DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **13**, Number **1**, June **2005**

Website: http://aimSciences.org

pp. **139–162**

TWO-PARAMETER FAMILIES OF IMPLICIT DIFFERENTIAL EQUATIONS

Farid Tari

ICMC-USP, Dept. de Matemática, Av. do Trabalhador São-Carlense, 400 Centro
Caixa Postal 668, CEP 13560-970, São Carlos (SP), Brazil

Abstract. We study in this paper local codimension 2 singularities of (first order) implicit differential equations $F(x, y, p) = 0$, where F is a germ of a smooth function, $p = \frac{dy}{dx}$, $F_p = 0$ and $F_{pp} \neq 0$ at the singular point. We obtain topological models of these singularities and deal with their bifurcations in generic 2-parameter families of equations.

1. **Introduction.** Let $F(x, y, p) = 0$ be an implicit differential equation (IDE), where F is a smooth function (i.e C^∞) in some domain in $\mathbb{R}^3$ and $p = \frac{dy}{dx}$. If $F_p(q_0) \neq 0$ at $q_0 = (x_0, y_0, p_0) \in \mathbb{R}^3$ (when not indicated otherwise, subscript denote partial differentiation), the above equation can be written locally (in a neighbourhood of q_0) in the form $p = g(x, y)$ and studied using the methods from the theory of ordinary differential equations.

When $F_p(q_0) = 0$, the equation may define more than one direction in a neighbourhood of (x_0, y_0). The cases that have been studied intensively are the IDEs that define at most two directions (except maybe at some isolated points). This can happen locally when:

$$F(x, y, p) = 0,\ F(q_0) = F_p(q_0) = 0,\ F_{pp}(q_0) \neq 0 \tag{1.1}$$

or if the equation is given in a quadratic form

$$a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2 = 0. \tag{1.2}$$

In fact, by the division theorem, any IDE (1.1) can be expressed in a quadratic form. A crucial difference between the two cases is that an IDE (1.1) defines locally at most two directions at each point close to (x_0, y_0), whereas in (1.2) the whole projective line is a solution at points where all the coefficients a, b, c vanish.

IDEs have extensive applications to differential geometry (see section 4), second order partial differential equations and control theory (see [13] and [22]). For example, asymptotic, characteristic and principal directions on a smooth surface in $\mathbb{R}^3$ are given by IDEs. Their study reveals a great deal of information about the geometry of the surface (see for example [3], [5], [8], [18], [19], [26]).

Our aim here is to study local codimension 2 singularities of IDEs (1.1) and their bifurcations in generic families. We deal in [23] with codimension 2 singularities of IDEs (1.2) when the coefficients a, b, c all vanish at a given point. We recall that the local configurations of all codimension ≤ 1 singularities of IDEs (1.1) are known. See [10], [11], [12], [21], [22] for the stable cases and [5], [14], [15], [22] for

2000 *Mathematics Subject Classification.* 34A09, 37G10, 37C10, 58K05, 53A05.
Key words and phrases. Implicit differential equations, singularities, bifurcations.
Research partially supported by PROCAD-CAPES 0092-01-0, Brazil.