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# MARTINGALE SOLUTIONS FOR STOCHASTIC NAVIER-STOKES EQUATIONS DRIVEN BY LÉVY NOISE

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Abstract. In this paper, we establish the solvability of martingale solutions for the stochastic Navier-Stokes equations with Itô-Lévy noise in bounded and unbounded domains in $\mathbb{R}^d, d = 2, 3$. The tightness criteria for the laws of a sequence of semimartingales is obtained from a theorem of Rebolledo as formulated by Metivier for the Lusin space valued processes. The existence of martingale solutions (in the sense of Stroock and Varadhan) relies on a generalization of Minty-Browder technique to stochastic case obtained from the local monotonicity of the drift term.

1. **Introduction.** Martingale solutions provide a characterization of the space-time statistical solutions for stochastic system. The concept of martingale solutions approach for finite dimensional diffusion processes was pioneered in the works of Stroock and Varadhan [35, 36] while this approach for some class of infinite-dimensional problems in Lusin spaces was formulated in Metivier [22] (see also, [40]). The stochastic Navier-Stokes equations (SNSEs) driven by Gaussian noise has been extensively studied over the past decades (see, for instance, [41, 3, 10, 31, 20, 9, 33]). For the history of the solvability of the deterministic Navier-Stokes equations, we refer the reader to Ladyzhenskaya [18]. Stochastic partial differential equations (SPDEs) driven by Lévy processes whose paths may contain random jump discontinuities of arbitrary size occurring at arbitrary random times are now being used in many different areas of applied sciences (see, [7, 16, 26]). Some of the earlier work related to martingale problem and finite dimensional Lévy processes, see, for example, [34, 14, 15, 12].

In this paper, we establish the existence of martingale solutions supported on the set of all solutions of the Navier-Stokes equations perturbed by Gaussian and Lévy noises. This study is motivated by practical engineering scenario where aerodynamic flow is often subjected to abrupt external disturbances due to structural and environmental disturbances. We use the semimartingale formulation involving the finite dimensional Galerkin approximation given by $X^n = A^n + M^n$ where $A^n$ is a process with finite variation and $M^n$ is a local martingale to obtain the tightness criteria. From a theorem of Rebolledo as given by Metivier [22] (see, also [13]) for

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the Lusin space valued càdlàg processes we obtain the tightness of the laws $P^n$ in terms of the tightness of the processes $A^n$ and $\lhd M^n \rhd$. Here $\lhd M^n \rhd$ is the right continuous increasing process with paths of finite variation (Meyer process) associated with $M^n$. We use the stochastic Minty-Browder argument to prove the continuity of the martingale on path space. Similar idea has been used for SPDEs with Gaussian noise in [22], SNSEs and stochastic magneto-hydrodynamic systems with Gaussian noise in Sritharan [31], Sritharan and Sundar [32] respectively. Since the sum of Stokes operator and inertia term are not globally monotone, a local monotonicity result to prove the existence of strong solutions for 2D stochastic Navier-Stokes equation perturbed by Gaussian noise has been devised in Menaldi and Sritharan [20]. By making use of the stochastic Minty-Browder argument involving the locally monotone operators, we prove the existence of martingale solutions for SNSEs with multiplicative Itô-Lévy noises in bounded and unbounded domains. Using the classical Yamada-Watanabe construction, we prove the uniqueness of the laws with suitable moment condition in $d = 3$. The recent paper Dong and Zhai [6] studies the existence of martingale solution for SNSEs with multiplicative jump noise coefficients in bounded domains. The tightness criteria used for probability measures in [6] is different from ours in the sense that we have utilized Metivier's technique in the Lusin spaces. Since we have applied the Minty stochastic lemma to avoid the need of compact embeddings in unbounded domain, this paper addresses an interesting problem even with additive noises in unbounded domain (see, Remark 4.2) in contrast to [6].

The paper is organized as follows. In section 2, we state the main assumptions of the noise coefficients and prove the local monotonicity of the drift term $\mathbf{A}+\mathbf{B}(\cdot)$ for any $d \geq 2$. In section 3, we establish higher order moment estimates of order $p \geq 2$ for the Galerkin approximations with approximate laws $\mathbb{P}^n$ being defined on $\mathbb{D}(0,T;\mathbb{V}')$ and show that these laws converge weakly to a probability measure $\mathbb{P}$. In section 4, we prove the well-posedness of the martingale problem by first showing that the weak limit $\mathbb{P}$ is indeed a martingale solution for the SNSEs using the Minty stochastic lemma. We then analyze these results in unbounded domain and prove the uniqueness of the laws.

## 2. Navier-Stokes equations with Itô-Lévy noise.

Let $\mathcal{O} \subset \mathbb{R}^d, d = 2,3$ be an arbitrary, possibly unbounded, open domain with smooth boundary $\partial\mathcal{O}$ if the domain has a boundary. Let $\mathbf{u} = \mathbf{u}(x,t)$ and $p = p(x,t)$ denote the velocity and pressure fields and $\mathbf{g} = \mathbf{g}(\cdot,\cdot) : \mathcal{O} \times (0,T) \to \mathbb{R}^d$ is an external body force. Let us consider the Navier-Stokes model perturbed by the Gaussian as well as Lévy type stochastic forces as follows:

$$\begin{aligned} d\mathbf{u} + (-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p)dt &= \mathbf{g}dt + \sigma(t,\mathbf{u})d\mathbf{W} \\ &+ \sum_{k=1}^{\infty} \int_{0<|z_k|_{\mathcal{Z}}<1} \phi_k\big(\mathbf{u}(x,t-), z_k\big)\widetilde{\pi}_k(dt, dz_k) \\ &+ \sum_{k=1}^{\infty} \int_{|z_k|_{\mathcal{Z}}\geq 1} \psi_k\big(\mathbf{u}(x,t-), z_k\big)\pi_k(dt, dz_k) \quad \text{in} \quad \mathcal{O}\times(0,T) \end{aligned} \tag{2.1}$$

with the incompressibility condition

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \mathcal{O}\times(0,T), \tag{2.2}$$

the non-slip boundary condition and the initial condition respectively:

$$\mathbf{u} \;=\; 0 \;\text{ on }\; \partial\mathcal{O}\times(0,T), \quad \mathbf{u}(x,0)=\mathbf{u}_0(x) \;\text{ in }\; \mathcal{O}. \tag{2.3}$$

One may also require the far-field condition

$$\mathbf{u}(x,t)\to 0 \text{ as } |x|\to\infty \text{ if } \mathcal{O} \text{ is unbounded.}$$

In (2.1), the parameter $\nu$ is the kinematic viscosity and $\mathbf{W}(\cdot)$ represents a space-time Gaussian noise term modelled as a Hilbert-space valued Wiener process that is independent of the compensated Poisson measure $\widetilde{\pi}_k(dt,dz_k)=\pi_k(dt,dz_k)-dt\mu_k(dz_k)$ for all $k=1,2\cdots$, where $\mu_k(\cdot)=\mathbb{E}(\pi_k(1,\cdot))$ is the intensity measure.

Now we proceed to state the main results of this paper. Let $\widetilde{\Omega}=\mathbb{D}(0,T;\mathbb{V}')_J\cap L^\infty(0,T;\mathbb{H})_{w^*}\cap\mathbb{L}^2(0,T;\mathbb{V})_w$ (see, before section 3.1 for the precise definitions of the topologies) be the path space with $\omega\in\widetilde{\Omega}$ denoting a generic point in $\widetilde{\Omega}$, where $\mathbb{D}(.;.)$ is the class of *càdlàg* functions from $[0,T]$ into $\mathbb{V}'$. *Càdlàg* functions are right continuous and have left limits at any point $t\in[0,T]$ (and for more details on this space, see, [7, 25]). Let $\widetilde{\mathscr{F}}$ be the $\sigma$-algebra of Borel subsets of the Lusin space $\widetilde{\Omega}$ endowed with supremum of the topologies. Let $\xi$ be the mapping from $[0,T]\times\widetilde{\Omega}\to\mathbb{V}'$ defined by $\xi(t,\omega):=\omega(t)$ and let $\widetilde{\mathscr{F}_t}$ be the canonical filtration generated by the functions $\xi(t,\omega)$ on $\widetilde{\Omega}$, that is, $\widetilde{\mathscr{F}_t}=\sigma\{\xi(s,\omega):0\le s\le t\}$ for all $t\in[0,T]$. We call the new measure $\mathbb{P}$ such that $P\circ\mathbf{u}^{-1}=\mathbb{P}$ is the law of the processes $\mathbf{u}$ defined on $(\widetilde{\Omega},\widetilde{\mathscr{F}},\widetilde{\mathscr{F}_t})$.

**Definition 2.1** (Martingale Problem)**.** Let $\mathscr{L}f(\cdot)$ be the generator as defined in (2.17). Then given an initial measure $\mathscr{P}$ on $\mathbb{H}$, a solution to $\mathscr{L}f$-martingale problem is a probability measure $\mathbb{P}:\mathcal{B}(\widetilde{\Omega})\to[0,1]$ on $(\widetilde{\Omega},\widetilde{\mathscr{F}},\widetilde{\mathscr{F}_t})$ such that $\mathbb{P}\{\xi(0)=\mathbf{u}_0\}=1$ and the process

$$\mathbf{M}_t^f:=f(\xi(t))-f(\xi(0))-\int_0^t\mathscr{L}f(\xi(s))ds, \quad \text{with} \quad f\in\mathcal{D}(\mathscr{L})$$

is a $\mathbb{R}$-valued locally square integrable $(\widetilde{\Omega},\widetilde{\mathscr{F}},\widetilde{\mathscr{F}_t},\mathbb{P})$-local càdlàg martingale.

The following is the first main theorem concerning the existence of martingale solutions.

**Theorem 2.1.** *Let $\mathcal{O}$ be an open domain in $\mathbb{R}^d, d=2,3$. Suppose the noise coefficients $\sigma$ and $\phi_k,\psi_k,k=1,2\cdots$ satisfy Assumptions 2.1 and 2.2 respectively. Then for a given initial probability measure $\mathscr{P}$ on $\mathbb{H}$ satisfying $\int_{\mathbb{H}}|x|^2d\mathscr{P}(x)<\infty$, there exists a martingale solution to the equation (2.1)(or the abstract form (2.10)).*

The uniqueness of martingale solutions is the uniqueness in the sense of probability laws. The solution of (2.1) is unique in the sense of probability law if whenever $\mathbf{u}(t)$ and $\mathbf{v}(t),t\ge0$ are two solutions such that the probability laws of $\mathbf{u}_0$ and $\mathbf{v}_0$ are the same, then the laws of $\mathbf{u}$ and $\mathbf{v}$ are the same. The pathwise uniqueness holds for (2.1) if $\mathbf{U}(t)=\{\mathbf{u}(t),\mathbf{u}_0,\mathbb{Q},\pi_k;t\ge0,k\ge1\}$ and $\mathbf{U}'(t)=\{\mathbf{u}'(t),\mathbf{u}_0',\mathbb{Q}',\pi_k';t\ge0,k\ge1\}$ are any two solutions defined on a same probability space $(\Omega,\mathscr{F},\mathscr{F}_t,P)$, then $\mathbf{u}_0=\mathbf{u}_0',\mathbb{Q}=\mathbb{Q}'$ and $\pi_k=\pi_k'$ imply $P\{\mathbf{U}(t)=\mathbf{U}'(t);t\ge0,k\ge1\}=1$.

The existence of a solution to the martingale problem is equivalent to that of a weak solution to the stochastic differential equations (see, [22]). Hence one may apply the Yamada-Watanabe construction ([42], see also [16]) to deduce that the existence of a weak solution together with the pathwise uniqueness property imply the uniqueness in law.

Thus, in essence we only need to prove the following result concerning the pathwise uniqueness of solutions for the system (2.1).

**Theorem 2.2.** *Let $\mathcal{O}$ be an open domain in $\mathbb{R}^d, d = 2, 3$. Let the noise coefficients $\sigma$ and $\phi_k, \psi_k, k = 1, 2\cdots$ satisfy Assumptions 2.1 and 2.2 respectively. Let $\mathbf{u}, \mathbf{v} \in \mathbb{D}(0, T; \mathbb{V}') \cap L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V})$ be the two paths defined on a same probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ with same $\mathbb{Q}$-Wiener process $\mathbf{W}$ and Poisson measure $\pi_k, k = 1, 2\cdots$ satisfying the system (2.1). Then there exist positive constants $C_\nu$ and $\bar{C}$ such that*

$$\mathbb{E}\Big(|\mathbf{u}(t) - \mathbf{v}(t)|^2 \exp\big\{ - C_\nu \int_0^t \|\mathbf{v}(s)\|^{4/(4-d)} ds\big\}\Big) \leq \exp(\bar{C}T)\mathbb{E}|\mathbf{u}(0) - \mathbf{v}(0)|^2. \tag{2.4}$$

*If the initial data $\mathbf{u}(0) = \mathbf{v}(0) = \mathbf{u}_0$, then*

*(i) For $d = 2$, the solution $\mathbf{u}$ is pathwise unique, that is, $\mathbf{u}(t) = \mathbf{v}(t), P$-a.s.*
*(ii) For $d = 3$, the solution $\mathbf{u}$ is pathwise unique under the additional condition $\mathbb{E}\int_0^T \|\mathbf{v}(s)\|^4 ds < \infty$.*

The proofs of Theorem 2.1 and 2.2 are given in section 4.

2.1. **Assumptions and semimartingale formulation.** Now we proceed for the semimartingale framework of the stochastic Navier-Stokes system (2.1)-(2.3) by making use of the following conventional notations. Let us denote $\mathcal{V} = \{\mathbf{v} \in C_0^\infty(\mathcal{O}) : \nabla\cdot\mathbf{v} = 0\}$. Let $\mathbb{H}$ and $\mathbb{V}$ be the completion of $\mathcal{V}$ in $\mathbb{L}^2(\mathcal{O})$ and $\mathbb{H}^1(\mathcal{O})$ norms respectively. In the case of bounded domains, we then get

$$\mathbb{H} := \{\mathbf{v} \in \mathbb{L}^2(\mathcal{O}; \mathbb{R}^d) : \nabla\cdot\mathbf{v} = 0 \ \ \mathbf{v}\cdot\mathbf{n}|_{\partial\mathcal{O}} = 0\} \tag{2.5}$$

with norm $\|\mathbf{v}\|_{\mathbb{H}} := \big(\int_{\mathcal{O}} |\mathbf{v}|^2 dx\big)^{1/2} = |\mathbf{v}|$, where $\mathbf{n}$ is the outward normal to $\partial\mathcal{O}$ and

$$\mathbb{V} := \{\mathbf{v} \in \mathbb{H}_0^1(\mathcal{O}; \mathbb{R}^d) : \nabla\cdot\mathbf{v} = 0\} \tag{2.6}$$

with norm $\|\mathbf{v}\|_{\mathbb{V}} := \big(\int_{\mathcal{O}} |\nabla\mathbf{v}|^2 dx\big)^{1/2} = \|\mathbf{v}\|$. The inner product in the Hilbert space $\mathbb{H}$ is denoted by $(\cdot,\cdot)$ and the induced duality, for instance between the spaces $\mathbb{V}$ and its dual $\mathbb{V}'$, by $\langle\cdot,\cdot\rangle$.

Let $P_{\mathbb{H}} : \mathbb{L}^2(\mathcal{O}) \to \mathbb{H}$ be the Helmholtz-Hodge (orthogonal) projection, then define the Stokes operator

$$\mathbf{A} : \mathcal{D}(\mathbf{A}) \to \mathbb{H} \quad \text{with} \quad \mathbf{A}\mathbf{v} = -P_{\mathbb{H}}\Delta\mathbf{v}, \tag{2.7}$$

where $\mathcal{D}(\mathbf{A}) = \mathbb{V} \cap \mathbb{H}^2(\mathcal{O}) = \{\mathbf{v} \in \mathbb{H}_0^1(\mathcal{O}) \cap \mathbb{H}^2(\mathcal{O}) : \nabla\cdot\mathbf{v} = 0\}$ and the nonlinear operator

$$\mathbf{B} : \mathcal{D}(\mathbf{B}) \subset \mathbb{H}\times\mathbb{V} \to \mathbb{H} \quad \text{with} \quad \mathbf{B}(\mathbf{u}, \mathbf{v}) = P_{\mathbb{H}}(\mathbf{u}\cdot\nabla\mathbf{v}). \tag{2.8}$$

Note that with the use of the Gelfand triple $\mathbb{V} \subset \mathbb{H} = \mathbb{H}' \subset \mathbb{V}'$, we may consider $\mathbf{A}$ as the mapping from $\mathbb{V}$ into $\mathbb{V}'$. Besides, setting $\mathbf{u} = (u_i), \mathbf{v} = (v_i)$ and $\mathbf{w} = (w_i)$, we can write

$$\langle\mathbf{A}\mathbf{u}, \mathbf{w}\rangle = \nu\sum_{i,j}\int_{\mathcal{O}} \partial_i u_j \partial_i w_j dx, \quad \text{where} \quad \partial_i u_j = \frac{\partial u_j}{\partial x_i}.$$

Define trilinear form $b(\cdot,\cdot,\cdot) : \mathbb{V}\times\mathbb{V}\times\mathbb{V} \to \mathbb{R}$ by the relation

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j}\int_{\mathcal{O}} u_i \partial_i v_j w_j dx$$

whence we can define the bilinear operator $\mathbf{B}(\cdot,\cdot):\mathbb{V}\times\mathbb{V}\to\mathbb{V}'$ such that

$$\langle \mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle = b(\mathbf{u},\mathbf{v},\mathbf{w}), \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{V}.$$

Integrating by parts in the previous equality, we also get

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}) \quad \text{and} \quad b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0,\ \forall\ \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{V}. \tag{2.9}$$

According to Helmholtz decomposition $\mathbb{L}^2(\mathcal{O})$ admits an orthogonal decomposition of a sum of two non-trivial subspaces such that $\mathbb{L}^2(\mathcal{O}) = \mathbb{H}(\mathcal{O})\oplus\mathbb{H}^\perp(\mathcal{O})$, where the space $\mathbb{H}^\perp$ is characterized by $\mathbb{H}^\perp(\mathcal{O}) = \{\mathbf{u}\in\mathbb{L}^2(\mathcal{O}) : \mathbf{u} = \nabla p,\ p\in\mathbb{H}^1(\mathcal{O})\}$.

Thus, taking into account of all the preceding observations along with the projection $P_{\mathbb{H}}$ applied to the system (2.1), we arrive at

$$\begin{aligned}\mathbf{u}(t) &= \mathbf{u}_0 + \int_0^t \Big( -\nu\mathbf{A}\mathbf{u}(s) - \mathbf{B}(\mathbf{u}(s)) + \mathbf{g}(s) \\ &\quad + \sum_{k=1}^\infty \int_{|z_k|\geq 1} \psi_k\big(\mathbf{u}(s), z_k\big)\mu_k(dz_k)\Big) ds + \mathbf{M}_t \quad \text{in} \quad \widetilde{\Omega} \\ \mathbf{u}(0) &= \mathbf{u}_0 \ \text{ in } \ \mathbb{H}\end{aligned} \tag{2.10}$$

where $\mathbf{B}(\mathbf{u}) = \mathbf{B}(\mathbf{u},\mathbf{u}),\ \mathbf{g}\in\mathbb{L}^2(0,T;\mathbb{V}')$ and the local martingale

$$\begin{aligned}\mathbf{M}_t &= \int_0^t \sigma(s,\mathbf{u}(s))d\mathbf{W}(s) + \int_0^t \sum_{k=1}^\infty \int_{|z_k|<1} \phi_k\big(\mathbf{u}(s-), z_k\big)\widetilde{\pi}_k(ds, dz_k) \\ &\quad + \int_0^t \sum_{k=1}^\infty \int_{|z_k|\geq 1} \psi_k\big(\mathbf{u}(s-), z_k\big)\widetilde{\pi}_k(ds, dz_k).\end{aligned} \tag{2.11}$$

Let $\mathbb{Q}$ be a positive, symmetric and trace class operator on $\mathbb{H}$. Then there is an orthonormal basis $\{e_k\}$ of $\mathbb{H}$ consisting of eigenvectors of $\mathbb{Q}$ such that $\mathbb{Q}e_k = \lambda_k e_k, k\in\mathbb{N}$. Here $\lambda_k$ is the eigenvalue corresponding to $\{e_k\}$ which is real and positive with

$$\text{tr}(\mathbb{Q}) = \sum_{k=1}^\infty \lambda_k < \infty \ \text{ and } \ \mathbb{Q}^{1/2}\mathbf{v} = \sum_{k=1}^\infty \sqrt{\lambda_k}(\mathbf{v}, e_k)e_k,\ \mathbf{v}\in\mathbb{H}.$$

Let $(\Omega,\mathscr{F},P)$ be a probability space equipped with an increasing family of subsigma fields $\{\mathscr{F}_t\}_{0\leq t\leq T}$ of $\mathscr{F}$ satisfying $(i)$ $\mathscr{F}_0$ contains all elements $E\in\mathscr{F}$ with $P(E) = 0$ $(ii)$ $\mathscr{F}_t = \mathscr{F}_{t+} = \bigcap_{s>t}\mathscr{F}_s$ for $0\leq t\leq T$. Then

**Definition 2.2.** A stochastic process $\{\mathbf{W}(t) : 0\leq t\leq T\}$ is said to be a $\mathbb{H}$-valued $\{\mathscr{F}_t\}$-adapted Wiener process with covariance operator $\mathbb{Q}$ if for each nonzero $h\in\mathbb{H}$, $|\mathbb{Q}^{1/2}h|^{-1}(\mathbf{W}(t), h)$ is a standard one-dimensional Wiener process and for each $h\in\mathbb{H}$, $(\mathbf{W}(t), h)$ is a $\mathscr{F}_t-$martingale.

The stochastic process $\{\mathbf{W}(t) : 0\leq t\leq T\}$ is a $\mathbb{H}$-valued Wiener process with covariance $\mathbb{Q}$ if and only if for arbitrary $t$, the process $\mathbf{W}(t)$ can be expressed as $\mathbf{W}(t) = \sum_{k=1}^\infty \sqrt{\lambda_k}\beta_k(t)e_k$, where $\beta_k(t), k\in\mathbb{N}$ are independent one dimensional Brownian motions on $(\Omega,\mathscr{F},P)$ and $\{e_k\}$ are the orthonormal basis functions, as explained above, of $\mathbb{H}$(see, [16], Page 93, Definition 3.2.1 and Theorem 3.2.2). Besides, the $\mathbb{Q}$-Wiener process $\mathbf{W}(t)$ satisfies $\mathbb{E}(\mathbf{W}(t)) = 0$ and $\text{Cov}(\mathbf{W}(t)) = t\mathbb{Q},\ t\geq 0$.

We denote by $\mathcal{L}_{HS}$, the space of all bounded linear operators $S:\mathbb{H}\to\mathbb{H}$ such that $\sum_{k=1}^\infty |S\mathbb{Q}^{1/2}e_k|^2 < \infty$, where $\{e_k\}$ is an orthonormal basis in $\mathbb{H}$ and $\|\cdot\|_{\mathcal{L}_{HS}}$

be the norm on $\mathcal{L}_{HS}$ which is given by

$$\begin{aligned}\sum_{k=1}^{\infty}|S\mathbb{Q}^{1/2}e_k|^2 &= \sum_{k=1}^{\infty}(\mathbb{Q}^{1/2}S^*S\mathbb{Q}^{1/2}e_k, e_k) = \mathrm{tr}(\mathbb{Q}^{1/2}S^*S\mathbb{Q}^{1/2}) \\ &= \mathrm{tr}(S\mathbb{Q}S^*) := \|S\|^2_{\mathcal{L}_{HS}}.\end{aligned}$$

The Gaussian noise coefficient mapping $\sigma : [0,T]\times\mathbb{H}\to\mathcal{L}_{HS}(\mathbb{H})$ is measurable from $([0,T]\times\mathbb{H}, \mathcal{B}([0,T]\times\mathbb{H}))$ into $(\mathcal{L}_{HS}(\mathbb{H}), \mathcal{B}(\mathcal{L}_{HS}(\mathbb{H}))$ satisfies the following:

**Assumption 2.1.** $[H1]$ *For all* $t\in[0,T]$, *there is a positive constant* $\widetilde{N}_1$ *such that*

$$\|\sigma(t,\mathbf{u})-\sigma(t,\mathbf{v})\|^2_{\mathcal{L}_{HS}} \le \widetilde{N}_1|\mathbf{u}-\mathbf{v}|^2, \quad \forall \mathbf{u},\mathbf{v}\in\mathbb{H}. \tag{2.12}$$

$[H2]$ *For all* $t\in[0,T]$, *there is a positive constant* $\widetilde{N}_2$ *satisfying the growth condition*

$$\|\sigma(t,\mathbf{u})\|^2_{\mathcal{L}_{HS}} \le \widetilde{N}_2(1+|\mathbf{u}|^2),\ \forall\mathbf{u}\in\mathbb{H}. \tag{2.13}$$

Next we state the assumptions on the jump noise coefficients. Let $\mathcal{Z}$ be a separable Banach space and $\mathbf{L}(t)_{t\ge0}$ be a $\mathcal{Z}-$valued Lévy process with jump $\Delta\mathbf{L}(t) := \mathbf{L}(t)-\mathbf{L}(t-)$ at $t\ge0$, then

$$\pi_k([0,t],\Gamma) = \#\{s\in[0,t] : \Delta\mathbf{L}(s)\in\Gamma\},\ \text{where } \Gamma\in\mathcal{B}(\mathcal{Z}\backslash\{0\})$$

is the Poisson random measure or jump measure associated with the Lévy process $\mathbf{L}(t)$. Here $\mathcal{B}(\mathcal{Z}\backslash\{0\})$ is the Borel $\sigma$-field, $\pi_k(dt,dz_k)$ is the random measure defined on $(\mathbb{R}^+\times(\mathcal{Z}\backslash\{0\}), \mathcal{B}(\mathbb{R}^+\times(\mathcal{Z}\backslash\{0\})))$, and $\mu_k(\cdot)$ is the intensity measure on $((\mathcal{Z}\backslash\{0\}), \mathcal{B}(\mathcal{Z}\backslash\{0\}).)$ Then the compensated Poisson random measure is defined by $\widetilde{\pi}_k(dt,dz_k) = \pi_k(dt,dz_k) - dt\mu_k(dz_k)$, for all $k=1,2\cdots$, where $dt\mu_k(dz_k)$ is the compensator of the Lévy process $\mathbf{L}(t)$ with Lesbegue measure $dt$. The intensity measure $\mu_k(\cdot)$ on $\mathcal{Z}$ satisfies the conditions $\mu_k(\{0\})=0$, for all $k=1,2,\cdots$ and

$$\begin{aligned}&\sum_{k=1}^{\infty}\int_{\mathcal{Z}}(1\wedge|z_k|^2)\mu_k(dz_k) < +\infty \quad \text{and} \\ &\sum_{k=1}^{\infty}\int_{|z_k|\ge1}|z_k|^p\mu_k(dz_k) < +\infty,\ \forall p\ge1.\end{aligned} \tag{2.14}$$

The process $\big(\int_{|z_k|<1}\phi_k\big(\mathbf{u}(t-),z_k\big)\widetilde{\pi}_k(t,dz_k), t\ge0, k\ge1\big)$ in (2.1) describing the sum of small jumps of size less than 1 is the compensated Poisson process while the process $\big(\int_{|z_k|\ge1}\psi_k\big(\mathbf{u}(t-),z_k\big)\pi_k(t,dz_k), t\ge0, k\ge1\big)$ describing the large jumps of size greater than or equal to 1 in (2.1) is a compound Poisson process (see, [2]).

The jump noise coefficients mappings $\phi_k,\psi_k : \mathbb{H}\times\mathcal{Z}\to\mathbb{H}, k=1,2,\cdots$ are measurable from $(\mathbb{H}\times\mathcal{Z}, \mathcal{B}(\mathbb{H}\times\mathcal{Z}))$ into $(\mathbb{H},\mathcal{B}(\mathbb{H}))$ satisfy the following

**Assumption 2.2.** $[H3]$ *For all* $t\in[0,T]$, *there is a positive constant* $N_2$ *such that*

$$\begin{aligned}&\sum_{k=1}^{\infty}\int_{|z_k|<1}|\phi_k(\mathbf{u},z_k)-\phi_k(\mathbf{v},z_k)|^2\mu_k(dz_k) \\ &\quad+\sum_{k=1}^{\infty}\int_{|z_k|\ge1}|\psi_k(\mathbf{u},z_k)-\psi_k(\mathbf{v},z_k)|^2\mu_k(dz_k) \le N_2|\mathbf{u}-\mathbf{v}|^2,\ \forall\mathbf{u},\mathbf{v}\in\mathbb{H}.\end{aligned} \tag{2.15}$$

[*H*4] *For all $p \geq 1$ and $t \in [0,T]$, there is a positive constant $N_3$ satisfying the growth condition*

$$\sum_{k=1}^{\infty}\int_{|z_k|<1}|\phi_k(\mathbf{u},z_k)|^p\mu_k(dz_k) + \sum_{k=1}^{\infty}\int_{|z_k|\geq 1}|\psi_k(\mathbf{u},z_k)|^p\mu_k(dz_k) \leq N_3(1+|\mathbf{u}|^p),\ \forall \mathbf{u}\in\mathbb{H}. \tag{2.16}$$

Next we state and prove some of the results which are used frequently in the rest of the paper. If $T_t$ is the transition semigroup of the Itô-Lévy process $\mathbf{u}(t)$ defined on a complete probability space $(\Omega, \mathscr{F}_t, P)$, then the generator $\mathscr{L}$ of $\mathbf{u}(t)$ can be defined on functions $f(\cdot):\mathbb{H}\to\mathbb{R}$ by

$$\mathscr{L}f = \lim_{t\downarrow 0}\frac{T_t f - f}{t} \text{ for each } f\in\mathcal{D}(\mathscr{L}),$$

where

$$\mathcal{D}(\mathscr{L}) := \{f:\mathbb{H}\to\mathbb{R} \text{ such that } \lim_{t\downarrow 0}\frac{T_t f - f}{t} \text{ exists }\}.$$

However, in the following definition we give the formal generator for some class of test functions.

**Definition 2.3** (Formal Generator)**.** For $f\in\mathcal{D}(\mathscr{L})$, the formal generator $\mathscr{L}f$ is given by

$$\begin{aligned}\mathscr{L}f(\mathbf{u}) = &-\Big\langle \nu\mathbf{A}\mathbf{u}+\mathbf{B}(\mathbf{u})-\mathbf{g}, \frac{\partial f}{\partial \mathbf{u}}\Big\rangle + \frac{1}{2}\mathrm{tr}\Big(\sigma(t,\mathbf{u})\mathbb{Q}\sigma^*(t,\mathbf{u})\frac{\partial^2 f}{\partial \mathbf{u}^2}\Big) \\ &+\sum_{k=1}^{\infty}\int_{|z_k|<1}\Big\{f(\mathbf{u}+\phi_k(\mathbf{u},z_k)) - f(\mathbf{u}) - \Big\langle \phi_k(\mathbf{u},z_k),\frac{\partial f}{\partial \mathbf{u}}\Big\rangle\Big\}\mu_k(dz_k) \\ &+\sum_{k=1}^{\infty}\int_{|z_k|\geq 1}\Big\{f(\mathbf{u}+\psi_k(\mathbf{u},z_k)) - f(\mathbf{u})\Big\}\mu_k(dz_k),\ \forall \mathbf{u}\in\mathcal{D}(\mathbf{A}).\end{aligned} \tag{2.17}$$

Finite dimensional Lévy generator can be found in [29, 2] and for an infinite dimensional case, see, [26]. Infinite dimensional Lévy- Khintchine ([38]) representation is an another form of expressing such generators. As an example for the functions $f\in\mathcal{D}(\mathscr{L})$, one may consider the following:

**Remark 2.1.** The test functions $f(\cdot)$ of the form

$$f(\mathbf{u}) := \varphi(\langle e_1,\mathbf{u}\rangle, \langle e_2,\mathbf{u}\rangle,\cdots,\langle e_m,\mathbf{u}\rangle),\quad \mathbf{u}\in\mathbb{H} \tag{2.18}$$

where $\varphi(\cdot):\mathbb{R}^m\to\mathbb{R}$ is a smooth function with compact support in $\mathbb{R}^m$, $e_k\in\mathcal{D}(\mathbf{A}), k=1,2,\cdots,m$.

**Lemma 2.1** (Burkholder-Davis-Gundy, [27, 26])**.** *For every fixed $p\geq 1$, there is a constant $C_p\in(0,\infty)$ such that for every real-valued square integrable càdlàg martingale $\mathbf{M}_t$ with $\mathbf{M}_0 = 0$, and for every $T\geq 0$,*

$$C_p^{-1}\mathbb{E}[\mathbf{M}]_T^{p/2} \leq \mathbb{E}\sup_{t\in[0,T]}|\mathbf{M}_t|^p \leq C_p\mathbb{E}[\mathbf{M}]_T^{p/2}, \tag{2.19}$$

*where $[\mathbf{M}]_t$ is the quadratic variation of $\mathbf{M}_t$ and the constant $C_p$ does not depend on the choice of $\mathbf{M}_t$.*

**Lemma 2.2** (Gagliardo-Nirenberg, [5]). *Let $\mathcal{O} \subset \mathbb{R}^d$ and $\mathbf{v} \in \mathbb{W}^{1,p}(\mathcal{O}), p \geq 1$. Then for every fixed numbers $q, r \geq 1$, there exists a constant $N > 0$ such that*

$$\|\mathbf{v}\|_{\mathbb{L}^r(\mathcal{O})} \leq N_{d,p,q} \|\nabla \mathbf{v}\|^{\lambda}_{\mathbb{L}^p(\mathcal{O})} \|\mathbf{v}\|^{1-\lambda}_{\mathbb{L}^q(\mathcal{O})}, \quad \lambda \in [0,1] \tag{2.20}$$

*where the numbers $p, q, r$ and $\lambda$ satisfy the relation*

$$\lambda = \Big(\frac{1}{q} - \frac{1}{r}\Big)\Big(\frac{1}{d} - \frac{1}{p} + \frac{1}{q}\Big)^{-1}.$$

The particular cases of Lemma 2.2 are the well known inequalities, due to Ladyzhenskaya ([17], Lemmas 1,2 and 3, Page 8-10), which are given below.

**Lemma 2.3.** *For $\mathbf{v} \in C_0^\infty(\mathbb{R}^d), d = 2, 3$, there exists a constant $L$ such that*

$$\|\mathbf{v}\|^4_{\mathbb{L}^4(\mathbb{R}^d)} \quad \leq \quad L\|\mathbf{v}\|^{4-d}_{\mathbb{L}^2(\mathbb{R}^d)} \|\nabla \mathbf{v}\|^d_{\mathbb{L}^2(\mathbb{R}^d)}, \tag{2.21}$$

*where $L = 2, 4$ respectively for $d = 2, 3$ and*

$$\|\mathbf{v}\|^6_{\mathbb{L}^6(\mathbb{R}^3)} \quad \leq \quad 48\|\nabla \mathbf{v}\|^6_{\mathbb{L}^2(\mathbb{R}^3)}. \tag{2.22}$$

Note that Lemma 2.3 holds true for all functions $\mathbf{v} \in \mathbb{H}^1_0(\mathcal{O})$. From (2.21), the nonlinear term $\mathbf{B}(\cdot)$ satisfies the estimate

$$\|\mathbf{B}(\mathbf{v})\|_{\mathbb{V}'(\mathcal{O})} \leq \|\mathbf{v}\|^2_{\mathbb{L}^4(\mathcal{O})} \leq L|\mathbf{v}|^{2-(d/2)}\|\mathbf{v}\|^{d/2}, \quad \forall \mathbf{v} \in \mathbb{V},\ d = 2, 3. \tag{2.23}$$

Next we prove the local monotonicity of the operator $\Theta(\mathbf{u}) + \lambda \mathbf{u}$, where $\Theta(\mathbf{u}) := \nu \mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u})$ and $\lambda > 0$ :

**Lemma 2.4.** *For a given $\rho > 0$ and $p > d$, let $B_r$ denote the ball $B_r = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\|_{\mathbb{L}^p(\mathcal{O})} \leq \rho\}$ and $\Theta(\cdot)$ be the nonlinear operator as above on $\mathbb{V}$. Then for any $\mathbf{u} \in \mathbb{V}, \mathbf{v} \in B_r$ and $\mathbf{w} = \mathbf{u} - \mathbf{v}$, there exists a $\lambda > 0$ such that the operator $\Theta(\mathbf{u}) + \lambda\mathbf{u}$ is monotone in $B_r$ :*

$$\langle \Theta(\mathbf{u}) - \Theta(\mathbf{v}), \mathbf{w}\rangle + \lambda|\mathbf{w}|^2 \geq \frac{\nu}{2}\|\mathbf{w}\|^2, \tag{2.24}$$

*where $\lambda = C_{p,d,\nu}\rho^{2p/(p-d)}$ and the constant $C_{p,d,\nu}$ is defined below.*

*Proof.* Let us first note that $\langle \mathbf{A}\mathbf{w}, \mathbf{w}\rangle = -\langle \Delta \mathbf{w}, P_{\mathbb{H}}\mathbf{w}\rangle = \|\mathbf{w}\|^2$. From the bilinear and trilinear forms defined in (2.9), we have

$$\langle \mathbf{B}(\mathbf{u}), \mathbf{w}\rangle = -b(\mathbf{u}, \mathbf{w}, \mathbf{u}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{w}) - b(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}).$$

Similarly $\langle \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle = -b(\mathbf{v}, \mathbf{w}, \mathbf{v})$ so that

$$\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle = -b(\mathbf{w}, \mathbf{w}, \mathbf{v}) = -\langle \mathbf{B}(\mathbf{w}), \mathbf{v}\rangle, \forall \mathbf{u} \in \mathbb{V} \text{ and } \mathbf{v} \in B_r.$$

If we apply Hölder's inequality with exponents $q = 2, r = 2p/(p-2)$ and fixed $p$,

$$|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle| \leq \|\mathbf{w}\|_{\mathbb{L}^{2p/(p-2)}(\mathcal{O})}\|\mathbf{w}\|\|\mathbf{v}\|_{\mathbb{L}^p(\mathcal{O})};$$

but the Gagliardo-Nirenberg inequality (2.20) with exponents $\widetilde{p} = \widetilde{q} = 2$ and $\widetilde{r} = 2p/(p-2)$, further gives

$$\|\mathbf{w}\|_{\mathbb{L}^{2p/(p-2)}(\mathcal{O})} \leq N_{d,2,2}\|\mathbf{w}\|^{d/p}|\mathbf{w}|^{1-(d/p)}.$$

The inequality

$$ab \leq \nu\frac{a^m}{2} + \frac{b^l}{l(\frac{\nu m}{2})^{l/m}} \quad \text{for any} \quad a, b \geq 0$$

with $l = 2p/(p-d)$ and $m = 2p/(p+d)$ leads to

$$|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle| \leq \frac{\nu}{2}\|\mathbf{w}\|^2 + C_{p,d,\nu}|\mathbf{w}|^2\|\mathbf{v}\|^{2p/(p-d)}_{\mathbb{L}^p(\mathcal{O})}, \tag{2.25}$$

where

$$C_{p,d,\nu} = \frac{(N_{d,2,2})^{2p/(p-d)}(p-d)}{2p\big(p\nu/(p+d)\big)^{(p+d)/(p-d)}}.$$

Therefore,

$$\langle \Theta(\mathbf{u}) - \Theta(\mathbf{v}), \mathbf{w}\rangle \geq \frac{\nu}{2}\|\mathbf{w}\|^2 - C_{p,d,\nu}\rho^{2p/(p-d)}|\mathbf{w}|^2. \tag{2.26}$$

This completes the proof. □

The following lemma is useful in proving the main result.

**Lemma 2.5.** *Let* $C_\mu = \sum_{k=1}^\infty \int_{|z_k|\geq 1}\mu_k(dz_k) < \infty$ *and* $\mathbf{u}_n \to \mathbf{u}$ *strongly in* $\mathbb{L}^2(0,T;\mathbb{H})$*. Then* $\bar{I}_n := \int_0^T \big(\sum_{k=1}^\infty \int_{|z_k|\geq 1}\psi_k(\mathbf{u}_n(t),z_k)\mu_k(dz_k), \mathbf{u}_n(t)\big)dt$ *converges to the integral* $I := \int_0^T \big(\sum_{k=1}^\infty \int_{|z_k|\geq 1}\psi_k(\mathbf{u}(t),z_k)\mu_k(dz_k), \mathbf{u}(t)\big)dt$.

*Proof.* Consider the integral

$$\begin{aligned} I_n &:= \bar{I}_n - I \\ &= \int_0^T \big(\sum_{k=1}^\infty \int_{|z_k|\geq 1}\psi_k(\mathbf{u}_n(t),z_k)\mu_k(dz_k), \mathbf{u}_n(t)-\mathbf{u}(t)\big)dt \\ &\quad + \int_0^T \big(\sum_{k=1}^\infty \int_{|z_k|\geq 1}\big(\psi_k(\mathbf{u}_n(t),z_k) - \psi_k(\mathbf{u}(t),z_k)\big)\mu_k(dz_k), \mathbf{u}(t)\big)dt \\ &:= \widehat{I}_n + \widetilde{I}_n. \end{aligned}$$

Note that

$$\begin{aligned} |\widehat{I}_n| &\leq \int_0^T \Big|\sum_{k=1}^\infty \int_{|z_k|\geq 1}\psi_k(\mathbf{u}_n(t),z_k)\mu_k(dz_k)\Big||\mathbf{u}_n(t)-\mathbf{u}(t)|dt \\ &\leq \Big(\int_0^T \Big|\sum_{k=1}^\infty \int_{|z_k|\geq 1}\psi_k(\mathbf{u}_n(t),z_k)\mu_k(dz_k)\Big|^2 dt\Big)^{1/2}\Big(\int_0^T |\mathbf{u}_n(t)-\mathbf{u}(t)|^2dt\Big)^{1/2} \\ &\leq \Big(\int_0^T C_\mu \sum_{k=1}^\infty \int_{|z_k|\geq 1}|\psi_k(\mathbf{u}_n(t),z_k)|^2\mu_k(dz_k)dt\Big)^{1/2}\Big(\int_0^T |\mathbf{u}_n(t)-\mathbf{u}(t)|^2dt\Big)^{1/2} \\ &\leq \sqrt{C_\mu N_3}\Big(T + \int_0^T |\mathbf{u}_n(t)|^2dt\Big)^{1/2}\Big(\int_0^T |\mathbf{u}_n(t)-\mathbf{u}(t)|^2dt\Big)^{1/2} \to 0 \end{aligned}$$

as $n\to\infty$ since the first integral is finite and the second integral tends to zero due to the strong convergence of $\mathbf{u}_n \to \mathbf{u}$ in $\mathbb{L}^2(0,T;\mathbb{H})$. Estimating the integral $\widetilde{I}_n$, we get

$$\begin{aligned} |\widetilde{I}_n| &\leq \Big(\int_0^T C_\mu \sum_{k=1}^\infty \int_{|z_k|\geq 1}|\psi_k(\mathbf{u}_n(t),z_k) - \psi_k(\mathbf{u}(t),z_k)|^2\mu_k(dz_k)dt\Big)^{1/2} \\ &\qquad \times\Big(\int_0^T |\mathbf{u}(t)|^2dt\Big)^{1/2} \\ &\leq \sqrt{C_\mu N_2}\Big(\int_0^T |\mathbf{u}_n(t)-\mathbf{u}(t)|^2dt\Big)^{1/2}\Big(\int_0^T |\mathbf{u}(t)|^2dt\Big)^{1/2} \to 0 \text{ as } n\to\infty, \end{aligned}$$

due to the similar reasoning as before. Thus, the conclusion of the lemma follows. □

2.2. **Quadratic variation and Meyer process of the martingale.** Let $\mathbf{u}$ be an adapted process in a stochastic basis $(\Omega, \mathscr{F}_t, P)$ with paths in $\mathbb{D}(0,T;\mathbb{V}') \cap L^\infty(0,T;\mathbb{H}) \cap \mathbb{L}^2(0,T;\mathbb{V})$. Let $\mathscr{M}^2_{loc}(\mathbb{H})$ be the class of all locally square integrable $\mathbb{H}$-valued martingales $\mathbf{M}_t = (\mathbf{M}(t), t \geq 0)$ with respect to $\mathscr{F}_t$. The following lemma establishes the quadratic variation of the martingale $\mathbf{M}_t$ given in (2.11). For more details regarding quadratic variation and Meyer process, one can refer to Chapter I of [22] and also [26].

**Lemma 2.6** (**Quadratic Variation Process**). *Let $\mathbf{M}_t \in \mathscr{M}^2_{loc}(\mathbb{H})$. Then there exists an increasing adapted càdlàg process $[\![\mathbf{M}]\!]_t$ with $[\![\mathbf{M}]\!]_0 = 0$ such that*

$$\begin{aligned}
[\![\mathbf{M}]\!]_t \quad &= \quad \int_0^t \sigma(s,\mathbf{u}(s))\mathbb{Q}\sigma^*(s,\mathbf{u}(s))ds \\
&\quad + \int_0^t \sum_{k=1}^\infty \int_{|z_k|<1} \phi_k \otimes \phi_k(\mathbf{u}(s-), z_k)\pi_k(dz_k, ds) \\
&\quad + \int_0^t \sum_{k=1}^\infty \int_{|z_k|\geq 1} \psi_k \otimes \psi_k(\mathbf{u}(s-), z_k)\pi_k(dz_k, ds). \qquad (2.27)
\end{aligned}$$

*Proof.* Let $\{e_i\}$ be an orthonormal basis in $\mathbb{H}$. Then the martingale $\mathbf{M}_t$ has a representation $\mathbf{M}_t = \sum_{i=1}^\infty (\mathbf{M}_t, e_i)e_i, \ \forall\ t \geq 0$. Setting $\mathbf{M}^i_t = (\mathbf{M}_t, e_i)$, we get that

$$\mathbb{E}|\mathbf{M}_t|^2 = \mathbb{E}\sum_{i=1}^\infty (\mathbf{M}^i_t)^2 = \sum_{i=1}^\infty \mathbb{E}(\mathbf{M}^i_t)^2 < \infty$$

whence $\mathbf{M}^i_t \in \mathscr{M}^2_{loc}(\mathbb{R}), \forall i$.

Since the quadratic variation process has to satisfy the identity $\big([\![\mathbf{M}]\!]_t e_i, e_j\big) = [\mathbf{M}^i, \mathbf{M}^j]_t$, we define the processes $[\![\mathbf{M}]\!]_t$ by (see, [22])

$$[\![\mathbf{M}]\!]_t = \sum_{i,j=1}^\infty [\mathbf{M}^i, \mathbf{M}^j]_t e_i \otimes e_j, \qquad (2.28)$$

(equivalently, $\big([\![\mathbf{M}]\!]_t \mathbf{p}, \mathbf{q}\big) = \sum_{i,j=1}^\infty [\mathbf{M}^i, \mathbf{M}^j]_t (e_i, \mathbf{p})(e_j, \mathbf{q}), \mathbf{p}, \mathbf{q} \in \mathbb{H}$), where $\{e_i \otimes e_j\}, i, j = 1, 2 \cdots$ is an orthonormal basis in $\mathbb{H}\hat{\otimes}\mathbb{H}$ which denotes the space $\mathbb{H} \otimes \mathbb{H}$ completed with respect to the Hilbert-Schmidt norm. The relation (2.28) can be written as

$$[\![\mathbf{M}]\!]_t = \sum_{i,j=1}^\infty \big\{[\mathbf{M}^i, \mathbf{M}^j]^c_t + \sum_{0\leq s\leq t} \Delta[\mathbf{M}^i, \mathbf{M}^j]_s\big\} e_i \otimes e_j, \qquad (2.29)$$

where we decomposed the infinite matrix valued process $[\mathbf{M}^i, \mathbf{M}^j]_t$ into continuous and jump parts of the martingale $\mathbf{M}_t$. Now we note that

$$\begin{aligned}
[\mathbf{M}^i, \mathbf{M}^j]^c_t \quad &= \quad \sum_{k,l=1}^\infty \int_0^t \big(\sigma(s,\mathbf{u})\mathbb{Q}^{1/2}e_k, e_i\big)\big(\sigma(s,\mathbf{u})\mathbb{Q}^{1/2}e_l, e_j\big) d[\beta_k, \beta_l]_s \\
&= \quad \sum_{k,l=1}^\infty \int_0^t \big(e_k, (\sigma(s,\mathbf{u})\mathbb{Q}^{1/2})^* e_i\big)\big(e_l, (\sigma(s,\mathbf{u})\mathbb{Q}^{1/2})^* e_j\big)\delta_{kl} ds \\
&= \quad \int_0^t \big((\sigma(s,\mathbf{u})\mathbb{Q}^{1/2})(\sigma(s,\mathbf{u})\mathbb{Q}^{1/2})^* e_i, e_j\big) ds, \qquad (2.30)
\end{aligned}$$

where $\beta_k, k = 1, 2, \cdots$ are real-valued independent Brownian motions defined on $(\Omega, \mathscr{F}_t, P)$ and the last equality follows from Parseval's identity.

Since the quadratic covariation process occur only at points where both processes have jumps, we have $\sum_{0\leq s\leq t}\Delta[\mathbf{M}^i,\mathbf{M}^j]_s=\sum_{0\leq s\leq t}\Delta\mathbf{M}^i_s\Delta\mathbf{M}^j_s$.

Noting that $\mathbf{L}(t)$ is a $\mathcal{Z}$-valued Lévy process, define $\mathcal{Z}_0:=\{z\in\mathcal{Z}\backslash\{0\}:|z|<1\}$, $\chi_1(s):=\chi_{\{\Delta L_k(s)\in\mathcal{Z}_0\}}$ and $\bar{\mathcal{Z}}_0:=\{z\in\mathcal{Z}\backslash\{0\}:|z|\geq 1\}$, $\chi_2(s):=\chi_{\{\Delta L_k(s)\in\bar{\mathcal{Z}}_0\}}$. Recalling the compensated measure $\widetilde{\pi}_k(dt,dz)=\pi_k(dt,dz_k)-dt\mu_k(dz_k)$, for all $k=1,2\cdots$, we define the corresponding compensated integral as

$$I_t:=\int_0^t\sum_{k=1}^\infty\int_{|z_k|<1}\phi_k(\mathbf{u}(s-),z_k)\widetilde{\pi}_k(ds,dz_k).$$

Then

$$\begin{aligned}\sum_{0\leq s\leq t}\Delta I_s &= \int_0^t\sum_{k=1}^\infty\int_{|z_k|<1}\phi_k(\mathbf{u}(s-),z_k)\pi_k(ds,dz_k)\\ &= \sum_{0\leq s\leq t}\sum_{k=1}^\infty\phi_k(\mathbf{u}(s),\Delta L_k(s))\chi_1(s).\end{aligned}$$

Moreover, $\sum_{0\leq s\leq t}|\Delta I_s|^2<\infty, P-a.s.$ Indeed, for $m>0$, define the stopping time

$$\tau_m=\inf\{t\geq 0:\int_0^t\sum_{k=1}^\infty\int_{|z_k|<1}|\phi_k(\mathbf{u}(s-),z_k)|^2\pi_k(dz_k,ds)\geq m\}$$

so that $\tau_m\to\infty$ as $m\to\infty$. Then

$$\begin{aligned}\mathbb{E}\sum_{0\leq s\leq t\wedge\tau_m}|\Delta I_s|^2 &= \mathbb{E}\sum_{0\leq s\leq t\wedge\tau_m}\sum_{k=1}^\infty|\phi_k(\mathbf{u}(s),\Delta L_k(s))|^2\chi_1(s)\\ &= \mathbb{E}\int_0^t\sum_{k=1}^\infty\int_{|z_k|<1}|\phi_k(\mathbf{u}(s-),z_k)|^2\pi_k(dz_k,ds)\\ &= \mathbb{E}\int_0^{t\wedge\tau_m}\sum_{k=1}^\infty\int_{|z_k|<1}|\phi_k(\mathbf{u}(s-),z_k)|^2\mu_k(dz_k)ds\leq C,\end{aligned}$$

where the finiteness follows from Assumption 2.2 on $\phi_k, k\geq 1$ and energy estimates proved in the next section. It shows that $P\big(\sum_{0\leq s\leq t}|\Delta I_s|^2\geq m\big)\leq C/m\to 0$ as $m\to\infty$.

Noting that $e_i\otimes e_j, i,j=1,2\cdots$ is an orthonormal basis in $\mathbb{H}\hat{\otimes}\mathbb{H}$, we have

$$\begin{aligned}&\sum_{0\leq s\leq t}\Delta\mathbf{M}^i_s\Delta\mathbf{M}^j_s\\ &= \sum_{0\leq s\leq t}\Big[\sum_{k=1}^\infty\big(\phi_k(\mathbf{u}(s),\Delta L_k(s))\chi_1(s),e_i\big)\big(\phi_k(\mathbf{u}(s),\Delta L_k(s))\chi_1(s),e_j\big)\\ &\quad+\sum_{k=1}^\infty\big(\psi_k(\mathbf{u}(s),\Delta L_k(s))\chi_2(s),e_i\big)\big(\psi_k(\mathbf{u}(s),\Delta L_k(s))\chi_2(s),e_j\big)\Big]\\ &= \sum_{0\leq s\leq t}\sum_{k=1}^\infty\big(\phi_k\otimes\phi_k(\mathbf{u}(s),\Delta L_k(s))\chi_1(s),e_i\otimes e_j\big)\\ &\quad+\sum_{0\leq s\leq t}\sum_{k=1}^\infty\big(\psi_k\otimes\psi_k(\mathbf{u}(s),\Delta L_k(s))\chi_1(s),e_i\otimes e_j\big).\end{aligned}$$

Thus, we rewrite the sum as follows

$$\sum_{0\le s\le t} \Delta\mathbf{M}_s^i \Delta\mathbf{M}_s^j \tag{2.31}$$
$$= \int_0^t \sum_{k=1}^{\infty} \int_{|z_k|<1} \big(\phi_k \otimes \phi_k(\mathbf{u}(s-), z_k), e_i \otimes e_j\big)\pi_k(ds, dz_k)$$
$$+ \int_0^t \sum_{k=1}^{\infty} \int_{|z_k|\ge 1} \big(\psi_k \otimes \psi_k(\mathbf{u}(s-), z_k), e_i \otimes e_j\big)\pi_k(ds, dz_k).$$

Combining the equations (2.29)-(2.31), one can conclude the proof. □

Since $\mathbf{M}_t \in \mathscr{M}^2_{loc}(\mathbb{H})$, it should be noted by the Doob-Meyer decomposition (see, [21, 26]) that there exits a unique predictable process the so-called *Meyer Process* denoted by $\ll \mathbf{M} \gg_t$ such that $([\![\mathbf{M}]\!]_t - \ll \mathbf{M} \gg_t)$ is a local martingale. Indeed, we will have the following interesting observation.

**Remark 2.2.** Since from the Remark 2.1, $\{e_i\}$ is an orthonormal basis in $\mathbb{H}$ and $(\rho, e_i) = \rho_i\ \forall i$, so we can take $f_i(\rho) = \rho_i$ for some $i$. Now for $f_i, f_if_j \in \mathcal{D}(\mathscr{L})$, let us set

$$\mathbf{b}(\rho) = \mathscr{L}f(\rho), \qquad \mathbf{a}(\rho) = \mathscr{L}\big(f(\rho)\otimes f(\rho)\big) - 2f(\rho)\otimes \mathscr{L}f(\rho) \tag{2.32}$$

and the stopping time $\tau_m = \inf\{t : |\mathbf{u}(t)| \ge m\}$. Let $\mathbf{u}$ be the solution of (2.1). Applying the Dynkin formula for all $f(\mathbf{u}) \in \mathcal{D}(\mathscr{L})$ with the stopping time $\tau_m$, we get that

$$f(\mathbf{u}(t\wedge\tau_m)) - f(\mathbf{u}(0)) - \int_0^{t\wedge\tau_m} \mathscr{L}f(\mathbf{u}(s)ds \quad \text{is a local martingale.}$$

Indeed, taking $f(\mathbf{u}) = \mathbf{u}$, we have

$$\mathbf{u}(t\wedge\tau_m) = \mathbf{u}(0) + \int_0^{t\wedge\tau_m} \mathbf{b}(\mathbf{u}(\mathbf{s}))ds + \mathbf{M}_{t\wedge\tau_m}, \tag{2.33}$$

where $\mathbf{M}_{t\wedge\tau_m}$ is a local martingale. Applying the same for $\mathbf{u}\otimes\mathbf{u}$, we get

$$\mathbf{u}\otimes\mathbf{u}(t\wedge\tau_m)$$
$$= \mathbf{u}\otimes\mathbf{u}(0) + 2\int_0^{t\wedge\tau_m} \mathbf{u}\otimes\mathbf{b}(\mathbf{u}(\mathbf{s}))ds + \int_0^{t\wedge\tau_m} \mathbf{a}(\mathbf{u}(\mathbf{s}))ds + \widetilde{\mathbf{M}}_{t\wedge\tau_m}, \tag{2.34}$$

where $\widetilde{\mathbf{M}}_{t\wedge\tau_m}$ is a local martingale. But the integration by parts formula applied to $\mathbf{u}\otimes\mathbf{u}$ together with (2.33), we also get

$$\mathbf{u}\otimes\mathbf{u}(t\wedge\tau_m) = \mathbf{u}\otimes\mathbf{u}(0) + 2\int_0^{t\wedge\tau_m} \mathbf{u}\otimes\mathbf{b}(\mathbf{u}(\mathbf{s}))ds \tag{2.35}$$
$$+2\int_0^{t\wedge\tau_m} \mathbf{u}(s)\otimes d\mathbf{M}_s + [\![\mathbf{M}]\!]_{t\wedge\tau_m},$$

where $[\![\mathbf{M}]\!]_{t\wedge\tau_m}$ denotes the process $[\mathbf{M}^i, \mathbf{M}^j]_{t\wedge\tau_m}$. Thus taking expectation in (2.35) and comparing it with (2.34), we see that

$$[\![\mathbf{M}]\!]_{t\wedge\tau_m} - \int_0^{t\wedge\tau_m} \mathbf{a}(\mathbf{u}(\mathbf{s}))ds, \quad \text{is a local martingale.} \tag{2.36}$$

Now we calculate the Meyer process associated with the martingale $\mathbf{M}_t$.

**Lemma 2.7 (Meyer Process).** *Let $\mathbf{M}_t \in \mathscr{M}^2_{loc}(\mathbb{H})$. Then there exists a uniquely defined predictable increasing process $\ll \mathbf{M} \gg_t$ with $\ll \mathbf{M} \gg_0 = 0$ such that*

$$\begin{aligned}\ll \mathbf{M} \gg_t &= \int_0^t \sigma(s, \mathbf{u}(s))\mathbb{Q}\sigma^*(s, \mathbf{u}(s))ds \\ &\quad + \int_0^t \sum_{k=1}^\infty \int_{|z_k|<1} \phi_k \otimes \phi_k(\mathbf{u}(s), z_k)\mu_k(dz_k)ds \\ &\quad + \int_0^t \sum_{k=1}^\infty \int_{|z_k|\geq 1} \psi_k \otimes \psi_k(\mathbf{u}(s), z_k)\mu_k(dz_k)ds. \qquad (2.37)\end{aligned}$$

*Proof.* Let $\{e_i\}$ be an orthonormal basis in $\mathbb{H}$ with each $e_i \in \mathcal{D}(\mathbf{A}), i = 1, 2, \cdots$ so that $\mathbf{u} = \sum_{i=1}^\infty \zeta_i e_i$. From (2.36), the Meyer process can be defined as $\ll \mathbf{M} \gg_t = \int_0^t \mathbf{a}(\mathbf{u}(\mathbf{s}))ds$. Now for $f \in \mathcal{D}(\mathscr{L})$ and $h \in \mathbb{H}$, note that $\left(\frac{\partial f}{\partial \mathbf{u}}, h\right) = \sum_{i=1}^\infty \frac{\partial f}{\partial \zeta_i}(e_i, h)$ and $\frac{\partial^2 f}{\partial \mathbf{u}^2}(h, h) = \sum_{i,j=1}^\infty \frac{\partial^2 f}{\partial \zeta_i \zeta_j}(e_i, h)(e_j, h)$. Setting $F(\mathbf{u}) := \nu \mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u}) - \mathbf{g}$, we write the generator in Definition 2.3 as follows

$$\begin{aligned}\mathscr{L} f &= -\sum_{i=1}^\infty \tilde{F}_i \frac{\partial f}{\partial \zeta_i} + \frac{1}{2}\sum_{i,j=1}^\infty \frac{\partial^2 f}{\partial \zeta_i \zeta_j}\big(\sigma(t, \mathbf{u})\mathbb{Q}\sigma^*(t, \mathbf{u})e_i, e_j\big) \qquad (2.38) \\ &\quad + \sum_{k=1}^\infty \int_{|z_k|<1} \Big\{f(\zeta + \phi_k(\mathbf{u}, z_k)) - f(\zeta) - \sum_{i=1}^\infty \frac{\partial f}{\partial \zeta_i}\tilde{\phi}_k^i\Big\}\mu_k(dz_k) \\ &\quad + \sum_{k=1}^\infty \int_{|z_k|\geq 1} \Big\{f(\zeta + \psi_k(\mathbf{u}, z_k)) - f(\zeta)\Big\}\mu_k(dz_k),\end{aligned}$$

where $\tilde{F}_i = (F(\mathbf{u}), e_i)$ and $\tilde{\phi}_k^i = \big(\phi_k(\mathbf{u}(t), z_k), e_i\big)$. In particular, taking $f_p(\zeta) = \zeta_p$ and $f_q(\zeta) = \zeta_q$, we obtain that

$$f(\zeta) \otimes \mathscr{L} f(\zeta) = J, \qquad (2.39)$$

where

$$\begin{aligned}J &= -\zeta \otimes \tilde{F} + \sum_{k=1}^\infty \int_{|z_k|<1} \big[\zeta \otimes \phi_k(\mathbf{u}, z_k)) - \zeta \otimes \tilde{\phi}_k\big]\mu_k(dz_k) \\ &\quad + \sum_{k=1}^\infty \int_{|z_k|\geq 1} \zeta \otimes \psi_k(\mathbf{u}, z_k))\mu_k(dz_k).\end{aligned}$$

Applying (2.38) to $\zeta_p \zeta_q$, we obtain

$$\begin{aligned}\mathscr{L}\big(f(\zeta) \otimes f(\zeta)\big) &= 2J + \sum_{p,q=1}^\infty \big(\sigma(t, \mathbf{u})\mathbb{Q}\sigma^*(t, \mathbf{u})e_p, e_q\big) \qquad (2.40) \\ &\quad + \sum_{k=1}^\infty \int_{|z_k|<1} \phi_k \otimes \phi_k(\mathbf{u}, z_k)\mu_k(dz_k) + \sum_{k=1}^\infty \int_{|z_k|\geq 1} \psi_k \otimes \psi_k(\mathbf{u}, z_k)\mu_k(dz_k).\end{aligned}$$

The proof follows from (2.39) and (2.40) and the definition of $\mathbf{a}(\cdot)$ in Remark 2.2. □

3. **A priori estimates and tightness of measures.** In this section, we obtain moment estimates of order $p \geq 2$ by applying the Itô calculus for the finite dimensional Galerkin approximation of the Navier-Stokes systems (2.10). Let $\{e_1, e_2, \cdots\}$ be the orthonormal basis in $\mathbb{H}$ included in $\mathbb{V}$ with each $e_i \in \mathcal{D}(\mathbf{A}), i = 1, 2 \cdots$. Let $\Pi^n$ be the orthogonal projection in $\mathbb{V}$ onto the space $\mathbb{V}^n := \text{span}\{e_1, e_2, \cdots, e_n\}$.

Then $\mathbf{u}^n(t) := \Pi^n \mathbf{u}(t) = \sum_{i=1}^n (\mathbf{u}(t), e_i)e_i$ solves the following finite dimensional Navier-Stokes equations

$$\begin{aligned} du^n(t) &= (-\nu\Pi^n \mathbf{A}\mathbf{u}^n(t) - \Pi^n \mathbf{B}(\mathbf{u}^n(t)) + \Pi^n \mathbf{g}(t))dt \\ &\quad + \sum_{k=1}^n \int_{|z_k|\geq 1} \psi_k^n\big(\mathbf{u}^n(t), z_k\big)\mu_k(dz_k)dt + d\mathbf{M}_t^n \end{aligned} \tag{3.1}$$

where the local martingale $\mathbf{M}_t^n$ with notations $\sigma^n = \Pi^n\sigma, \mathbf{W}^n = \Pi^n\mathbf{W}, \phi_k^n = \Pi^n\phi_k$ and $\psi_k^n = \Pi^n\psi_k$ is given by

$$\begin{aligned} \mathbf{M}_t^n &= \int_0^t \sigma^n(s, \mathbf{u}^n(s))d\mathbf{W}^n(s) + \int_0^t \sum_{k=1}^n \int_{|z_k|<1} \phi_k^n\big(\mathbf{u}^n(s-), z_k\big)\widetilde{\pi}_k(ds, dz_k) \\ &\quad + \int_0^t \sum_{k=1}^n \int_{|z_k|\geq 1} \psi_k^n\big(\mathbf{u}^n(s-), z_k\big)\widetilde{\pi}_k(ds, dz_k). \end{aligned} \tag{3.2}$$

Since the càdlàg process $\mathbf{u}^n$ solves the system (3.1) in $\mathbb{V}^n$ with initial condition $\Pi^n\mathbf{u}_0$ and $\mathbb{V}^n \subset \mathbb{H} \subset \mathbb{V}'$, the laws $\mathbb{P}^n$ of these finite dimensional approximations can be considered as defined on $\mathbb{D}(0, T; \mathbb{V}')$ satisfying the properties $\mathbb{P}^n\{\xi(0) = \Pi^n\mathbf{u}_0\} = 1$ and the process

$$\mathbf{M}_t^f = f(\xi(t)) - f(\xi(0)) - \int_0^t \mathscr{L}f(\xi(s))ds \tag{3.3}$$

is a $\mathbb{R}$-valued locally square integrable $\mathbb{P}^n$-local càdlàg martingale, where $\mathscr{L}f(\cdot)$ stands for the restriction of formal generator (2.17) associated with the finite dimensional system (3.1). Then the following theorem gives the uniform bounds for the measures $\mathbb{P}^n$ on $\widetilde{\Omega}$.

**Theorem 3.1.** *Let $\mathbf{g} \in \mathbb{L}^2(0, T; \mathbb{V}')$ and $\mathbb{E}|\xi(0)|^p = \int_{\mathbb{H}} |x|^p d\mathscr{P}(x) < \infty$, $\forall p \geq 2$. Suppose the noise coefficients $\sigma^n$ and $\phi_k^n, \psi_k^n, k = 1, 2\cdots$ satisfy Assumptions 2.1 and 2.2 respectively. Assume that $\mathbf{M}_t^f$ defined in (3.3) is a $\mathbb{H}^n$-valued square integrable $\mathbb{P}^n$-local càdlàg martingale. Then $\mathbb{P}^n$ is supported in $\widetilde{\Omega}$ and*

$$\mathbb{E}^{\mathbb{P}^n}|\xi(t)|^2 + \nu\mathbb{E}^{\mathbb{P}^n}\int_0^t \|\xi(s)\|^2 ds \leq \widehat{C}(1 + \mathscr{E}(\xi_0, \mathbf{g})), \tag{3.4}$$

*with $\widehat{C} = \big[1 + \exp(NT)(NT + 1)\big](1 \vee NT), N = (1 + \widetilde{N}_2 + CN_3)$, where*

$$\mathscr{E}(\xi_0, \mathbf{g}) = \mathbb{E}|\xi(0)|^2 + \frac{1}{\nu}\int_0^T \|\mathbf{g}(t)\|_{\mathbb{V}'}^2 dt.$$

*Besides,*

$$\mathbb{E}^{\mathbb{P}^n} \sup_{0\leq s\leq t} |\xi(s)|^2 + \nu\mathbb{E}^{\mathbb{P}^n}\int_0^t \|\xi(s)\|^2 ds \leq \tilde{C}(1 + \mathscr{E}(\xi_0, \mathbf{g})), \tag{3.5}$$

*where $\tilde{C} = 2[1 + \exp(18NT)(18NT + 1)](1 \vee 9NT)$. Moreover, for all $p \geq 2$*

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^n} \sup_{0\leq s\leq t} |\xi(s)|^p + \nu\mathbb{E}^{\mathbb{P}^n}\int_0^t |\xi(s)|^{p-2}\|\xi(s)\|^2 ds \\ &\quad \leq C(p, T, \widetilde{N}_2, N_3, \nu)\Big[1 + \mathbb{E}|\xi(0)|^p + \Big(\int_0^T \|\mathbf{g}(t)\|_{\mathbb{V}'}^2 dt\Big)^{p/2}\Big]. \end{aligned} \tag{3.6}$$

*Proof.* For $m > 0$, define the stopping time $\tau_m = \inf\{t \geq 0 : |\xi(t)|^2 + \nu \int_0^t \|\xi(s)\|^2 ds \geq m\}$, where by convention, $\tau_m = T$ if the set is empty. Applying the finite dimensional Itô formula (see, [27, 2, 26]) to $|\xi(t)|^2$ with a note of $(\Pi^n \mathbf{B}(\xi), \xi) = (\mathbf{B}(\xi), \Pi^n \xi) = 0$, $\mathbb{P}^n - a.s.$, we get

$$|\xi(t \wedge \tau_m)|^2 + 2\nu \int_0^{t\wedge\tau_m} \|\xi(s)\|^2 ds = |\xi(0)|^2 \tag{3.7}$$
$$+2 \int_0^{t\wedge\tau_m} \Big\langle \sum_{k=1}^n \int_{|z_k|\geq 1} \psi_k^n\big(\xi(s), z_k\big)\mu_k(dz_k) + \Pi^n \mathbf{g}(s), \xi(s) \Big\rangle ds$$
$$+[\mathbf{M}^n]_{t\wedge\tau_m} + 2\int_0^{t\wedge\tau_m} (\xi(s), d\mathbf{M}_s^n).$$

Here we note that $2\langle \xi, \Pi^n \mathbf{g}\rangle = 2\langle \Pi^n \xi, \mathbf{g}\rangle \leq \nu\|\xi\|^2 + \frac{1}{\nu}\|\mathbf{g}\|^2_{\mathbb{V}'}$ and

$$2\int_0^{t\wedge\tau_m} \Big\langle \sum_{k=1}^n \int_{|z_k|\geq 1} \psi_k^n\big(\xi(s), z_k\big)\mu_k(dz_k), \xi(s)\Big\rangle ds$$
$$\leq \int_0^{t\wedge\tau_m} |\xi(s)|^2 ds + \int_0^{t\wedge\tau_m} \Big| \sum_{k=1}^n \int_{|z_k|\geq 1} \psi_k^n\big(\xi(s), z_k\big)\mu_k(dz_k)\Big|^2 ds$$
$$\leq \int_0^{t\wedge\tau_m} |\xi(s)|^2 ds + C_\mu \int_0^{t\wedge\tau_m} \sum_{k=1}^n \int_{|z_k|\geq 1} |\psi_k^n\big(\xi(s), z_k\big)|^2 \mu_k(dz_k) ds$$
$$\leq (C_\mu N_3 + 1)\int_0^{t\wedge\tau_m} (1 + |\xi(s)|^2) ds,$$

where $C_\mu = \sum_{k=1}^\infty \int_{|z_k|\geq 1} \mu_k(dz_k)$. The expectation of the local martingale is zero, gives

$$\mathbb{E}^{\mathbb{P}^n}|\xi(t\wedge\tau_m)|^2 + \nu\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m} \|\xi(s)\|^2 ds \leq \mathbb{E}|\xi(0)|^2 + \frac{1}{\nu}\int_0^{t\wedge\tau_m} \|\mathbf{g}(s)\|^2_{\mathbb{V}'} ds$$
$$+(C_\mu N_3 + 1)\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m} (1 + |\xi(s)|^2) ds + \mathbb{E}^{\mathbb{P}^n}[\mathbf{M}^n]_{t\wedge\tau_m}. \tag{3.8}$$

Since $([\mathbf{M}^n]_{t\wedge\tau_m} - \lhd \mathbf{M}^n \rhd_{t\wedge\tau_m})_{t\geq 0}$ is a local martingale,

$$\mathbb{E}^{\mathbb{P}^n}[\mathbf{M}^n]_{t\wedge\tau_m} = \mathbb{E}^{\mathbb{P}^n} \lhd \mathbf{M}^n \rhd_{t\wedge\tau_m} = \mathbb{E}^{\mathbb{P}^n} \mathrm{tr} \ll \mathbf{M}^n \gg_{t\wedge\tau_m}.$$

Besides, from Lemma 2.7, we have

$$\mathbb{E}^{\mathbb{P}^n} \mathrm{tr} \ll \mathbf{M}^n \gg_{t\wedge\tau_m} = \mathbb{E}^{\mathbb{P}^n} \int_0^{t\wedge\tau_m} \mathrm{tr}\big(\sigma^n(s, \xi(s))\mathbb{Q}\sigma^{*n}(s, \xi(s))\big) ds$$
$$+\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m} \Big( \sum_{k=1}^n \int_{|z_k|<1} |\phi_k^n(\xi(s), z_k)|^2 \mu_k(dz_k)$$
$$+\sum_{k=1}^n \int_{|z_k|\geq 1} |\psi_k^n(\xi(s), z_k)|^2 \mu_k(dz_k)\Big) ds.$$

Using Assumptions 2.1 and 2.2, we arrive at

$$\mathbb{E}^{\mathbb{P}^n} \mathrm{tr} \ll \mathbf{M}^n \gg_{t\wedge\tau_m} \leq (\widetilde{N}_2 + N_3)\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m} (1 + |\xi(s)|^2) ds$$
$$= (\widetilde{N}_2 + N_3)\Big(\mathbb{E}^{\mathbb{P}^n}(t\wedge\tau_m) + \int_0^{t\wedge\tau_m} \mathbb{E}^{\mathbb{P}^n}|\xi(s)|^2 ds\Big). \tag{3.9}$$

The last equality follows from the stochastic Fubini theorem. Thus, dropping the second integral in the left hand side of (3.8) and using Gronwall's inequality, we get

$$\mathbb{E}^{\mathbb{P}^n}|\xi(t\wedge\tau_m)|^2 \leq C_1\big(1+\mathscr{E}(\xi_0,\mathbf{g})\big), \tag{3.10}$$

where $C_1 = \exp(NT)(1\vee NT)$. Also substitutions of (3.9) and (3.10) into (3.8) give rise to

$$\nu\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m}\|\xi(s)\|^2 ds \leq C_2\big(1+\mathscr{E}(\xi_0,\mathbf{g})\big), \tag{3.11}$$

where $C_2 = (1+NT\exp(NT))(1\vee NT)$. But taking supremum up to time $t\wedge\tau_m$ in (3.7) and then applying expectation leads to

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^n}\sup_{s\leq t\wedge\tau_m}|\xi(s)|^2 + \nu\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m}\|\xi(s)\|^2 ds \leq \mathbb{E}^{\mathbb{P}^n}|\xi(0)|^2 \\ +\frac{1}{\nu}\int_0^{T\wedge\tau_m}\mathbb{E}^{\mathbb{P}^n}\|\mathbf{g}(t)\|^2_{\mathbb{V}'}dt + (C_\mu N_3+1)\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m}(1+|\xi(s)|^2)ds \\ +\mathbb{E}^{\mathbb{P}^n}[\mathbf{M}^n]_{t\wedge\tau_m} + 2\mathbb{E}^{\mathbb{P}^n}\sup_{s\leq t\wedge\tau_m}|\widetilde{\mathbf{M}}^n_s|,\end{aligned} \tag{3.12}$$

where $\widetilde{\mathbf{M}}^n_s = \int_0^s(\xi(r), d\mathbf{M}^n_r)$. Using the Burkholder-Davis-Gundy (BDG) inequality followed by Lemma 2.7, we have

$$\begin{aligned}2\mathbb{E}^{\mathbb{P}^n}\sup_{s\leq t\wedge\tau_m}|\widetilde{\mathbf{M}}^n_s| &\leq 2\sqrt{2}\mathbb{E}^{\mathbb{P}^n}[\widetilde{\mathbf{M}}^n]^{1/2}_{t\wedge\tau_m} = 2\sqrt{2}\mathbb{E}^{\mathbb{P}^n}\Big\{\int_0^{t\wedge\tau_m}(\xi(s), d[\![\mathbf{M}^n]\!]_s\xi(s))\Big\}^{1/2} \\ &\leq 2\sqrt{2}\mathbb{E}^{\mathbb{P}^n}\Big\{\int_0^{t\wedge\tau_m}|\xi(s)|^2\mathrm{tr}\big(\sigma^n(s,\xi(s))\mathbb{Q}\sigma^{*n}(s,\xi(s))\big)ds\Big\}^{1/2} \\ &\quad +2\sqrt{2}\mathbb{E}^{\mathbb{P}^n}\Big\{\int_0^{t\wedge\tau_m}|\xi(s)|^2\Big(\sum_{k=1}^n\int_{|z_k|<1}|\phi^n_k(\xi(s-),z_k)|^2\pi_k(dz_k,ds) \\ &\quad +\sum_{k=1}^n\int_{|z_k|\geq 1}|\psi^n_k(\xi(s-),z_k)|^2\pi_k(dz_k,ds)\Big)\Big\}^{1/2}.\end{aligned}$$

Applying Young's inequality, we further estimate

$$\begin{aligned}&\leq 2\mathbb{E}^{\mathbb{P}^n}\Big(\big\{\sup_{s\leq t\wedge\tau_m}|\xi(s)|^2\big\}^{1/2}\big\{2\int_0^{t\wedge\tau_m}\|\sigma^n(s,\xi(s))\|^2_{\mathcal{L}_{HS}}ds\big\}^{1/2}\Big) \\ &\quad +2\mathbb{E}^{\mathbb{P}^n}\Big(\Big\{\sup_{s\leq t\wedge\tau_m}|\xi(s)|^2\Big\}^{1/2}\Big\{2\int_0^{t\wedge\tau_m}\sum_{k=1}^n\int_{|z_k|<1}|\phi^n_k(\xi(s-),z_k)|^2\pi_k(dz_k,ds) \\ &\quad +\int_0^{t\wedge\tau_m}\sum_{k=1}^n\int_{|z_k|\geq 1}|\psi^n_k(\xi(s-),z_k)|^2\pi_k(dz_k,ds)\Big\}^{1/2}\Big).\end{aligned}$$

And from Assumptions 2.1 and 2.2, we finally have

$$\begin{aligned}
2\mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_m}|\widetilde{\mathbf{M}}^n_s| &\le \frac{1}{2}\mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_m}|\xi(s)|^2+8\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m}\|\sigma^n(s,\xi(s))\|^2_{\mathcal{L}_{HS}}ds\\
&\quad+8\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m}\Big(\sum_{k=1}^n\int_{|z_k|<1}|\phi^n_k(\xi(s),z_k)|^2\mu_k(dz_k)\\
&\quad+\sum_{k=1}^n\int_{|z_k|\ge1}|\psi^n_k(\xi(s),z_k)|^2\mu_k(dz_k)\Big)ds\\
&\le \frac{1}{2}\mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_m}|\xi(s)|^2+8(\widetilde{N}_2+N_3)\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m}(1+|\xi(s)|^2)ds. \qquad (3.13)
\end{aligned}$$

Substitution of (3.9) and (3.13) into (3.12) leads to

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_m}|\xi(s)|^2+2\nu\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m}\|\xi(s)\|^2ds\le 2\mathbb{E}|\xi(0)|^2 \qquad (3.14)\\
&\quad+\frac{2}{\nu}\int_0^T\mathbb{E}^{\mathbb{P}^n}\|\mathbf{g}(t)\|^2_{\mathbb{V}'}dt+18N\Big(T+\int_0^t\mathbb{E}^{\mathbb{P}^n}\sup_{r\le s\wedge\tau_m}|\xi(r)|^2ds\Big).
\end{aligned}$$

Again by Gronwall's inequality

$$\mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_m}|\xi(s)|^2\le C_3(1+\mathscr{E}(\xi_0,\mathbf{g})), \qquad (3.15)$$

where $C_3=2\exp(18NT)(1\vee 9NT)$ and

$$\mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_m}|\xi(s)|^2+\nu\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_m}\|\xi(s)\|^2ds\le C_4(1+\mathscr{E}(\xi_0,\mathbf{g})), \qquad (3.16)$$

where $C_4=2[1+\exp(18NT)+18NT\exp(18NT)](1\vee 9NT)$. Now we define the set

$$\widetilde{\Omega}_m=\{\omega\in\widetilde{\Omega}:|\xi(t)|^2+\nu\int_0^t\|\xi(s)\|^2ds<m\},$$

then from (3.16), one obtains

$$\begin{aligned}
&\int_{\widetilde{\Omega}_m}\Big(|\xi(t)|^2+\nu\int_0^t\|\xi(s)\|^2ds\Big)d\mathbb{P}^n(\omega)\\
&\quad+\int_{\widetilde{\Omega}\backslash\widetilde{\Omega}_m}\Big(|\xi(t)|^2+\nu\int_0^t\|\xi(s)\|^2ds\Big)d\mathbb{P}^n(\omega)\le C_4(1+\mathscr{E}(\mathbf{u}_0,\mathbf{g})).
\end{aligned}$$

Here the first integral has bounded integrand and therefore recalling the fact that

$$|\xi(t)|^2+\nu\int_0^t\|\xi(s)\|^2ds\ge m \text{ in } \widetilde{\Omega}\backslash\widetilde{\Omega}_m,$$

we get $\mathbb{P}^n\{\Omega\backslash\widetilde{\Omega}_m\}\le C_5/m$. Besides for any $t\le T$, we note that

$$\mathbb{P}^n\{\omega\in\widetilde{\Omega}:\tau_m<t\}=\mathbb{P}^n\Big\{|\xi(t)|^2+\nu\int_0^t\|\xi(s)\|^2ds\ge m\Big\}\le\frac{C_5}{m}$$

whence $\limsup_{m\to\infty}\mathbb{P}^n\{\omega\in\widetilde{\Omega}:\tau_m<t\}=0$. Therefore, $\tau_m\to t$ as $m\to\infty$ and hence passing the limit in (3.10), (3.11) and (3.16), one can arrive at (3.4) and (3.5).

Next we obtain the higher order moment estimates of order $p \geq 2$. For fixed $\tilde{m} > 0$, define the stopping time

$$\tau_{\tilde{m}} = \inf\big\{t \geq 0 : |\xi(t)|^p + \int_0^t |\xi(s)|^{p-2}\|\xi(s)\|^2 ds \geq \tilde{m}\big\}.$$

Applying the finite dimensional Itô formula (see, [2], Page 251, Theorem 4.4.7) to $|\xi(t)|^p$, we get

$$\begin{aligned} |\xi(t)|^p + p\nu \int_0^t |\xi(s)|^{p-2}\|\xi(s)\|^2 ds &= \int_0^t p|\xi(s)|^{p-2}\langle \xi(s), \Pi^n \mathbf{g}(s)\rangle ds \\ &+ |\xi(0)|^p + \int_0^t p|\xi(s)|^{p-2}\big(\sigma^n(s,\xi(s)),\xi(s)\big) d\mathbf{W}^n(s) \qquad (3.17)\\ &+ \frac{1}{2}\int_0^t \{p(p-2)|\xi(s)|^{p-4}\xi(s)\otimes\xi(s) + p|\xi(s)|^{p-2}\} d[\mathbf{M}^n]_s^c + \sum_{i=1}^3 J_{i,t} \end{aligned}$$

where $[\mathbf{M}^n]_t^c$ denotes the quadratic variation process corresponding to the continuous part of the martingale $\mathbf{M}_t^n$ and

$$\begin{aligned} J_{1,t} &= \int_0^t \sum_{k=1}^n \int_{|z_k|\geq 1} \Big\{|\xi(s-) + \psi_k^n(\xi(s-), z_k)|^p - |\xi(s-)|^p\Big\}\pi_k(dz_k, ds), \\ J_{2,t} &= \int_0^t \sum_{k=1}^n \int_{|z_k|<1} \Big\{|\xi(s-) + \phi_k^n(\xi(s-), z_k)|^p - |\xi(s-)|^p\Big\}\widetilde{\pi}_k(dz_k, ds), \\ J_{3,t} &= \int_0^t \sum_{k=1}^n \int_{|z_k|<1} \Big\{|\xi(s) + \phi_k^n(\xi(s), z_k)|^p - |\xi(s)|^p \\ &\quad - p|\xi(s)|^{p-2}\big(\xi(s), \phi_k^n(\xi(s), z_k)\big)\Big\}\mu_k(dz_k) ds. \end{aligned}$$

Taking supremum up to time $t \wedge \tau_{\tilde{m}}$ and then expectation, we have the following estimates for the terms on the right hand of (3.17). First applying the Cauchy-Schwarz inequality followed by Young's inequality, we get

$$\begin{aligned} p\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_{\tilde{m}}} |\xi(s)|^{p-2}\langle\xi(s), \Pi^n\mathbf{g}(s)\rangle ds &\leq \frac{p\nu}{2}\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_{\tilde{m}}} |\xi(s)|^{p-2}\|\xi(s)\|^2 ds \\ &+ \mathbb{E}^{\mathbb{P}^n}\Big(\sup_{s\leq t\wedge\tau_{\tilde{m}}} |\xi(s)|^{(p-2)}\frac{p}{2\nu}\int_0^{t\wedge\tau_{\tilde{m}}} \|\mathbf{g}(s)\|_{\mathbb{V}'}^2 ds\Big) \\ &\leq \frac{p\nu}{2}\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_{\tilde{m}}} |\xi(s)|^{p-2}\|\xi(s)\|^2 ds + \frac{1}{8}\mathbb{E}^{\mathbb{P}^n}\sup_{s\leq t\wedge\tau_{\tilde{m}}}|\xi(s)|^p \\ &+ C_5(p,\nu)\Big(\int_0^T \|\mathbf{g}(t)\|_{\mathbb{V}'}^2 dt\Big)^{p/2}. \qquad (3.18) \end{aligned}$$

Using the BDG inequality and Young's inequality, we get

$$\begin{aligned} & p\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} \Big| \int_0^s |\xi(s)|^{p-2}\big(\sigma^n(s,\xi(s)),\xi(s)\big)d\mathbf{W}^n(s)\Big| \qquad (3.19)\\ \le\ & C(p)\mathbb{E}^{\mathbb{P}^n}\Big\{\int_0^{t\wedge\tau_{\tilde m}} |\xi(s)|^{2(p-1)}\|\sigma^n(s,\xi(s))\|^2_{\mathcal{L}_{HS}}ds\Big\}^{1/2}\\ \le\ & C(p)\mathbb{E}^{\mathbb{P}^n}\Big(\Big\{\sup_{s\le t\wedge\tau_{\tilde m}}|\xi(s)|^{2(p-1)}\Big\}^{1/2}\Big\{\widetilde N_2\int_0^{t\wedge\tau_{\tilde m}}(1+|\xi(s)|^2)ds\Big\}^{1/2}\Big)\\ \le\ & \frac18\mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_{\tilde m}}|\xi(s)|^p + C_6(p,\widetilde N_2,T)\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_{\tilde m}}(1+|\xi(s)|^p)ds \end{aligned}$$

where we have also used Assumption 2.1. From (2.30), we have

$$\begin{aligned} & \frac12\mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_{\tilde m}}\Big|\int_0^s\{p(p-2)|\xi(r)|^{p-4}\xi(r)\otimes\xi(r)+p|\xi(r)|^{p-2}\}d[\mathbf{M}^n]^c_r\Big| \qquad (3.20)\\ \le\ & \frac{p(p-1)}{2}\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_{\tilde m}}|\xi(s)|^{p-2}\|\sigma^n(s,\xi(s))\|^2_{\mathcal{L}_{HS}}ds\\ \le\ & \frac18\mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_{\tilde m}}|\xi(s)|^p + C_7(p,\widetilde N_2)\Big(\mathbb{E}^{\mathbb{P}^n}(t\wedge\tau_{\tilde m})+\int_0^{t\wedge\tau_{\tilde m}}\mathbb{E}^{\mathbb{P}^n}|\xi(s)|^pds\Big). \end{aligned}$$

The basic inequality $(a+b)^p\le 2^{p-1}(a^p+b^p)$ for all $p\ge1$ and $a,b\ge0$ leads to

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_{\tilde m}}|J_{1,s}|\\ \le\ & 2^{p-1}\mathbb{E}^{\mathbb{P}^n}\sup_{s\le t\wedge\tau_{\tilde m}}\int_0^s\sum_{k=1}^n\int_{|z_k|\ge1}\Big\{|\xi(r-)|^p+|\psi_k^n(\xi(r-),z_k)|^p\Big\}\pi_k(dz_k,dr)\\ \le\ & 2^{p-1}\mathbb{E}^{\mathbb{P}^n}\int_0^{t\wedge\tau_{\tilde m}}\sum_{k=1}^n\int_{|z_k|\ge1}\Big\{|\xi(s)|^p+|\psi_k^n(\xi(s),z_k)|^p\Big\}\mu_k(dz_k)ds\\ \le\ & C_8(p,N_3)\Big(\mathbb{E}^{\mathbb{P}^n}(t\wedge\tau_{\tilde m})+\int_0^{t\wedge\tau_{\tilde m}}\mathbb{E}^{\mathbb{P}^n}|\xi(s)|^pds\Big). \qquad (3.21) \end{aligned}$$

Besides, the integrals $J_{2,t}$ and $J_{3,t}$ can be written as

$$\begin{aligned} J_{2,t}+J_{3,t} =\ & \int_0^t\sum_{k=1}^n\int_{|z_k|<1}\Big\{|\xi(s-)+\phi_k^n(\xi(s-),z_k)|^p-|\xi(s-)|^p\\ & -p|\xi(s-)|^{p-2}\big(\xi(s-),\phi_k^n(\xi(s-),z_k)\big)\Big\}\pi_k(dz_k,ds)\\ & +\int_0^t\sum_{k=1}^n\int_{|z_k|<1}p|\xi(s-)|^{p-2}\big(\xi(s-),\phi_k^n(\xi(s-),z_k)\big)\widetilde\pi_k(dz_k,ds)\\ :=\ & J_{4,t}+J_{5,t}. \end{aligned}$$

For $\mathbf{a},\mathbf{b}\in\mathbb{H}^n$, we obtain from Taylor's formula that

$$\big||\mathbf{a}+\mathbf{b}|^p-|\mathbf{a}|^p-p|\mathbf{a}|^{p-2}(\mathbf{a},\mathbf{b})\big|\le C(p)(|\mathbf{a}|^{p-2}|\mathbf{b}|^2+|\mathbf{b}|^p).$$

So, taking $\mathbf{a} = \xi(s-)$ and $\mathbf{b} = \phi_k^n(\xi(s-), z_k)$, we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} |J_{4,s}| &\le C(p)\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} \int_0^s \sum_{k=1}^n \int_{|z_k|<1} \Big\{|\xi(r-)|^{(p-2)} \\
&\quad \times|\phi_k^n(\xi(r-), z_k)|^2 + |\phi_k^n(\xi(r-), z_k)|^p\Big\}\pi_k(dz_k, dr) \\
&\le C(p)\mathbb{E}^{\mathbb{P}^n}\Big(\sup_{s\le t\wedge\tau_{\tilde m}} |\xi(s)|^{p-2} \int_0^{t\wedge\tau_{\tilde m}} \sum_{k=1}^n \int_{|z_k|<1} |\phi_k^n(\xi(s), z_k)|^2\mu_k(dz_k)ds\Big) \\
&\quad +C(p)E^{\mathbb{P}^n} \int_0^{t\wedge\tau_{\tilde m}} \sum_{k=1}^n \int_{|z_k|<1} |\phi_k^n(\xi(s), z_k)|^p\mu_k(dz_k)ds.
\end{aligned}$$

Applying Assumption 2.2, we further estimate the above integral as

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} |J_{4,s}| &\le \frac{1}{8}\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} |\xi(s)|^p + C(p)\mathbb{E}^{\mathbb{P}^n}\Big\{N_3\int_0^{t\wedge\tau_{\tilde m}} (1 + |\xi(s)|^2)ds\Big\}^{p/2} \\
&\quad +C(p, N_3)\mathbb{E}^{\mathbb{P}^n} \int_0^{t\wedge\tau_{\tilde m}} (1 + |\xi(s)|^p)ds \\
&\le \frac{1}{8}\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} |\xi(s)|^p + C_9(p, N_3, T)\mathbb{E}^{\mathbb{P}^n} \int_0^{t\wedge\tau_{\tilde m}} (1 + |\xi(s)|^p)ds. \quad (3.22)
\end{aligned}$$

Using the BDG inequality(see, [11]) and Young's inequality, we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} |J_{5,s}| &\le C\mathbb{E}^{\mathbb{P}^n} \lhd J_5\rhd^{1/2}_{t\wedge\tau_{\tilde m}} \le C(p)\mathbb{E}^{\mathbb{P}^n}\Big(\Big\{\sup_{s\le t\wedge\tau_{\tilde m}} |\xi(s)|^{2(p-1)}\Big\}^{1/2} \\
&\quad \times\Big\{\int_0^{t\wedge\tau_{\tilde m}} \sum_{k=1}^n \int_{|z_k|<1} |\phi_k^n(\xi(s), z_k)|^2\mu_k(dz_k)ds\Big\}^{1/2}\Big) \\
&\le \frac{1}{8}\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} |\xi(s)|^p + C(p)\mathbb{E}^{\mathbb{P}^n}\Big\{N_3\int_0^{t\wedge\tau_{\tilde m}} (1 + |\xi(s)|^2)ds\Big\}^{p/2} \\
&\le \frac{1}{8}\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} |\xi(s)|^p + C_{10}(p, N_3, T)\mathbb{E}^{\mathbb{P}^n} \int_0^{t\wedge\tau_{\tilde m}} (1 + |\xi(s)|^p)ds. \quad (3.23)
\end{aligned}$$

Thus, the estimates (3.18)-(3.23) lead to

$$\begin{aligned}
&\frac{3}{8}\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} |\xi(s)|^p + \frac{p\nu}{2}\mathbb{E}^{\mathbb{P}^n} \int_0^{t\wedge\tau_{\tilde m}} |\xi(s)|^{p-2}\|\xi(s)\|^2ds \quad (3.24) \\
&\qquad \le \mathbb{E}|\xi(0)|^p + C_5(p, \nu)\Big(\int_0^T \|\mathbf{g}(t)\|^2_{\mathbb{V}'}dt\Big)^{p/2} \\
&\qquad +C_{11}(p, T, \widetilde{N}_2, N_3)\big(1 + \int_0^t \mathbb{E}^{\mathbb{P}^n} \sup_{r\le s\wedge\tau_{\tilde m}} |\xi(r)|^pds\big).
\end{aligned}$$

By Gronwall's inequality, we have

$$\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde m}} |\xi(s)|^p \le \bar{C}(p, T, \widetilde{N}_2, N_3, \nu)\Big[1 + \mathbb{E}|\xi(0)|^p + \Big(\int_0^T \|\mathbf{g}(t)\|^2_{\mathbb{V}'}dt\Big)^{p/2}\Big], \quad (3.25)$$

where $\bar{C} = 8\exp(C_{11})(1 \vee C_5 \vee C_{11})$. Hence,

$$\mathbb{E}^{\mathbb{P}^n} \sup_{s\le t\wedge\tau_{\tilde{m}}} |\xi(s)|^p + p\nu\mathbb{E}^{\mathbb{P}^n} \int_0^{t\wedge\tau_{\tilde{m}}} |\xi(s)|^{p-2}\|\xi(s)\|^2 ds \tag{3.26}$$
$$\le \quad C(p,T,\widetilde{N}_2,N_3,\nu)\Big[1+\mathbb{E}|\xi(0)|^p + \Big(\int_0^T \|\mathbf{g}(t)\|^2_{\mathbb{V}'}dt\Big)^{p/2}\Big],$$

where $C = 8[1+T\exp(C_{11})](1 \vee C_5 \vee C_{11})$. By the argument similar to the case of $p=2$, we also get that $\tau_{\tilde{m}} \to t$ as $\tilde{m}\to\infty$ and hence passing the limit in (3.26), we can conclude the proof. □

Now we prove the tightness of the probability measures $\mathbb{P}^n$ on $\widetilde{\Omega}$. Let us recall the topologies associated with the path space $\widetilde{\Omega}$. A topological space that is metrizable as a complete separable metric space is said to be Polish. A topological space is a Lusin if and only if it is homeomorphic to a topological space which is a Borel subset of a Polish space (see, [28]). A Radon measure is a measure on the Borel sigma algebra of a topological space that is locally finite and inner regular. Every Borel measure on a Lusin space is a Radon measure.

Taking the path space $\widetilde{\Omega} = \mathbb{D}(0,T;\mathbb{V}')_J \cap L^\infty(0,T;\mathbb{H})_{w^*} \cap \mathbb{L}^2(0,T;\mathbb{V})_w$ into account, we call $\mathcal{T}_1 := \mathbb{D}(0,T;\mathbb{V}')_J$, where $J$ denotes the extended Skorohod topology (see, [22]), $\mathcal{T}_2 := L^\infty(0,T;\mathbb{H})_{w^*}$, where $w^*$ denotes the weak-star topology and $\mathcal{T}_3 := \mathbb{L}^2(0,T;\mathbb{V})_w$, where $w$ denotes the weak topology and $\mathcal{T}_4$ as the strong topology of $\mathbb{L}^2(0,T;\mathbb{H})$. Note that the spaces $\mathbb{D}(0,T;\mathbb{V}')_J$, $L^\infty(0,T;\mathbb{H})_{w^*}$ and $\mathbb{L}^2(0,T;\mathbb{V})_w$ are completely regular and continuously embedded in $\mathbb{L}^2(0,T;\mathbb{V}')_w$. Let $\mathcal{T}$ be the supremum of four topologies, that is, $\mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2 \vee \mathcal{T}_3 \vee \mathcal{T}_4$. Then from Proposition 1, Page 63 of [22], it is clear that the intersection of these three spaces $\widetilde{\Omega}$ endowed with the topology $\mathcal{T}$ is a Lusin space.

Radon probability measures $P^n$ on a completely regular topological space $E$, that is, a topological space which is Hausdorff separated and whose topology can be defined by a set $\{d_\alpha, \alpha \in Z\}$ of pseudo-distances, is said to converge weakly to a Radon probability measure $P$ if $\lim_{n\to\infty}\int_{\widetilde{\Omega}} F dP^n = \int_{\widetilde{\Omega}} F dP$, $\forall F \in C_b(\widetilde{\Omega})$.

The tightness condition needed for the Prokhorov-Varadarajan theorem is that for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset E$ such that $\sup_n P^n(E\backslash K_\varepsilon) \le \varepsilon$ ([39, 22, 28]):

**Theorem 3.2.** *If the bounded Radon measures $P^n$ on a completely regular topological space $E$ satisfy $\sup_n P^n(E) < \infty$ and $P^n$ are tight, then the measures $P^n$ are relatively weakly compact in the set of bounded positive Radon measures.*

A sufficient condition for tightness of the laws $P^n$ of a semimartingale $X^n$ due to Rebolledo (see, [22], Page 66, which makes use of the Aldous condition [1]) is stated below. Let $S$ be a locally convex topological vector space with $\{d_\alpha, \alpha \in Z\}$ a filtering family of semi-norms defining its locally convex structure. Let $\Pi_\alpha$ be the canonical projection $S \to S_\alpha$. Assume that the normed space $S_\alpha$ associated with each $d_\alpha$ are separable Hilbert spaces. Let $X^n$ be the sequence of $S$-valued processes and let $X^{n,\alpha} = \Pi_\alpha X^n$ denote the $S_\alpha$ valued projections. Then

**Theorem 3.3.** *For each $\alpha$, let $X^{n,\alpha}$ be the semimartingale which is of the form $X^{n,\alpha}(t) = \Pi_\alpha X^n(0) + A^{n,\alpha}(t) + M^{n,\alpha}(t)$ where $A^{n,\alpha}$ is a process with finite variation and $M^{n,\alpha}$ is a square integrable martingale. If the laws $P^n$ of $X^{n,\alpha}$ satisfy the following two conditions,* *[T] For every t in some denumerable subset $I_0$ of $[0,T]$ and for every $\varepsilon > 0$, there exists a compact subset $K_\varepsilon$ of $S$ and a denumerable subset $Z_\varepsilon$ of $Z$ such that on $K_\varepsilon$, the families $\{\delta_\alpha : \alpha \in Z\}$ and $\{\delta_\alpha : \alpha \in Z_\varepsilon\}$ of pseudo-distances are equivalent and*

$$\sup_{t\in I_0}\sup_n P^n\{\omega(t) : \omega(t) \notin K_\varepsilon\} \le \varepsilon$$

*[A] For each $\alpha$, the processes $A^{n,\alpha}$ and $\lhd M^{n,\alpha}\rhd$ satisfy the Aldous condition, namely, for each $N > 0$, $\varepsilon > 0$ and $\eta > 0$, there exist $r > 0$ and $n_0$ such that*

$$\sup_{n\ge n_0}\sup_{0\le s\le r} P^n\Big\{d_\alpha\big(\omega(\tau_n + s), \omega(\tau_n)\big) \ge \eta\Big\} \le \varepsilon,$$

*for all stopping times $\tau_n$ with $\tau_n \le N$,*

*then the laws $P^n$ of the semimartingale $X^{n,\alpha}$ are tight.*

For more details of Theorem 3.3, in particular, for the definition of the metric $\delta_\alpha$ one can refer to [22], Page 64.

3.1. **The case of bounded domain.** If the domain $\mathcal{O}$ is bounded, the embeddings $\mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V}'$ are compact and dense, we also have the following compactness result (see, [22], Page 112, Lemma 2).

**Lemma 3.1.** *Let $K$ be a subset of $\mathbb{L}^2(0,T;\mathbb{H})$ which is included in a compact set of $\mathbb{L}^2(0,T;\mathbb{V}')$ and $\sup_{\mathbf{u}\in K}\int_0^T \|\mathbf{u}(t)\|_{\mathbb{V}}^2 dt < \infty$. Then $K \subset \mathbb{L}^2(0,T;\mathbb{H})$ is relatively compact.*

We are ready to state and prove the tightness of the measures $\mathbb{P}^n$ on the Lusin space $(\widetilde{\Omega}, \widetilde{\mathscr{F}_t})$ with the topology $\mathcal{T}$.

**Proposition 3.1.** *The sequence of probability measures $\mathbb{P}^n$ on $(\widetilde{\Omega}, \widetilde{\mathscr{F}_t})$ having support in $L^\infty(0,T;\mathbb{H})_{w^*} \cap \mathbb{L}^2(0,T;\mathbb{V})_w$ are tight on $\mathbb{D}(0,T;\mathbb{V}')_J$ and satisfy the uniform bound*

$$\mathbb{E}^{\mathbb{P}^n}\sup_{0\le t\le T}|\xi(t)|^2 + \int_0^T \mathbb{E}^{\mathbb{P}^n}\|\xi(t)\|^2 dt < \infty. \tag{3.27}$$

*Proof.* It is easy to see from the *a priori* estimate (3.5),

$$\mathbb{E}^{\mathbb{P}^n}\Big(\sup_{0\le t\le T}|\xi(t)|^2 + \nu\int_0^T \|\xi(t)\|^2 dt\Big) \le N, \tag{3.28}$$

where $N > 0$ is a constant independent of $n$, whence the tightness of $\mathbb{P}^n$ achieved in $L^\infty(0,T;\mathbb{H})_{w^*}$ and $\mathbb{L}^2(0,T;\mathbb{V})_w$. Indeed for $\varepsilon > 0$, taking $N_\varepsilon := N/\varepsilon$ and considering $\widetilde{\Omega}_\varepsilon = \{\omega \in \widetilde{\Omega} : \sup_{0\le t\le T}|\xi(t)|^2 \le N_\varepsilon\}$, we obtain from (3.28) that

$$\int_{\widetilde{\Omega}_\varepsilon}\sup_{0\le t\le T}|\xi(t)|^2 d\mathbb{P}^n(\omega) + \int_{\widetilde{\Omega}\setminus\widetilde{\Omega}_\varepsilon}\sup_{0\le t\le T}|\xi(t)|^2 d\mathbb{P}^n(\omega) \le N$$

whence $\mathbb{P}^n(\omega \in \widetilde{\Omega} : \sup_{0\le t\le T}|\xi(t)|^2 > N_\varepsilon) \le \varepsilon$. At the same time the set $K_\varepsilon^2 = \{\omega \in \widetilde{\Omega} : \sup_{0\le t\le T}|\xi(t)|^2 \le N_\varepsilon\}$ is relatively compact in $L^\infty(0,T;\mathbb{H})$ for the topology $\mathcal{T}_2$ and $\mathbb{P}^n(K_\varepsilon^2) \ge 1 - \varepsilon$. Similarly from (3.28) there exists

$$K_\varepsilon^3 = \{\omega \in \widetilde{\Omega} : \int_0^T \|\xi(t)\|^2 dt \le N_\varepsilon\} \text{ in } \widetilde{\Omega}$$

relatively compact in $\mathbb{L}^2(0,T;\mathbb{V})$ for the topology $\mathcal{T}_3$ such that $\mathbb{P}^n(K_\varepsilon^3) \ge 1 - \varepsilon$.

Now we establish the sufficient conditions for tightness of the laws $\mathbb{P}^n$ in $\mathbb{D}(0,T;\mathbb{V}')_J$ by using Theorem 3.3.

*Condition [T]:* Note that $\mathbb{P}^n\circ\xi(t)^{-1}$ form a tight sequence in $\mathbb{V}'$. In fact, from the energy estimate (3.4), $\mathbb{E}^{\mathbb{P}^n}|\xi(t)|^2\leq C$ and so for each $\varepsilon>0$ there exists a $C_\varepsilon>0$ such that $K_\varepsilon=\{\omega\in\widetilde{\Omega}:|\xi(t)|^2\leq C_\varepsilon\}$ and $\mathbb{P}^n(\widetilde{\Omega}\backslash K_\varepsilon)\leq\varepsilon$ in $\mathbb{H}_w$. Since the embedding $\mathbb{H}\hookrightarrow\mathbb{V}'$ is compact, the ball $K_\varepsilon$ is relatively compact in $\mathbb{V}'$ which confirms the tightness of marginal distribution of $\xi(t)$ in $\mathbb{V}'$. This proves one of the conditions ([T]) of Theorem 3.3.

*Condition [A]:* The Aldous condition ([A]) given in Theorem 3.3 can be established in our context using the Chebyshev inequality as follows(see, [13]): For any given $\varepsilon>0$, there exists $\delta$ such that for all stopping times $\tau_m\leq T-\delta$, the following hold:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}&\Big\|\nu\mathbf{A}\xi(s)+\mathbf{B}(\xi(s))+\mathbf{g}(s)\\&+\sum_{k=1}^n\int_{|z_k|\geq1}\psi_k^n(\xi(s),z_k)\mu_k(dz_k)\Big\|_{\mathbb{V}'}ds\leq\varepsilon\end{aligned}\tag{3.29}$$

$$\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}d(\mathrm{tr}\ll\mathbf{M}^n\gg_s)\leq\varepsilon.\tag{3.30}$$

Let us do the term by term verification of the estimate (3.29) using the energy estimate (3.28) and the Cauchy-Schwarz inequality. First, we note that

$$\begin{aligned}\nu\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}\|\mathbf{A}\xi(s)\|_{\mathbb{V}'}ds&\leq\nu\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}\|\xi(s)\|ds\\&\leq\nu\Big\{\mathbb{E}^{\mathbb{P}^n}\Big(\int_{\tau_m}^{\tau_m+\delta}\|\xi(s)\|ds\Big)^2\Big\}^{1/2}\\&\leq\nu\delta^{1/2}\Big\{\mathbb{E}^{\mathbb{P}^n}\int_0^\delta\|\xi(\tau_m+s)\|^2ds\Big\}^{1/2}\leq\widetilde{N}\delta^{1/2}.\end{aligned}\tag{3.31}$$

Then for $\mathcal{O}\subset\mathbb{R}^d,d=2,3$, we use Ladyzhenskaya's inequalities (2.21) to get

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}\|\mathbf{B}(\xi(s))\|_{\mathbb{V}'}ds\leq C_1\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}|\xi(s)|^{2-(d/2)}\|\xi(s)\|^{d/2}ds\\&\leq C_1\delta^p\mathbb{E}^{\mathbb{P}^n}\Big(\sup_{t\in[0,T]}|\xi(t)|^{2p}\Big\{\int_{\tau_m}^{\tau_m+\delta}\|\xi(s)\|^2ds\Big\}^{d/4}\Big)\\&\leq C_1\delta^p\mathbb{E}^{\mathbb{P}^n}\Big(p\sup_{t\in[0,T]}|\xi(t)|^2+(1-p)\int_0^\delta\|\xi(\tau_m+s)\|^2ds\Big)\leq\widetilde{N}\delta^p,\end{aligned}\tag{3.32}$$

where $p=1-(d/4)$. From Assumption 2.2, we have

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}\Big\|\sum_{k=1}^n\int_{|z_k|\geq1}\psi_k^n(\xi(s),z_k)\mu_k(dz_k)\Big\|_{\mathbb{V}'}ds\\&\leq CC_\mu\sqrt{N_3}\delta^{1/2}\mathbb{E}^{\mathbb{P}^n}\Big\{\int_{\tau_m}^{\tau_m+\delta}(1+|\xi(s)|^2)ds\Big\}^{1/2}\\&\leq CC_\mu\sqrt{N_3}\delta\Big\{1+\mathbb{E}^{\mathbb{P}^n}\sup_{t\in[0,T]}|\xi(t)|^2\Big\}^{1/2}\leq\widetilde{N}\delta,\end{aligned}\tag{3.33}$$

where $C_\mu = \big(\sum_{k=1}^n \int_{|z_k|\geq 1} \mu_k(dz_k)\big)^{1/2}$. Next we verify the estimate (3.30) using the Assumptions 2.1, 2.2 and the Meyer process which we obtained in Lemma 2.7 as follows

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^n} \int_{\tau_m}^{\tau_m+\delta} d(\mathrm{tr} \ll \mathbf{M}^n \gg_s) &= \mathbb{E}^{\mathbb{P}^n} \int_{\tau_m}^{\tau_m+\delta} \mathrm{tr}\big(\sigma^n(s,\xi(s))\mathbb{Q}\sigma^{*n}(s,\xi(s))\big) ds \\
&\quad + \mathbb{E}^{\mathbb{P}^n} \int_{\tau_m}^{\tau_m+\delta} \sum_{k=1}^n \int_{|z_k|<1} |\phi_k^n(\xi(s),z_k)|^2 \mu_k(dz_k) ds \\
&\quad + \mathbb{E}^{\mathbb{P}^n} \int_{\tau_m}^{\tau_m+\delta} \sum_{k=1}^n \int_{|z_k|\geq 1} |\psi_k^n(\xi(s),z_k)|^2 \mu_k(dz_k) ds \\
&\leq (\widetilde{N}_2 + N_3)\mathbb{E}^{\mathbb{P}^n} \int_{\tau_m}^{\tau_m+\delta} (1+|\xi(s)|^2) ds \\
&\leq (\widetilde{N}_2 + N_3)\delta(1+\mathbb{E}^{\mathbb{P}^n} \sup_{t\in[0,T]} |\xi(t)|^2) \leq \widetilde{N}\delta. \qquad (3.34)
\end{aligned}$$

Eventually, the estimates (3.31)-(3.34) establish the Aldous condition [A] of Theorem 3.3. Thus conditions [T] and [A] show that there exists $K_\varepsilon^1$ in $\widetilde{\Omega}$ relatively compact for the topology $\mathcal{T}_1 := \mathbb{D}(0,T;\mathbb{V}')_J$ such that $\mathbb{P}^n(K_\varepsilon^1) \geq 1-\varepsilon$.

Finally, we obtain the tightness of the measures $\mathbb{P}^n$ in $\mathbb{L}^2(0,T;\mathbb{H})$ for the strong topology $\mathcal{T}_4$. In view of Lemma 3.1, we generate a subset $K_\varepsilon^4$ from $\mathbb{L}^2(0,T;\mathbb{H}) \cap \mathbb{D}(0,T;\mathbb{V}')$ such that it is bounded in $\mathbb{L}^2(0,T;\mathbb{H})$ and relatively compact in $\mathbb{L}^2(0,T;\mathbb{V}')$. Taking the embedding $\mathbb{L}^2(0,T;\mathbb{V}) \hookrightarrow \mathbb{L}^2(0,T;\mathbb{H})$ into account, we can use the tightness of $\mathbb{P}^n$ in $\mathbb{L}^2(0,T;\mathbb{V})$ for the topology $\mathcal{T}_3$ so that for each $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that

$$\mathbb{P}^n\{\omega \in \widetilde{\Omega} : \int_0^T \|\xi(t)\|^2 dt \leq N_\varepsilon\} \geq 1-\varepsilon.$$

From the existence of the set $K_\varepsilon^1$ for $\mathcal{T}_1$, define

$$K_\varepsilon^4 = K_\varepsilon^1 \cap \{\xi(t) \in \mathbb{L}^2(0,T;\mathbb{H}) : \int_0^T \|\xi(t)\|^2 dt \leq N_\varepsilon\}.$$

Then for each $\varepsilon > 0$, the set $K_\varepsilon^4$ in $\mathbb{L}^2(0,T;\mathbb{H}) \cap \mathbb{D}(0,T;\mathbb{V}')$ satisfies $\mathbb{P}^n(K_\varepsilon^4) \geq 2(1-\varepsilon)$. Besides, the embeddings $\mathbb{L}^2(0,T;\mathbb{H}) \cap \mathbb{D}(0,T;\mathbb{V}') \hookrightarrow \mathbb{L}^2(0,T;\mathbb{V}')$ being compact, we can conclude from Lemma 3.1 that the set $K_\varepsilon^4$ is relatively compact in $\mathbb{L}^2(0,T;\mathbb{H})$.

Hitherto, we have shown that the Radon measures $\mathbb{P}^n$ are tight in the topologies $\mathcal{T}_i, i = 1,2,3,4$. Since the spaces $\mathbb{D}(0,T;\mathbb{V}')_J$, $L^\infty(0,T;\mathbb{H})_{w^*}$, $\mathbb{L}^2(0,T;\mathbb{V})_w$, and $\mathbb{L}^2(0,T;\mathbb{H})$ are continuously embedded in $\mathbb{L}^2(0,T;\mathbb{V}')_w$, it follows from Proposition 1, Page 63 of [22] that the measures $\mathbb{P}^n$ are tight in the Lusin space $\widetilde{\Omega}$ endowed with the topology $\mathcal{T}$. The proof is thus completed. □

3.2. **The case of unbounded domain.** When the domain $\mathcal{O}$ is unbounded the embeddings $\mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V}'$ are not compact and hence the tightness properties established in the previous section on the spaces $\mathbb{D}(0,T;\mathbb{V}')$ and $\mathbb{L}^2(0,T;\mathbb{H})$ are no longer valid. This leads us to prove the tightness of the measures $\mathbb{P}^n$ in the weak topology of $\mathbb{H}$, namely $\mathbb{H}_w$.

Let $\mathcal{T}_5$ be the topology induced by $\mathbb{D}(0,T;\mathbb{H}_w)$. Since $\mathbb{D}(0,T;\mathbb{V}')\cap L^\infty(0,T;\mathbb{H}) \subset \mathbb{D}(0,T;\mathbb{H}_w)$(see, [23]), we can take $\widetilde{\mathcal{T}} = \mathcal{T}_3 \vee \mathcal{T}_5$, where $\mathcal{T}_3$ is the weak topology of

$\mathbb{L}^2(0,T;\mathbb{V})$. The path space $\widetilde{\Omega}$ as a Borel subset of $\mathbb{D}(0,T;\mathbb{H}_w)\cap\mathbb{L}^2(0,T;\mathbb{V})_w$, it forms a Lusin space for the topology $\widetilde{\mathcal{T}}$. Then, we have the following result.

**Proposition 3.2.** *The sequence of probability measures $\mathbb{P}^n$ on $(\widetilde{\Omega},\widetilde{\mathscr{F}_t})$ are tight for the Lusin topology $\widetilde{\mathcal{T}}$.*

*Proof.* It is easy to see from the arguments of Proposition 3.1 that the measures $\mathbb{P}^n$ are tight in $\mathbb{L}^2(0,T;\mathbb{V})_w$ and the marginal distributions of $\xi(t),\forall t\in[0,T]$ are tight in $\mathbb{H}_w$.

To complete the tightness in $\mathbb{D}(0,T;\mathbb{H}_w)$, we need to prove the following weak form of the Aldous condition (noting that the set $\mathbb{V}$ is dense in $\mathbb{H}$) for every $v\in\mathbb{V},\varepsilon>0$, there exists $\delta$ such that for all stopping times $\tau_m\le T-\delta$, the following hold:

$$\Big|\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}\big\langle\nu\mathbf{A}\xi(s)+\mathbf{B}(\xi(s))+\sum_{k=1}^{n}\int_{|z_k|\ge1}\psi_k^n\big(\xi(s),z_k\big)\mu_k(dz_k),v\big\rangle ds\Big|\le\varepsilon \tag{3.35}$$

$$\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}\big\langle v,d\ll\mathbf{M}^n\gg_s v\big\rangle\le\varepsilon. \tag{3.36}$$

For every $v\in\mathbb{V}$, we obtain from (3.31) that

$$\nu\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}|\langle\mathbf{A}\xi(s),v\rangle|ds\le\nu\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}\|\mathbf{A}\xi(s)\|_{\mathbb{V}'}ds\|v\|\le\widetilde{N}\delta^{1/2}\|v\|. \tag{3.37}$$

Similar estimates hold true for the nonlinear term $\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}|\langle\mathbf{B}(\xi(s)),v\rangle|ds$ and the compensating integral

$$\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}\Big|\big\langle\sum_{k=1}^{n}\int_{|z_k|\ge1}\psi_k^n\big(\xi(s),z_k\big)\mu_k(dz_k),v\big\rangle\Big|ds.$$

Besides, from (3.34) note that

$$\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}\big\langle v,d\ll\mathbf{M}^n\gg_s v\big\rangle\le\mathbb{E}^{\mathbb{P}^n}\int_{\tau_m}^{\tau_m+\delta}d(\mathrm{tr}\ll\mathbf{M}^n\gg_s)|v|^2\le\widetilde{N}\delta|v|^2.$$

The preceding estimates prove the tightness of $\mathbb{P}^n$ in $\mathbb{D}(0,T;\mathbb{H}_w)$ for $\mathcal{T}_5$ and thereby one can conclude that the measures $\mathbb{P}^n$ are tight on $\widetilde{\Omega}$ for the Lusin topology $\widetilde{\mathcal{T}}$. □

4. **Martingale problem for Navier-Stokes equations with Lévy noise.** In the case unbounded domain (see, Remark 4.1), the lack of compact embeddings affects the continuity of the martingales $\mathbf{M}_t^f$ on $\widetilde{\Omega}$ and hence one cannot conclude to the martingale property of any limit $\mathbb{P}$ of the sequence $\mathbb{P}^n$. In order to overcome this difficulty, we use the so-called Minty stochastic lemma (see, [40, 22, 31]) which is classical in the case of deterministic evolution equations with monotone operators.

Now we prove the main theorem.

*Proof of Theorem 2.1: (The domain $\mathcal{O}$ is bounded in $\mathbb{R}^2$).* First note that the notion of $\mathbf{M}_t^f$ is an $(\widetilde{\Omega},\widetilde{\mathscr{F}_t},\mathbb{P})$-martingale can be stated equivalently by

$$\mathbb{E}^{\mathbb{P}}[\Phi(\mathbf{M}_t^f-\mathbf{M}_s^f)]=0,\ \text{for all } t\ge s \tag{4.1}$$

and for all $\Phi\in C_b(\widetilde{\Omega})$ such that $\Phi$ is $\widetilde{\mathscr{F}_s}$-measurable. In fact, to make the above conclusion, we naturally use the fact that for every $n$

$$\mathbb{E}^{\mathbb{P}^n}[\Phi(\mathbf{M}_t^f-\mathbf{M}_s^f)]=0,\ \text{for all } t\ge s. \tag{4.2}$$

But the tightness of the measures $\mathbb{P}^n$ on $\widetilde{\Omega}$ and $\mathbb{P}(\widetilde{\Omega}) = 1$ (follows from the energy estimates in Theorem 3.1, see, for example, [22], Page 102) are not sufficient to conclude (4.1) from (4.2) since $\mathbf{M}_t^f$ is not continuous on $\widetilde{\Omega}$. So we proceed as follows.

The boundedness of $\Theta(\xi) := \nu \mathbf{A}\xi + \mathbf{B}(\xi)$ in $\mathbb{L}^2(0,T;\mathbb{V}')$ ensures the continuity of $\Theta(\cdot)$ from $\mathbb{L}^2(0,T;\mathbb{V})$ to $\mathbb{L}^2(0,T;\mathbb{V}')_w$ and so $\Theta(\cdot)$ is Borel measurable from $\widetilde{\Omega} \to \mathbb{L}^2(0,T;\mathbb{V}')_w$. Now we define the image of $\mathbb{P}^n$ under the map $\xi \to (\xi, \Theta(\xi))$ as $\widehat{\mathbb{P}}^n(S) := \mathbb{P}^n\{\omega \in \widetilde{\Omega}; (\omega, \Theta(\omega)) \in S\}$, for $S \in \mathcal{B}(\widetilde{\Omega} \times \mathbb{L}^2(0,T;\mathbb{V}')_w)$.

On $\widehat{\Omega} := \widetilde{\Omega} \times \mathbb{L}^2(0,T;\mathbb{V}')_w$, consider the canonical right-continuous filtration $\widehat{\mathcal{G}}_t$ and canonical processes $\xi(t,\omega,\mathbf{v}) = \omega(t)$ and $\chi(t,\omega,\mathbf{v}) = \mathbf{v}(t)$. Since the measures $\mathbb{P}^n$ are tight on $\widetilde{\Omega}$ with Lusin topology, we note from Theorem 3.1 that $\mathbb{E}^{\mathbb{P}^n} \int_0^T \|\Theta(\xi(t))\|_{\mathbb{V}'}^2 dt \leq C$, and hence there exists a $N_\varepsilon > 0$ such that the following holds:

$$\sup_n \widehat{\mathbb{P}}^n \Big\{ (\omega, \mathbf{v}) \in \widetilde{\Omega} \times \mathbb{L}^2(0,T;\mathbb{V}')_w; \ \int_0^T \|\mathbf{v}(t)\|_{\mathbb{V}'}^2 dt > N_\varepsilon \Big\} \leq \varepsilon.$$

Since the set $\{\mathbf{v} : \int_0^T \|\mathbf{v}(t)\|_{\mathbb{V}'}^2 \leq N_\varepsilon\}$ is compact in $\mathbb{L}^2(0,T;\mathbb{V}')_w$, the measures $\widehat{\mathbb{P}}^n$ are tight on $\widehat{\Omega}$. Then the sequence of probability measures $\widehat{\mathbb{P}}^n$ on $\widehat{\Omega}$ satisfy the following:

[$N_1$] $\widehat{\mathbb{P}}^n\{(\omega,\mathbf{v}) \in \widehat{\Omega} : \Theta(\omega) = \mathbf{v}\} = 1$.

[$N_2$] For every $(\omega,\mathbf{v}) \in \widehat{\Omega}$ and for any $f \in \mathcal{D}(\mathscr{L})$, the process $\widehat{\mathbf{M}}_t^f$ on $\widehat{\Omega}$ defined by $\widehat{\mathbf{M}}_t^f(\omega,\mathbf{v}) := f(\xi(t,\omega,\mathbf{v})) - f(\xi(0,\omega,\mathbf{v})) - \int_0^t \mathscr{L} f(s,\xi(s,\omega,\mathbf{v}))ds$ is a $\mathbb{R}$-valued locally square integrable $(\widehat{\Omega}, \widehat{\mathcal{G}}_t, \widehat{\mathbb{P}}^n)$-local càdlàg martingale.

It is easy to show that $(\widehat{\mathbf{M}}_t^f, \widehat{\mathbb{P}}^n)$ is square integrable. Let $\{e_k\}, k = 1, 2, \cdots, n$ be an orthonormal basis in $\mathbb{H}^n$. Since $|\widehat{\mathbf{M}}_t^f|^2 - \lhd \widehat{\mathbf{M}}^f \rhd_t$ is a martingale,

$$\begin{aligned}
\mathbb{E}^{\widehat{\mathbb{P}}^n} |\widehat{\mathbf{M}}_t^f|^2 &= \sum_{k=1}^n \mathbb{E}^{\widehat{\mathbb{P}}^n} (\widehat{\mathbf{M}}_t^f, e_k)^2 = \sum_{k=1}^n \mathbb{E}^{\widehat{\mathbb{P}}^n} \lhd (\widehat{\mathbf{M}}^f, e_k), (\widehat{\mathbf{M}}^f, e_k) \rhd_t \\
&= \sum_{k=1}^n \mathbb{E}^{\widehat{\mathbb{P}}^n} (e_k, \ll \widehat{\mathbf{M}}^f \gg_t e_k) = \mathbb{E}^{\widehat{\mathbb{P}}^n} (\mathrm{tr} \ll \widehat{\mathbf{M}}^f \gg_t)
\end{aligned}$$

and so

$$\begin{aligned}
\mathbb{E}^{\widehat{\mathbb{P}}^n} |\widehat{\mathbf{M}}_t^f|^2 &= \mathbb{E}^{\widehat{\mathbb{P}}^n} \int_0^t \|\sigma^n(s,\xi(s))\|_{\mathcal{L}_{HS}}^2 ds \\
&\quad + \mathbb{E}^{\widehat{\mathbb{P}}^n} \int_0^t \sum_{k=1}^n \int_{|z_k|<1} |\phi_k^n(\xi(s),z_k)|^2 \mu_k(dz_k) ds \\
&\quad + \mathbb{E}^{\widehat{\mathbb{P}}^n} \int_0^t \sum_{k=1}^n \int_{|z_k|\geq 1} |\psi_k^n(\xi(s),z_k)|^2 \mu_k(dz_k) ds. \qquad (4.3) \\
&\leq (\widetilde{N}_2 + N_3) \mathbb{E}^{\widehat{\mathbb{P}}^n} \int_0^t (1 + |\xi(s)|^2) ds \leq C(1 + \mathscr{E}(\xi(0),\mathbf{g})) < \infty,
\end{aligned}$$

where we used the uniform bound (3.4). For $\epsilon > 0$, making use of (4.3), we obtain

$$\begin{aligned}
\mathbb{E}^{\widehat{\mathbb{P}}^n} |\widehat{\mathbf{M}}_t^f|^{1+\epsilon} &\leq \big\{\mathbb{E}^{\widehat{\mathbb{P}}^n}(1)^p\big\}^{1/p} \big\{\mathbb{E}^{\widehat{\mathbb{P}}^n} |\widehat{\mathbf{M}}_t^f|^2\big\}^{(1+\epsilon)/2} \qquad (4.4) \\
&\leq \big\{C(1 + \mathscr{E}(\xi(0),\mathbf{g}))\big\}^{(1+\epsilon)/2} < \infty,
\end{aligned}$$

where $p = 2/(1-\epsilon)$. By Fatou's lemma

$$\mathbb{E}^{\widehat{\mathbb{P}}}|\widehat{\mathbf{M}}_t^f|^{1+\epsilon} \leq \left\{C(1+\mathscr{E}(\xi(0),\mathbf{g}))\right\}^{(1+\epsilon)/2}$$

so that $\mathbb{E}^{\widehat{\mathbb{P}}}|\widehat{\mathbf{M}}_t^f|^2 < \infty$. Thus we have also shown that $(\widehat{\mathbf{M}}_t^f, \widehat{\mathbb{P}})$ is square integrable.

Since the sequence of measures $\{\widehat{\mathbb{P}}^n\}$ are tight on $\widehat{\Omega}$, there exist a subsequence of $\{\widehat{\mathbb{P}}^n\}$ converging weakly to a measure $\widehat{\mathbb{P}}$. At this point, we need the following fundamental lemma.

**Lemma 4.1** (Minty Stochastic Lemma)**.** *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^2$. Let $\widehat{\mathbb{P}}^n$ be the sequence of probability measures on $\widehat{\Omega}$ satisfying $[N_1]$ and $[N_2]$. Suppose Assumptions 2.1 and 2.2 hold true. Assume that the measures $\widehat{\mathbb{P}}^n$ converge weakly to a measure $\widehat{\mathbb{P}}$ on $\widehat{\Omega}$ such that $[N_2]$ holds for $\widehat{\mathbb{P}}$. Then $[N_1]$ also holds true for $\widehat{\mathbb{P}}$, that is,*

$$\widehat{\mathbb{P}}\{(\omega,\mathbf{v}) \in \widehat{\Omega} : \Theta(\xi(\omega,\mathbf{v})) = \chi(\omega,\mathbf{v})\} = 1. \tag{4.5}$$

*Proof.* Let $\zeta(\cdot,\cdot,t)$ be the continuous function of the form $\zeta(\omega,\mathbf{v},t) = \sum_{i=1}^k \varphi_i(\omega,\mathbf{v}, t)e_i$ with $e_i \in \mathbb{V}$ which form a dense set in the space $\mathbb{L}^2(\widehat{\Omega};\mathbb{L}^2(0,T;\mathbb{V}))$, where $\varphi_i(\cdot,\cdot,t)$ are continuous in $\widehat{\Omega}$ with paths in $L^2(0,T)$. Here we restrict to the function $\zeta(\omega,\mathbf{v},t) = \varphi(\omega,\mathbf{v},t)e_0, e_0 \in \mathbb{V}$.

For each given $\zeta(\cdot,\cdot,t)$ and

$$\rho(t) := \frac{27}{\nu^3}\int_0^t \|\zeta(s)\|^4_{\mathbb{L}^4(\mathcal{O})} ds,$$

let us define

$$\begin{aligned}\Psi(\omega,\mathbf{v}) \quad := \quad & 2\int_0^T e^{-\rho(t)}\langle\chi(\omega,\mathbf{v},t) - \Theta(\zeta(\omega,\mathbf{v},t)), \xi(\omega,t) - \zeta(\omega,\mathbf{v},t)\rangle dt \\ & + \int_0^T e^{-\rho(t)}\dot{\rho}(t)|\xi(\omega,t) - \zeta(\omega,\mathbf{v},t)|^2 dt.\end{aligned} \tag{4.6}$$

But in view of $[N_1]$ and Lemma 2.4, we get

$$\int_{\widehat{\Omega}} \Psi(\omega,\mathbf{v}) d\widehat{\mathbb{P}}^n(\omega,\mathbf{v}) \geq 0. \tag{4.7}$$

We decompose $\Psi$ into $\Psi_1$ and $\Psi_2$ as follows

$$\Psi_1 = 2\int_0^T e^{-\rho(t)}\langle\chi(t),\xi(t)\rangle dt + \int_0^T e^{-\rho(t)}\dot{\rho}(t)|\xi(t)|^2 dt - \int_0^T e^{-\rho(t)} d[\widehat{\mathbf{M}}]_t \tag{4.8}$$

and

$$\begin{aligned}\Psi_2 \quad = \quad & -2\int_0^T e^{-\rho(t)}\langle\chi(t) - \Theta(\zeta(t)), \zeta(t)\rangle dt \qquad (4.9) \\ & -2\int_0^T e^{-\rho(t)}\langle\Theta(\zeta(t)),\xi(t)\rangle dt + \int_0^T e^{-\rho(t)}\dot{\rho}(t)|\zeta(t)|^2 dt \\ & -2\int_0^T e^{-\rho(t)}\dot{\rho}(t)\langle\zeta(t),\xi(t)\rangle dt + \int_0^T e^{-\rho(t)} d[\widehat{\mathbf{M}}]_t,\end{aligned}$$

where $\widehat{\mathbf{M}}$ is the local martingale with quadratic variation

$$
\begin{aligned}
[\widehat{\mathbf{M}}]_t = \mathrm{tr}([\![\widehat{\mathbf{M}}]\!]_t) &= \int_0^t \|\sigma(\xi(s))\|^2_{\mathcal{L}_{HS}} ds \\
&+ \int_0^t \sum_{k=1}^\infty \int_{|z_k|<1} |\phi_k(\xi(s), z_k)|^2 \pi_k(dz_k, ds) \\
&+ \int_0^t \sum_{k=1}^\infty \int_{|z_k|\geq 1} |\psi_k(\xi(s), z_k)|^2 \pi_k(dz_k, ds).
\end{aligned}
$$

Now we prove the continuity of $\Psi_1(\omega, \mathbf{v})$ in the Lusin topology of $\widehat{\Omega}$. Applying the Itô formula for Hilbert space valued local semimartingale (see, [26]) to $e^{-\rho(t)}|\xi(t)|^2$, we get

$$
\begin{aligned}
e^{-\rho(t)}|\xi(t)|^2 &= |\xi(0)|^2 - \int_0^t e^{-\rho(s)}\dot{\rho}(s)|\xi(s)|^2 ds - 2\int_0^t e^{-\rho(s)}\langle \chi(s), \xi(s)\rangle ds \\
&+2\int_0^t e^{-\rho(s)}\Big\langle \xi(s), \mathbf{g}(s) + \sum_{k=1}^\infty \int_{|z_k|\geq 1} \psi_k(\xi(s), z_k)\mu_k(dz_k)\Big\rangle ds \\
&+2\int_0^t e^{-\rho(s)}(\xi(s), d\widehat{\mathbf{M}}_s) + \int_0^t e^{-\rho(s)} d[\widehat{\mathbf{M}}]_s.
\end{aligned}
$$

Since $\widehat{\mathbf{M}}(\omega, \mathbf{v})$ is a local $\widehat{\mathbb{P}}$-martingale, it has zero averages and therefore taking

$$
\begin{aligned}
\widehat{\Psi}_1(\omega, \mathbf{v}) &= |\xi(0)|^2 - e^{-\rho(T)}|\xi(T)|^2 + 2\int_0^T e^{-\rho(s)}\langle \xi(s), \mathbf{g}(s)\rangle ds \\
&+2\int_0^T e^{-\rho(s)}\Big\langle \xi(s), \sum_{k=1}^\infty \int_{|z_k|\geq 1} \psi_k(\xi(s), z_k)\mu_k(dz_k)\Big\rangle ds,
\end{aligned}
$$

we arrive at

$$
\int_{\widehat{\Omega}} \widehat{\Psi}_1(\omega, \mathbf{v}) d\widehat{\mathbb{P}}(\omega, \mathbf{v}) = \int_{\widehat{\Omega}} \Psi_1(\omega, \mathbf{v}) d\widehat{\mathbb{P}}(\omega, \mathbf{v}). \tag{4.10}
$$

Besides, $\widehat{\mathbf{M}}(\omega, \mathbf{v})$ is a local $\widehat{\mathbb{P}}^n$-martingale ($[N_2]$), we also note that

$$
\int_{\widehat{\Omega}} \widehat{\Psi}_1(\omega, \mathbf{v}) d\widehat{\mathbb{P}}^n(\omega, \mathbf{v}) = \int_{\widehat{\Omega}} \Psi_1(\omega, \mathbf{v}) d\widehat{\mathbb{P}}^n(\omega, \mathbf{v}). \tag{4.11}
$$

From Lemma 2.5, the jump integral in $\widehat{\Psi}_1(\omega, \mathbf{v})$ is continuous in the Lusin topology of $\widehat{\Omega}$. Moreover, $\widehat{\Psi}_1(\omega, \mathbf{v})$ is upper semicontinuous on $\widehat{\Omega}$ for the Lusin topology $\mathcal{T}$(that is, due to the topologies $\mathcal{T}_3$ and $\mathcal{T}_4$)

$$
\limsup_{n\to\infty} \int_{\widehat{\Omega}} \widehat{\Psi}_1(\omega, \mathbf{v}) d\widehat{\mathbb{P}}^n(\omega, \mathbf{v}) \leq \int_{\widehat{\Omega}} \widehat{\Psi}_1(\omega, \mathbf{v}) d\widehat{\mathbb{P}}(\omega, \mathbf{v}). \tag{4.12}
$$

The integral with quadratic variation $[\widehat{\mathbf{M}}]_t$ in $\Psi_2(\omega, \mathbf{v})$ is continuous in the Lusin topology of $\widehat{\Omega}$ (in particular, due to the strong topology $\mathcal{T}_4$) and the integral involving $\Theta(\cdot)$ is continuous due to the continuity of $\varphi(\cdot, \cdot, t)$ on $\widehat{\Omega}$. Therefore,

$$
\lim_{n\to\infty} \int_{\widehat{\Omega}} \Psi_2(\omega, \mathbf{v}) d\widehat{\mathbb{P}}^n(\omega, \mathbf{v}) = \int_{\widehat{\Omega}} \Psi_2(\omega, \mathbf{v}) d\widehat{\mathbb{P}}(\omega, \mathbf{v}). \tag{4.13}
$$

From (4.7), (4.12) and (4.13), we arrive at

$$
\begin{aligned}
0 \le \limsup_{n\to\infty} \int_{\widehat{\Omega}} \Psi(\omega,\mathbf{v}) d\widehat{\mathbb{P}}^n(\omega,\mathbf{v}) &= \limsup_{n\to\infty} \int_{\widehat{\Omega}} (\widehat{\Psi}_1(\omega,\mathbf{v}) + \Psi_2(\omega,\mathbf{v})) d\widehat{\mathbb{P}}^n(\omega,\mathbf{v}) \\
&\le \int_{\widehat{\Omega}} (\Psi_1(\omega,\mathbf{v}) + \Psi_2(\omega,\mathbf{v})) d\widehat{\mathbb{P}}(\omega,\mathbf{v}).
\end{aligned}
$$

Eventually, we have shown that

$$
\int_{\widehat{\Omega}} \Psi(\omega,\mathbf{v}) d\widehat{\mathbb{P}}(\omega,\mathbf{v}) \ge 0. \tag{4.14}
$$

Now taking $\zeta(\omega,\mathbf{v},t) = \xi(\omega,t) - \lambda\mathbf{w}(\omega,\mathbf{v},t)$, where $\lambda > 0$ and $\mathbf{w}(\omega,\mathbf{v},t)$ is a bounded continuous mapping from $\widehat{\Omega}$ to $\mathbb{L}^2(0,T;\mathbb{V})$, we obtain

$$
\begin{aligned}
2\lambda \int_{\widehat{\Omega}} \Big( \int_0^T e^{-\rho(t)} \langle \chi(\omega,\mathbf{v},t) - \Theta(\xi(\omega,t) - \lambda\mathbf{w}(\omega,\mathbf{v},t)), \mathbf{w}(\omega,\mathbf{v},t)\rangle dt \Big) d\widehat{\mathbb{P}}(\omega,\mathbf{v}) \\
+\lambda^2 \int_{\widehat{\Omega}} \Big( \int_0^T e^{-\rho(t)} \dot{\rho}(t) |\mathbf{w}(\omega,\mathbf{v},t)|^2 dt \Big) d\widehat{\mathbb{P}}(\omega,\mathbf{v}) \ge 0. 
\end{aligned} \tag{4.15}
$$

Since $\Theta$ is hemicontinuous, $\langle \Theta(\xi - \lambda\mathbf{w}), \mathbf{w}\rangle \to \langle \Theta(\xi), \mathbf{w}\rangle$ as $\lambda \to 0$; indeed,

$$
\langle \Theta(\xi - \lambda\mathbf{w}), \mathbf{w}\rangle \;=\; \langle \Theta(\xi), \mathbf{w}\rangle - \lambda\langle \mathbf{A}\mathbf{w} + \mathbf{B}(\xi,\mathbf{w}) + \mathbf{B}(\mathbf{w},\xi), \mathbf{w}\rangle + \lambda^2\langle \mathbf{B}(\mathbf{w}), \mathbf{w}\rangle.
$$

Dividing (4.15) by $\lambda$, taking $\lambda \to 0$ and applying the dominated convergence theorem, we get

$$
\int_{\widehat{\Omega}} \Big( \int_0^T e^{-\rho(t)} \langle \chi(\omega,\mathbf{v},t) - \Theta(\xi(\omega,t)), \mathbf{w}(\omega,\mathbf{v},t)\rangle dt \Big) d\widehat{\mathbb{P}}(\omega,\mathbf{v}) \ge 0. \tag{4.16}
$$

Moreover, taking $\zeta(\omega,\mathbf{v},t) = \xi(\omega,t) + \lambda\mathbf{w}(\omega,\mathbf{v},t)$, we can get the reverse inequality of (4.16) so that

$$
\int_{\widehat{\Omega}} \Big( \int_0^T e^{-\rho(t)} \langle \chi(\omega,\mathbf{v},t) - \Theta(\xi(\omega,t)), \mathbf{w}(\omega,\mathbf{v},t)\rangle dt \Big) d\widehat{\mathbb{P}}(\omega,\mathbf{v}) = 0. \tag{4.17}
$$

Since (4.17) holds true for each bounded continuous function $\mathbf{w}$, the conclusion (4.5) is achieved. □

Now we come back to the proof of main theorem. The Lemma 4.1 indeed show that $\widehat{\Phi}(\widehat{\mathbf{M}}_t^f - \widehat{\mathbf{M}}_s^f) \in C(\widehat{\Omega})$, for any $\widehat{\mathcal{G}_s}$-measurable function $\widehat{\Phi} \in C_b(\widehat{\Omega})$. In order to pass the limit, we need the following.

**Lemma 4.2** (see, [31]). *Let $\widehat{\Omega}$ be a Lusin space and $\widehat{\mathbb{P}}^n$ be the sequence of probability measures on $\widehat{\Omega}$ converging weakly to a measure $\widehat{\mathbb{P}}$ as $n \to \infty$. Let $g \in C(\widehat{\Omega})$ and $\sup_n \mathbb{E}^{\widehat{\mathbb{P}}^n}[|g|^{1+\epsilon}] \le C$ for some $\epsilon > 0$. Then $\mathbb{E}^{\widehat{\mathbb{P}}^n}(g) \to \mathbb{E}^{\widehat{\mathbb{P}}}(g)$ as $n \to \infty$.*

Taking the estimate (4.4) and Lemma 4.1 into account, we can conclude from Lemma 4.2 that

$$
\lim_{n\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}^n}[\widehat{\Phi}(\widehat{\mathbf{M}}_t^f - \widehat{\mathbf{M}}_s^f)] = \mathbb{E}^{\widehat{\mathbb{P}}}[\widehat{\Phi}(\widehat{\mathbf{M}}_t^f - \widehat{\mathbf{M}}_s^f)] = 0, \text{ for all } t \ge s,
$$

whence $\widehat{\mathbf{M}}_t^f$ is a locally square integrable càdlàg $(\widehat{\Omega}, \widehat{\mathcal{G}}_t, \widehat{\mathbb{P}})$-local martingale. From the last equality and disintegration theorem for Radon measures on Lusin spaces, we can conclude that

$$\int_{\widehat{\Omega}} \widehat{\Phi}(\omega, \mathbf{v})[\widehat{\mathbf{M}}_t^f(\omega, \mathbf{v}) - \widehat{\mathbf{M}}_s^f(\omega, \mathbf{v})] d\widehat{\mathbb{P}}(\omega, \mathbf{v})$$
$$= \int_{\widetilde{\Omega}} \widehat{\Phi}(\omega, \Theta(\omega))[\mathbf{M}_t^f(\omega) - \mathbf{M}_s^f(\omega)] d\mathbb{P}(\omega),$$

and so $\mathbb{E}^{\mathbb{P}}[\widehat{\Phi}(\mathbf{M}_t^f - \mathbf{M}_s^f)] = 0$ for any $\widetilde{\mathscr{F}_s}$ measurable function $\widehat{\Phi} \in C_b(\widetilde{\Omega})$ and $t \geq s$. This completes the proof for 2D-case. □

Now let us prove Theorem 2.1 for 3D bounded domain.

*Proof of Theorem 2.1: (The domain $\mathcal{O}$ is bounded in $\mathbb{R}^3$).* In view of the estimate (2.23), it appears that Minty Stochastic Lemma 4.1 does not hold for 3D-case. Indeed, from the energy estimate (Theorem 3.1) and (2.23) for $d = 3$, it is clear that $\Theta(\cdot)$ exists only in the space $\mathbb{L}^{4/3}(0, T; \mathbb{V}')$ and so it does't make sense to define (4.6). In view of 2D-case, we need the following lemma for the continuity of the martingale $\mathbf{M}_t^f$ on $\widetilde{\Omega}$ and rest of the proof of existence of martingale solution follows from the similar arguments of 2D-case. □

**Lemma 4.3.** *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^3$. Let $f$ be the tame function as in Remark 2.1 with $\varphi(\cdot) \in C_0^\infty(\mathbb{R}^m)$ and $\theta_k \in \mathcal{D}(\mathbf{A}), k = 1, \cdots, m$. If $\mathbf{u}_n \to \mathbf{u}$ in $\widetilde{\Omega}$ for the Lusin topology $\mathcal{T}$, then $\mathbf{M}_t^f(\mathbf{u}_n) \to \mathbf{M}_t^f(\mathbf{u})$ on $\widetilde{\Omega}, \forall t \in [0, T]$.*

*Proof.* For simplicity, let us restrict the proof to the case $f(\mathbf{u}(t)) = \varphi(\langle \mathbf{u}(t), \theta_0 \rangle), \theta_0 \in \mathcal{D}(\mathbf{A})$. Let $\mathbf{u}_n \to \mathbf{u}$ for the Lusin topology $\mathcal{T}$. By suppressing the time dependence of $\mathbf{u}(t)$, we write

$$\begin{aligned}
\mathbf{M}_t^f(\mathbf{u}_n) &= \varphi(\langle \mathbf{u}_n, \theta_0 \rangle) - \varphi(\langle \mathbf{u}_n(0), \theta_0 \rangle) \\
&+ \int_0^t \big\langle \nu \mathbf{A}\mathbf{u}_n + \mathbf{B}(\mathbf{u}_n) - \mathbf{g}, \varphi'(\langle \mathbf{u}_n, \theta_0 \rangle)\theta_0 \big\rangle ds \\
&- \frac{1}{2} \int_0^t \varphi''(\langle \mathbf{u}_n, \theta_0 \rangle) \mathrm{tr}\big(\sigma(s, \mathbf{u}_n)\mathbb{Q}\sigma^*(s, \mathbf{u}_n)\theta_0 \otimes \theta_0\big) ds \\
&- \int_0^t \sum_{k=1}^\infty \int_{|z_k|<1} \big\{\varphi\big(\langle \mathbf{u}_n + \phi_k(\mathbf{u}_n, z_k), \theta_0 \rangle\big) - \varphi(\langle \mathbf{u}_n, \theta_0 \rangle) \\
&- \langle \phi_k(\mathbf{u}_n, z_k), \varphi'(\langle \mathbf{u}_n, \theta_0 \rangle)\theta_0) \rangle \big\} \mu_k(dz_k) ds \\
&- \int_0^t \sum_{k=1}^\infty \int_{|z_k|\geq 1} \big\{\varphi\big(\langle \mathbf{u}_n + \psi_k(\mathbf{u}_n, z_k), \theta_0 \rangle\big) - \varphi(\langle \mathbf{u}_n, \theta_0 \rangle)\big\} \mu_k(dz_k) ds.
\end{aligned} \tag{4.18}$$

Let $\mathbf{M}_t^f(\mathbf{u}_n) := \sum_{i=1}^6 I_{n,i}$ and $\mathbf{M}_t^f(\mathbf{u}) := \sum_{i=1}^6 I_i$. Then we need to show that $|\mathbf{M}_t^f(\mathbf{u}_n) - \mathbf{M}_t^f(\mathbf{u})| \to 0$ as $n \to \infty$. The continuity of $I_{n,1}$ and $I_{n,2}$ follow easily from the continuity of $\varphi$. Note that

$$\int_0^T \big\langle \nu \mathbf{A}\mathbf{u}_n, \varphi'(\langle \mathbf{u}_n, \theta_0 \rangle)\theta_0 \big\rangle dt = -\int_0^T \nu \big\langle \mathbf{u}_n(s), \Delta(\varphi'(\langle \mathbf{u}_n, \theta_0 \rangle)\theta_0) \big\rangle dt$$
$$\to -\int_0^T \nu \big\langle \mathbf{u}, \Delta(\varphi'(\langle \mathbf{u}, \theta_0 \rangle)\theta_0) \big\rangle dt = \int_0^T \big\langle \nu \mathbf{A}\mathbf{u}, \varphi'(\langle \mathbf{u}, \theta_0 \rangle)\theta_0 \big\rangle dt,$$

as $\mathbf{u}_n \xrightarrow{s} \mathbf{u}$ in $\mathbb{L}^2(0,T;\mathbb{H})$. Observe that

$$\begin{aligned}&\langle \mathbf{B}(\mathbf{u}_n), \varphi'(\langle \mathbf{u}_n, \theta_0\rangle)\theta_0\rangle - \langle \mathbf{B}(\mathbf{u}), \varphi'(\langle \mathbf{u}, \theta_0\rangle)\theta_0\rangle \\ &\quad = b(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n, \varphi'(\langle \mathbf{u}_n, \theta_0\rangle)\theta_0) + b(\mathbf{u}, \mathbf{u} - \mathbf{u}_n, \varphi'(\langle \mathbf{u}_n, \theta_0\rangle)\theta_0) \\ &\quad + b(\mathbf{u}, \mathbf{u}, (\varphi'(\langle \mathbf{u}_n, \theta_0\rangle) - \varphi'(\langle \mathbf{u}, \theta_0\rangle))\theta_0).\end{aligned}$$

From (2.23), we obtain that

$$\begin{aligned}\int_0^T |b(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n, \varphi'(\langle \mathbf{u}_n, \theta_0\rangle)\theta_0)|dt &\leq \int_0^T |\varphi'(\langle \mathbf{u}_n, \theta_0\rangle)||b(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n, \theta_0)|dt \\ \leq\ & C\|\varphi'\|_{L^\infty(\mathbb{R})}|\theta_0|^{1/4}\|\theta_0\|^{3/4}\int_0^T |\mathbf{u}_n - \mathbf{u}|^{1/4}\|\mathbf{u}_n - \mathbf{u}\|^{3/4}\|\mathbf{u}_n\|dt \\ \leq\ & C\|\varphi'\|_{L^\infty(\mathbb{R})}|\theta_0|^{1/4}\|\theta_0\|^{3/4}\Big(\int_0^T |\mathbf{u}_n - \mathbf{u}|^2 dt\Big)^{1/8} \\ & \times\Big(\int_0^T \|\mathbf{u}_n - \mathbf{u}\|^2 dt\Big)^{3/8}\Big(\int_0^T \|\mathbf{u}_n\|^2 dt\Big)^{1/2} \to 0 \text{ as } n \to \infty\end{aligned}$$

since $\mathbf{u}_n \xrightarrow{s} \mathbf{u}$ in $\mathbb{L}^2(0,T;\mathbb{H})$ and last two integrals are uniformly bounded. Making use of Lemma 2.2 and the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^6$ of (2.22), we get

$$\begin{aligned}\int_0^T |b(\mathbf{u}, \mathbf{u} - \mathbf{u}_n, \varphi'(\langle \mathbf{u}_n, \theta_0\rangle)\theta_0)|dt &\leq \int_0^T |\varphi'(\langle \mathbf{u}_n, \theta_0\rangle)||b(\mathbf{u}, \theta_0, \mathbf{u}_n - \mathbf{u})|dt \\ \leq\ & \|\varphi'\|_{L^\infty(\mathbb{R})}\int_0^T \|\mathbf{u}\|_{\mathbb{L}^6}\|\nabla\theta_0\|_{\mathbb{L}^3}|\mathbf{u}_n - \mathbf{u}|dt \\ \leq\ & C\|\varphi'\|_{L^\infty(\mathbb{R})}\|\theta_0\|^{1/2}|\mathbf{A}\theta_0|^{1/2}\int_0^T \|\mathbf{u}\||\mathbf{u}_n - \mathbf{u}|dt \\ \leq\ & C\|\varphi'\|_{L^\infty(\mathbb{R})}\|\theta_0\|^{1/2}|\mathbf{A}\theta_0|^{1/2}\Big(\int_0^T \|\mathbf{u}\|^2 dt\Big)^{1/2}\Big(\int_0^T |\mathbf{u}_n - \mathbf{u}|^2 dt\Big)^{1/2} \to 0,\end{aligned}$$

as $n \to \infty$ and

$$\begin{aligned}&\Big|\int_0^T b(\mathbf{u}, \mathbf{u}, (\varphi'(\langle \mathbf{u}_n, \theta_0\rangle) - \varphi'(\langle \mathbf{u}, \theta_0\rangle))\theta_0)dt\Big| \\ &\quad = \Big|\int_0^T \Big[\int_0^1 \frac{d}{dr}\Big(\varphi'\big(r\langle \mathbf{u}_n(t), \theta_0\rangle + (1-r)\langle \mathbf{u}(t), \theta_0\rangle\big)\Big)dr\Big] b(\mathbf{u}, \theta_0, \mathbf{u})dt\Big| \\ &\quad \leq C\|\varphi''\|_{L^\infty(\mathbb{R})}\|\theta_0\|^{1/2}|\mathbf{A}\theta_0|^{1/2}\int_0^T |\langle \mathbf{u}_n - \mathbf{u}, \theta_0\rangle|\|\mathbf{u}\||\mathbf{u}|dt \\ &\quad \leq C\|\varphi''\|_{L^\infty(\mathbb{R})}|\theta_0|\|\theta_0\|^{1/2}|\mathbf{A}\theta_0|^{1/2}\sup_{t\in[0,T]}|\mathbf{u}(t)|\Big(\int_0^T \|\mathbf{u}\|^2 dt\Big)^{1/2} \\ &\quad\quad \times\Big(\int_0^T |\mathbf{u} - \mathbf{u}_n|^2 dt\Big)^{1/2} \to 0 \text{ as } n \to \infty.\end{aligned}$$

By the argument similar to the above integral, we have

$$\Big|\int_0^T [\varphi'(\langle \mathbf{u}_n, \theta_0\rangle) - \varphi'(\langle \mathbf{u}, \theta_0\rangle)]\langle \mathbf{g}, \theta_0\rangle dt\Big| \leq \|\varphi''\|_{L^\infty(\mathbb{R})}|\theta_0|\|\theta_0\|\int_0^T |\mathbf{u}_n - \mathbf{u}|\|\mathbf{g}\|_{\mathbb{V}'}dt$$

since $\left(\int_0^T \|\mathbf{g}(t)\|_{\mathbb{V}'}^2 dt\right)^{1/2} < \infty$, the last integral also tends to zero.

Next, we note that

$$
\begin{aligned}
|I_{n,4} - I_4| \le \frac{1}{2}\int_0^T |\varphi''(\langle \mathbf{u}_n, \theta_0\rangle(t))| \big| \|\sigma(t, \mathbf{u}_n(t))\|^2_{\mathcal{L}_{HS}} - \|\sigma(t, \mathbf{u}(t))\|^2_{\mathcal{L}_{HS}} \big| dt \\
+\frac{1}{2}\int_0^T |\varphi''(\langle \mathbf{u}_n(t), \theta_0\rangle) - \varphi''(\langle \mathbf{u}(t), \theta_0\rangle)| \|\sigma(t, \mathbf{u}(t))\|^2_{\mathcal{L}_{HS}} dt \\
\le \quad \frac{1}{2}\|\varphi''\|_{L^\infty(\mathbb{R})}\int_0^T \big| \|\sigma(t, \mathbf{u}_n(t))\|^2_{\mathcal{L}_{HS}} - \|\sigma(t, \mathbf{u}(t))\|^2_{\mathcal{L}_{HS}} \big| dt \\
+\frac{1}{2}|\theta_0| T^{1/2} \|\varphi'''\|_{L^\infty(\mathbb{R})} \widetilde{N}_2 (1 + \sup_{t\in[0,T]} |\mathbf{u}(t)|^2) \Big( \int_0^T |\mathbf{u}_n - \mathbf{u}|^2 dt \Big)^{1/2}.
\end{aligned}
$$

Here both integrals on the right hand side tend to zero due to the continuity $[H1]$ (in Assumption 2.1) on the first integral while $\mathbf{u}_n \xrightarrow{s} \mathbf{u}$ in $\mathbb{L}^2(0,T;\mathbb{H})$ on the second integral.

Finally, since $\varphi \in C_0^\infty(\mathbb{R})$, we have for any $p, q \in \mathbb{R}$

$$
\Phi(p,q) := \varphi(p+q) - \varphi(p) - \varphi'(p) q = q^2 \int_0^1 (1-r)\varphi''(p+rq) dr
$$

so that for any $p_1, p_2 \in \mathbb{R}$

$$
\begin{aligned}
|\Phi(p_1, q) - \Phi(p_2, q)| \quad &= \quad \Big| q^2 \int_0^1 (1-r)[\varphi''(p_1 + rq) - \varphi''(p_2 + rq)] dr \Big| \\
&\le \quad \|\varphi'''\|_{L^\infty(\mathbb{R})} q^2 |p_1 - p_2|.
\end{aligned}
$$

Besides, for any $q_1, q_2 \in \mathbb{R}$, one can also obtain that

$$
|\Phi(p, q_1) - \Phi(p, q_2)| \le \|\varphi''\|_{L^\infty(\mathbb{R})} (|q_1| + |q_2|) |q_1 - q_2|.
$$

Taking $p_1 = \langle \mathbf{u}_n, \theta_0\rangle, q = q_1 = \langle \phi_k(\mathbf{u}_n, z_k), \theta_0\rangle$ and $p = p_2 = \langle \mathbf{u}, \theta_0\rangle$ and $q_2 = \langle \phi_k(\mathbf{u}, z_k), \theta_0\rangle$, we can obtain that

$$
\begin{aligned}
|I_{n,5} - I_5| \quad \le \quad & \int_0^T \sum_{k=1}^\infty \int_{|z_k|<1} \big\{ |\Phi(\langle \mathbf{u}_n, \theta_0\rangle, \langle \phi_k(\mathbf{u}_n, z_k), \theta_0\rangle) \\
& - \Phi(\langle \mathbf{u}, \theta_0\rangle, \langle \phi_k(\mathbf{u}_n, z_k), \theta_0\rangle)| \big\} \mu_k(dz_k) dt \\
& + \int_0^T \sum_{k=1}^\infty \int_{|z_k|<1} \big\{ |\Phi(\langle \mathbf{u}, \theta_0\rangle, \langle \phi_k(\mathbf{u}_n, z_k), \theta_0\rangle) \\
& - \Phi(\langle \mathbf{u}, \theta_0\rangle, \langle \phi_k(\mathbf{u}, z_k), \theta_0\rangle)| \big\} \mu_k(dz_k) dt.
\end{aligned}
$$

From the preceding estimates and Assumption 2.2, we have

$$
\begin{aligned}
|I_{n,5} - I_5| \leq \|\varphi'''\|_{L^\infty(\mathbb{R})}|\theta_0|^3 \int_0^T |\mathbf{u}_n - \mathbf{u}| \sum_{k=1}^\infty \int_{|z_k|<1} |\phi_k(\mathbf{u}_n, z_k)|^2 \mu_k(dz_k) dt \\
+ \|\varphi''\|_{L^\infty(\mathbb{R})}|\theta_0|^2 \int_0^T \sum_{k=1}^\infty \int_{|z_k|<1} \big(|\phi_k(\mathbf{u}_n, z_k)| \\
+ |\phi_k(\mathbf{u}, z_k)|\big)|\phi_k(\mathbf{u}_n, z_k) - \phi_k(\mathbf{u}, z_k)|\mu_k(dz_k)dt \\
\leq \Bigg[\|\varphi'''\|_{L^\infty(\mathbb{R})}|\theta_0|^3 N_3\big(1 + \sup_{t\in[0,T]}|\mathbf{u}_n(t)|^2\big)\sqrt{T} + 2T\|\varphi''\|_{L^\infty(\mathbb{R})}|\theta_0|^2 \\
\times \sqrt{N_2 N_3}\Big(2T + \int_0^T (|\mathbf{u}|^2 + |\mathbf{u}_n|^2)dt\Big)^{1/2}\Bigg]\Big(\int_0^T |\mathbf{u}_n - \mathbf{u}|^2 dt\Big)^{1/2}.
\end{aligned}
$$

Thus, $|I_{n,5} - I_5| \to 0$ as $n \to \infty$ due to the strong convergence of $\mathbf{u}_n \to \mathbf{u}$ in $\mathbb{L}^2(0,T;\mathbb{H})$. We can similarly establish the continuity of the integral $I_{n,6}$ and conclude the proof. □

4.1. **Existence of martingale solutions in unbounded domain.** As we noticed in section 3.3 that in the case of unbounded $\mathcal{O}$, we have the tightness of measures only in $\mathbb{D}(0,T;\mathbb{H}_w)$ and it is not sufficient to obtain the tightness in the strong topology $\mathcal{T}_4$ of $\mathbb{L}^2(0,T;\mathbb{H})$. But in order to prove the Minty Stochastic Lemma 4.1, we need the tightness in $\mathcal{T}_4$, in particular, to validate the continuity of the noise terms. The existence of martingale solutions for the SNSEs (2.1) in unbounded domain $\mathcal{O}$, at least, in $\mathbb{R}^2$ without any compactness argument may be addressed as follows.

**Remark 4.1.** If the noise terms are additive, that is, the noise coefficients are of the form $\sigma(t,\mathbf{u}) = \sigma(t)$, $\phi_k(\mathbf{u}, z_k) = \phi_k(z_k)$ and $\psi_k(\mathbf{u}, z_k) = \psi_k(z_k), k = 1, 2, \cdots$, the proof of Lemma 4.1, and hence the existence of martingale solutions in 2D-case, still hold since we have handled the nonlinearity coming from the drift term by using the local monotonicity argument.

If the noise terms are multiplicative (once again for the case of unbounded $\mathcal{O}$) to get the stochastic Minty-Browder technique to work, we need to assume that the mapping $\mathbf{u} \to \|\sigma(t,\mathbf{u})\|^2_{\mathcal{L}_{HS}}$ is continuous for the weak topology of $\mathbb{H}$ along with assumptions for the jump noise coefficients (see, for instance, [22]).

**Remark 4.2.** We note however that the existence of strong solutions for 2D stochastic Navier-Stokes equations with multiplicative Gaussian noise (see, [33]) and Itô-Lévy noise (see, [8]) in unbounded domain can be handled by local monotonicity method without any weak continuity assumptions on the noise coefficients.

We solve the unbounded case by extending Theorem 2.1 for bounded $\mathcal{O}$ as follows. We cut the unbounded domain $\mathcal{O}$ into a sequence of bounded domains $\mathcal{O}_i, i = 1, 2\cdots$ and construct martingale solutions $\mathbb{P}_i, i = 1, 2, \cdots$ for the SNSEs (2.1) in each of these bounded domains $\mathcal{O}_i$ and then show that in the limit the martingale solution $\mathbb{P}$ for $\mathcal{O}$ exists. The construction we use here is a standard procedure and can be found for example in [17] for various fluid flow problems in unbounded domain and in [30] to prove the existence of generalized solutions for Navier-Stokes equations in unbounded channel like domains.

*Proof of Theorem 2.1: (The domain $\mathcal{O}$ is unbounded).* Let $\mathcal{O}_i, i = 1, 2 \cdots$ be the nested, open bounded subsets $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots$ of $\mathcal{O}$ and $\cup_{i=1}^{\infty} \mathcal{O}_i = \mathcal{O}$. We may take $\mathcal{O}_i$ as $\mathcal{O} \cap \{|x| \leq r_i\}$. In each of these subdomains, we solve the following problem $(\mathbf{u}_i, p_i) : \mathcal{O}_i \times [0, T] \to \mathbb{R}^d \times \mathbb{R}$ such that

$$\begin{aligned} d\mathbf{u}_i \quad + \quad & (-\nu\Delta\mathbf{u}_i + \mathbf{u}_i \cdot \nabla\mathbf{u}_i + \nabla p_i)dt = \mathbf{g}dt + \sigma(t, \mathbf{u}_i)d\mathbf{W} \\ & + \sum_{k=1}^{\infty} \int_{0<|z_k|_{\mathcal{Z}}<1} \phi_{k,i}\big(t, \mathbf{u}_i(t-), z_k\big)\widetilde{\pi}_k(dt, dz_k) \\ & + \sum_{k=1}^{\infty} \int_{|z_k|_{\mathcal{Z}}\geq 1} \psi_{k,i}\big(t, \mathbf{u}_i(t-), z_k\big)\pi_k(dt, dz_k) \quad \text{in} \quad \mathcal{O}_i \times (0, T) \end{aligned} \tag{4.19}$$

with

$$\begin{aligned} \nabla \cdot \mathbf{u}_i &= 0 \quad \text{in} \quad \mathcal{O}_i \times (0, T), \\ \mathbf{u}_i &= 0 \quad \text{on} \quad \partial\mathcal{O}_i \times (0, T), \quad \mathbf{u}_i(x, 0) = \mathbf{u}_0(x) \quad \text{in} \quad \mathcal{O}_i. \end{aligned}$$

This leads to a semimartingale formulation analogous to (2.10) and then by the Galerkin approximation, we get the finite dimensional system similar to (3.1) in each of these subdomains $\mathcal{O}_i$. Let $\mathbb{P}_i^n$ be the measures on $\mathbb{D}(0, T; \mathbb{V}')$ associated with the finite dimensional processes $\mathbf{u}_i^n$. In view of the proof of Theorem 3.1, there exist constants $\widetilde{C}$ and $C$ independent of the size of the domains $\mathcal{O}_i$ satisfying

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}_i^n} \sup_{0\leq s\leq t} |\xi(s)|^2 + \nu\mathbb{E}^{\mathbb{P}_i^n} \int_0^t \|\xi(s)\|^2 ds \\ & \quad \leq \widetilde{C}\Big(T, \widetilde{N}_2, N_3, \nu, \mathbb{E}|\xi(0)|^2, \int_0^T \|\mathbf{g}(t)\|_{\mathbb{V}'}^2 dt\Big), \end{aligned} \tag{4.20}$$

and for all $p \geq 2$

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}_i^n} \sup_{0\leq s\leq t} |\xi(s)|^p + \nu\mathbb{E}^{\mathbb{P}_i^n} \int_0^t |\xi(s)|^{p-2}\|\xi(s)\|^2 ds \\ & \quad \leq C\Big(p, T, \widetilde{N}_2, N_3, \nu, \mathbb{E}|\xi(0)|^p, \big(\int_0^T \|\mathbf{g}(t)\|_{\mathbb{V}'}^2 dt\big)^{p/2}\Big). \end{aligned} \tag{4.21}$$

Following the proof of Proposition 3.1, we can show that the measures $\mathbb{P}_i^n$ are tight in the Lusin space $\widetilde{\Omega}$ endowed with the topology $\mathcal{T}$ by extending the flow field to zero outside $\mathcal{O}_i$. In order to conclude that any limit measure $\mathbb{P}$ of the sequence $\mathbb{P}_i^n$ is a solution of the martingale problem on $\mathcal{O}$, it is sufficient show that $\mathbf{M}_t^f(\xi)$ is continuous on $\widetilde{\Omega}$ and this follows from similar arguments we used earlier. □

### 4.2. **Pathwise uniqueness of solutions and uniqueness of the martingale solutions.**

*Proof of Theorem 2.2.* Let $\mathbf{u}$ and $\mathbf{v}$ be the two solutions of (2.10). For $m > 0$, define

$$\tau_m^1 = \inf\{t \leq T; |\mathbf{u}(t)|^2 \geq m\} \quad \text{and} \quad \tau_m^2 = \inf\{t \leq T; |\mathbf{v}(t)|^2 \geq m\}.$$

Let us take $\tau_m = \tau_m^1 \wedge \tau_m^2$. Define the set $\Omega_m = \{\omega \in \Omega : |\mathbf{u}(t)|^2 < m\}$. Then we obtain from the energy estimate that

$$\int_{\Omega_m} |\mathbf{u}(t)|^2 dP(\omega) + \int_{\Omega\setminus\Omega_m} |\mathbf{u}(t)|^2 dP(\omega) \leq C,$$

for some constant $C > 0$. So $P(\Omega\backslash\Omega_m) = P\{\omega \in \Omega : |\mathbf{u}(t)|^2 \geq m\} \leq \frac{C}{m}$. This further leads to

$$P\{\tau_m < T\} \leq P\big\{|\mathbf{u}(t)|^2 \geq m\big\} \vee P\big\{|\mathbf{v}(t)|^2 \geq m\big\} \leq \frac{C}{m}.$$

Hence $\limsup_{m\to\infty} P\{\tau_m < T\} = 0$ and so $\tau_m \to T$ as $m \to \infty$, a.s. For notation simplicity, define $\mathbf{w} = \mathbf{u}-\mathbf{v}, \widetilde{\sigma} = \sigma(t,\mathbf{u})-\sigma(t,\mathbf{v}), \widetilde{\phi}_k = \phi_k(\mathbf{u}(t-), z_k)-\phi_k(\mathbf{v}(t-), z_k)$ and $\widetilde{\psi}_k = \psi_k(\mathbf{u}(t-), z_k) - \psi_k(\mathbf{v}(t-), z_k), k = 1, 2\cdots$ . Then, we have

$$\mathbf{w}(t) = \mathbf{w}(0) - \int_0^t \Big([\Theta(\mathbf{u}(s)) - \Theta(\mathbf{v}(s))] - \sum_{k=1}^{\infty}\int_{|z_k|\geq 1} \widetilde{\psi}_k \mu_k(dz_k)\Big)ds + \bar{\mathbf{M}}_t,$$

where $\Theta(\mathbf{u}) = \nu\mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u})$ and the local martingale

$$\bar{\mathbf{M}}_t = \int_0^t \widetilde{\sigma} d\mathbf{W}(s) + \int_0^t \Big(\sum_{k=1}^{\infty}\int_{|z_k|<1} \widetilde{\phi}_k \widetilde{\pi}_k(ds, dz_k) + \sum_{k=1}^{\infty}\int_{|z_k|\geq 1} \widetilde{\psi}_k \widetilde{\pi}_k(ds, dz_k)\Big)ds.$$

Applying the Itô formula (see, [26]) to $|\mathbf{w}(t)|^2$, we get

$$\begin{aligned}|\mathbf{w}(t)|^2 &= |\mathbf{w}(0)|^2 - 2\int_0^t \langle\Theta(\mathbf{u}(s)) - \Theta(\mathbf{v}(s)), \mathbf{w}(s)\rangle ds \\ &+2\int_0^t \Big\langle \sum_{k=1}^{\infty}\int_{|z_k|\geq 1} \widetilde{\psi}_k \mu_k(dz_k), \mathbf{w}(s)\Big\rangle ds + [\bar{\mathbf{M}}]_t + 2\int_0^t (\mathbf{w}(s), d\bar{\mathbf{M}}_s).\end{aligned}$$

Using the estimate (2.21), note that

$$\begin{aligned}|\langle\mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle| &= |b(\mathbf{w}, \mathbf{v}, \mathbf{w})| \leq \|\mathbf{w}\|^2_{\mathbb{L}^4(\mathcal{O})}\|\mathbf{v}\| \\ &\leq C|\mathbf{w}|^{(4-d)/2}\|\mathbf{w}\|^{d/2}\|\mathbf{v}\| \\ &\leq \frac{\nu}{2}\|\mathbf{w}\|^2 + C_\nu\|\mathbf{v}\|^{4/(4-d)}|\mathbf{w}|^2, \quad \text{for} \quad d = 2, 3.\end{aligned}$$

So from Assumption 2.2, we arrive at

$$\begin{aligned}|\mathbf{w}(t)|^2 &\leq |\mathbf{w}(0)|^2 - \nu\int_0^t \|\mathbf{w}(s)\|^2 ds + 2C_\nu\int_0^t \|\mathbf{v}(s)\|^{4/(4-d)}|\mathbf{w}(s)|^2 ds \\ &+(1 + C_\mu N_2)\int_0^t |\mathbf{w}(s)|^2 ds + \mathrm{tr}[\![\bar{\mathbf{M}}]\!]_t + 2\int_0^t (\mathbf{w}(s), d\bar{\mathbf{M}}_s),\end{aligned}$$

where $C_\mu = \sum_{k=1}^{\infty}\int_{|z_k|\geq 1} \mu_k(dz_k)$. Taking $\rho(t) = 2C_\nu\int_0^t \|\mathbf{v}(s)\|^{4/(4-d)}ds, d = 2, 3$ the Itô formula again applied to $e^{-\rho(t)}|\mathbf{w}(t)|^2$ along with the preceding estimate give

$$\begin{aligned}e^{-\rho(t)}|\mathbf{w}(t)|^2+\nu\int_0^t e^{-\rho(s)}\|\mathbf{w}(s)\|^2 ds &\leq |\mathbf{w}(0)|^2 + (1 + C_\mu N_2)\int_0^t e^{-\rho(s)}|\mathbf{w}(s)|^2 ds \\ &+ \int_0^t e^{-\rho(s)} d(\mathrm{tr}[\![\bar{\mathbf{M}}]\!]_s) + 2\int_0^t e^{-\rho(s)}(\mathbf{w}(s), d\bar{\mathbf{M}}_s). \qquad (4.22)\end{aligned}$$

Let $\tau_m$ be the stopping time localizing the martingale $\int_0^t e^{-\rho(s)}(\mathbf{w}(s), d\bar{\mathbf{M}}_s)$. From Assumptions 2.1, 2.2 and Lemma 2.7, we note that

$$\begin{aligned}
\mathbb{E}\int_0^{t\wedge\tau_m} & e^{-\rho(s)}d(\mathrm{tr}[\![\bar{\mathbf{M}}]\!]_s) = \mathbb{E}\int_0^{t\wedge\tau_m} e^{-\rho(s)}d(\mathrm{tr} \ll \bar{\mathbf{M}} \gg_s) \\
= \quad & \mathbb{E}\int_0^{t\wedge\tau_m} e^{-\rho(s)}\mathrm{tr}(\widetilde{\sigma}\mathbb{Q}\widetilde{\sigma}^*)ds \\
& +\mathbb{E}\int_0^{t\wedge\tau_m} e^{-\rho(s)}\Big(\sum_{k=1}^{\infty}\int_{|z_k|<1}|\widetilde{\phi}|^2\mu_k(dz_k) + \sum_{k=1}^{\infty}\int_{|z_k|\geq 1}|\widetilde{\psi}|^2\mu_k(dz_k)\Big)ds \\
\leq \quad & (\widetilde{N}_1 + N_2)\mathbb{E}\int_0^{t\wedge\tau_m} e^{-\rho(s)}|\mathbf{w}(s)|^2 ds.
\end{aligned}$$

Taking into account that the expectation of the local martingale is zero, we have

$$\mathbb{E}[e^{-\rho(t\wedge\tau_m)}|\mathbf{w}(t\wedge\tau_m)|^2] \leq \mathbb{E}|\mathbf{w}(0)|^2 + \bar{C}\int_0^t \mathbb{E}[e^{-\rho(s\wedge\tau_m)}|\mathbf{w}(s\wedge\tau_m)|^2]ds,$$

where $\bar{C} = (1 + \widetilde{N}_1 + N_2 + C_\mu N_2)$. Therefore, by Gronwall's inequality, we deduce that

$$\mathbb{E}[e^{-\rho(t\wedge\tau_m)}|\mathbf{w}(t\wedge\tau_m)|^2] \leq \exp(\bar{C}T)\mathbb{E}|\mathbf{w}(0)|^2.$$

Eventually taking the conditions (i) and (ii) into account, the data $\mathbf{w}(0) = 0$ leads to $\mathbf{w}(t\wedge\tau_m) = 0$ a.s. But the fact that $\tau_m \to T$ as $m \to \infty$ gives $\mathbf{w}(t) = 0 \implies \mathbf{u}(t) = \mathbf{v}(t)$ for all $t \in [0, T]$ a.s. Hence the proof. □

5. **Concluding remarks.** We remark here that mathematically similar problems such as MHD equations (Chandrasekhar [4], Sritharan and Sundar [32]), regularized Navier-Stokes equations (Ou and Sritharan [24]), tamed Navier-Stokes equations (Leray [19], Sritharan and Sundar [33]) and combustion models (Temam [37]) etc. can fit in the abstract mathematical model (2.10) and hence this paper establishes the solvability of martingale problems for all these classes of problems subjected to Lévy noise.

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