mathbb{D}} e^u\right)^{-1} \leq 1 + \frac{1}{16\pi}\|\nabla u\|_2^2, u \geq 0. \tag{11}$$

Theorem 3.2. *There exists a constant $C > 0$ such that for every $u \in H_0^1(\mathbb{D})$, $u \geq 0$,*

$$\log\left(\int_{\mathbb{D}} \frac{e^u - 1 - u}{(1-r^2)^2} dx\right) \leq C + \frac{1}{16\pi}\|\nabla u\|_2^2. \tag{12}$$

Note that since we do not know the optimal value of the constant C, we do not have to include the term corresponding to $\left(\frac{1}{\pi}\int_{\mathbb{D}} e^u\right)^{-1}$, since it is bounded by 1.

Proof. The proof is similar to that of Theorem 3.1 and we only sketch the main points. Reduction to the radial functions uses the same argument as Theorem 3.1. The estimate of the integral over $r \in [0, \frac{1}{2}]$ follows immediately from (11). To estimate the integral over $[\frac{1}{2}, 1)$ note that, on this interval, $\frac{1}{1-r^2} \leq \frac{2}{3}\frac{1}{1-r}$, and since $u \geq 0$, that $e^u - 1 - u \leq u^2$. Thus we obtain

$$\int_{\frac{1}{2}}^1 \frac{e^u - 1 - u}{(1-r^2)^2} r dr \leq \frac{4}{9}\int_{\frac{1}{2}}^1 \frac{u^2 e^u}{(1-r)^2} r dr. \tag{13}$$

Note now that by Lemma 2.1,

$$u(r) \leq (2\pi)^{-\frac{1}{2}}\|\nabla u\|_2\sqrt{\log\frac{1}{r}} \leq (2\pi)^{-\frac{1}{2}}\|\nabla u\|_2\sqrt{\log 2}.$$

Thus, for any $\epsilon > 0$ there exists C_ϵ such that for all $r \in [\frac{1}{2}, 1]$,

$$u(r) \leq \epsilon\|\nabla u\|_2^2 + C_\epsilon.$$

Substituting this estimate into the right hand side of (13), we obtain, using the usual Hardy inequality with the distance from the boundary,

$$\int_{\frac{1}{2}}^1 \frac{e^u - 1 - u}{(1-r^2)^2} r dr \leq \frac{4}{9} e^{\epsilon\|\nabla u\|_2^2 + C_\epsilon}\int_{\frac{1}{2}}^1 \frac{u^2}{(1-r)^2} r dr \leq \frac{16}{9\cdot 2\pi}\|\nabla u\|_2^2 e^{\epsilon\|\nabla u\|_2^2 + C_\epsilon}.$$

Choosing a suitable ϵ we conclude that

$$\log\left(\int_{\frac{1}{2}\leq|x|<1} \frac{e^u - 1 - u}{(1-r^2)^2} dx\right) \leq C + 2\log(\|\nabla u\|_2) + \epsilon\|\nabla u\|_2^2 + C_\epsilon \leq \frac{\|\nabla u\|_2^2}{16\pi} + \hat{C},$$

which, combined with the estimate for the integral over $|x| \leq \frac{1}{2}$, gives (12). □

4. **Non-existence of a perfect critical nonlinearity.** We verify first what invariance requirements have to be satisfied by a non-negative function F so that the functional

$$J(u) = \int_{\mathbb{D}} F(|x|, u) dx$$

will be invariant with respect to Möbius shifts or actions of nonlinear dilations.

Lemma 4.1. *Let η_ζ be as in (7) and let $F \in C^1((0,\infty)\times\mathbb{R})$ be a non-negative function. If the functional*

$$J(u) = \int_{\mathbb{D}} F(|x|, u) dx$$

is continuous on $H_0^1(\mathbb{D})$ *and satisfies*

$$J(u \circ \eta_\zeta) = J(u) \tag{14}$$

for all $u \in H_0^1(\mathbb{D})$ *and* $\zeta \in \mathbb{D}$*, then*

$$F(r,u) = \frac{G(u)}{(1-r^2)^2}. \tag{15}$$

for some function G*.*

Proof. Let us use the complex numbers notation for points in the unit disk. Consider (14) with real-valued $\zeta = t$, that is, with $\eta_t(x) = \frac{z-t}{1-tz}$. Assume for the sake of simplicity that $F(r,u) = \frac{G(r^2,u)}{(1-r^2)^2}$. Then, by invariance of the Riemannian measure on $\mathbb{H}^2$ with respect to Möbius transformations,

$$J(u \circ \eta_t) = \int_{\mathbb{D}} G(|\eta_{-t} z|^2, u(z)) dxdy.$$

Explicit calculation of the derivative gives then, due to (14),

$$\frac{d}{dt} J(u \circ \eta_t)|_{t=0} = \int_{\mathbb{D}} 2x(1-r^2)\partial_1 G(r^2,u) \frac{dxdy}{(1-r^2)^2},$$

Restricting now our consideration to those functions u whose support lies in the right half-disk, we conclude that $\partial_1 G(x^2+y^2, u(x,y)) = 0$ for $(x,y) \in \mathbb{D}$, $x > 0$, which implies (15). □

Lemma 4.2. *Let* h_s *be as in (3), and let* $F \in C^1((0,\infty) \times \mathbb{R})$ *be a non-negative function. If the functional*

$$J(u) = \int_{\mathbb{D}} F(|x|, u) dx$$

is continuous on $H_0^1(\mathbb{D})$ *and satisfies*

$$J(h_s u) = J(u) \tag{16}$$

for all $u \in H_{0,r}^1(\mathbb{D})$ *and* $s > 0$*, then*

$$F(r,u) = H\left(\frac{u}{(\log \frac{1}{r})^{\frac{1}{2}}}\right) \tag{17}$$

with some function H*.*

Proof. Consider (16) with radial functions. Assume for the sake of simplicity that $F(r,u) = H\left(\frac{u}{(\log \frac{1}{\rho})^{\frac{1}{2}}}\right)$. Evaluation of $J(h_s u)$ by the change of variable $r = \rho^{\frac{1}{s}}$ gives

$$J(h_s u) = \int s^{-1} H\left(\rho^{\frac{1}{s}}, \frac{u(\rho)}{(\log \frac{1}{\rho})^{\frac{1}{2}}}\right) \rho^{2/s-2} \rho d\rho.$$

Evaluating $\frac{dJ(h_s u)}{ds}$ at $s = 1$ and using the argument analogous to that in Lemma 4.1, we conclude that $\partial_1 H = 0$ and (17) follows. □

An immediate conclusion of Lemmas 4.1 and 4.2 follows.

Corollary 1. *Let $F \in C^1((0,\infty)\times\mathbb{R})$ be a non-negative function satisfying both invariance requirements (14) and (16). Then F=0.*

This statement does not mean that in the two-dimensional Sobolev space there is no perfectly critical nonlinearity - that is, that there is no continuous invariant functional that dominates the Trudinger-Moser functional and lacks weak continuity at any point. It means merely that one cannot find a nontrivial functional with required prthatoperties that has the form $\int_{\mathbb{D}} F(|x|,u)$. We still can postulate the following

Conjecture 1. *There is a continuous convex functional $J(u)$ on $H_0^1(\mathbb{D})$, bounded for $\|\nabla u\|_2 \le 1$, and satisfying the following requirements:*

(a) $J(u\circ\eta_\zeta) = J(u)$ for all $\zeta\in\mathbb{D}$,
(b) $J(u\circ h_s) = J(u)$ for all $s>0$ and radial u,
(c) the functional J lacks weak continuity at any point,
(d) the functional J induces an Orlicz space where Trudinger-Moser functional is continuous and bounded on every bounded set.

In this conjecture, condition (b) is subject to further interpretation. In particular, gradient norm-preserving nonlinear dilations can be defined for more general (but not all) functions in $H_0^1(\mathbb{D})$ by the formula $h_s u(z) = s^{-1/2}u(z^s)$. If one drops (b) altogether, this conjecture is satisfied by the functional (8) of Mancini and Sandeep.

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