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SUBCLASSES OF HERGLOTZ-NEVANLINNA MATRIX–VALUED FUNCTIONS AND LINEAR SYSTEMS

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Abstract. We study Herglotz-Nevanlinna matrix–valued functions that are linear-fractional transformations of the transfer matrix–valued functions of some type of linear, stationary dynamical systems. We introduce the family $\mathcal{F}^k$ of Herglotz-Nevanlinna matrix–valued functions N such that the associated kernel $\frac{N(z)-N(w)^*}{z-w^*} - \frac{N(w)^*N(z)}{k}$ is positive in the open upper half–plane. We characterize these functions in terms of Schur functions. This allows us to solve the bitangential interpolation problem in the class $\mathcal{F}^k$. Finally, we consider an associated automorphism of rational unitary functions using state space methods.

1. **Introduction.** In the theory of linear stationary dynamical systems [16] an important role is played by the systems of the form

$$\begin{cases} (A - zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases}$$

where φ_- is an input vector, φ_+ is an output vector, A is the main operator in the state space, and K is the channel operator. This system is usually expressed as an operator colligation (a so called Brodskiĭ-Livšic operator colligation) of the form

$$\Theta = \begin{pmatrix} A & K & J \\ \mathcal{H} & & \mathcal{E} \end{pmatrix}.$$

Here A is a bounded linear operator acting in a Hilbert space $\mathcal{H}$, $J = J^* = J^{-1}$ is a bounded linear operator in a finite dimensional Hilbert space $\mathcal{E}$ and K is a bounded linear operator from $\mathcal{E}$ into $\mathcal{H}$. The operator A is assumed to be close to a selfadjoint operator in the sense that $\mathrm{Im}\, A = KJK^*$. The corresponding transfer operator-valued function (characteristic operator-valued function) is defined by

$$W_\Theta(z) = I - 2iK^*(A - zI)^{-1}KJ. \tag{1}$$

The characteristic operator-valued function of the form (1) is a unitary invariant of the nonselfadjoint operator A [11] and admits the normalization $\lim_{z\to\infty} W_\Theta(z) = I$ at ∞.

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The linear-fractional transformation of the transfer operator-valued function of the form

$$V_\Theta(z) = i[W_\Theta(z) - I][W_\Theta(z) + I]^{-1}J = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I]J \qquad (2)$$

is a Herglotz-Nevanlinna operator-valued function.

In this paper we pursue our study of certain subclasses of the family of Herglotz–Nevanlinna matrix–valued that are linear-fractional transformations of the transfer matrix–valued functions of the above mentioned stationary dynamical systems and develop the interpolation theory in these subclasses; see [9], [8], [13], [14], [15] and [6]. Recall first that $\mathbb{C}^{n\times n}$–valued function is a Herglotz–Nevanlinna function if it has a positive imaginary part in the open upper half–plane $\mathbb{C}_+$. As it is well known (see [18], [9]), such a matrix–valued function admits a representation of the form

$$N(z) = A + Bz + \int_{\mathbb{R}} d\mu(t)\left\{\frac{1}{t-z} - \frac{t}{t^2+1}\right\} \qquad (3)$$

where $A \in \mathbb{C}^{n\times n}$ is a self–adjoint matrix, $B \in \mathbb{C}^{n\times n}$ is a nonnegative matrix and $d\mu$ is a positive matrix–valued measure such that $\int_{\mathbb{R}} \frac{d\ \mathrm{Tr}\ \mu(t)}{t^2+1} < \infty$. It follows from the representation (3) that

$$\frac{N(z) - N(w)^*}{z - w^*} = \frac{N(z) - N(w)^*}{z - w^*} = B + \int_{\mathbb{R}} \frac{d\mu(t)}{(t-z)(t-w^*)} \qquad (4)$$

is positive in the sense of reproducing kernels in $\mathbb{C}_+$. (We recall that a function $K(z,w)$ is positive in a set Ω if for every choice of integers $p \in \mathbb{N}$ and $w_1, \ldots, w_p \in \Omega$, the $p \times p$ block matrix with ij block entry $K(w_j, w_i)$ is nonnegative). Note that extending the function N to the lower half plane via (3), that is, via $N(z^*)^* = N(z)$, the kernel (4) is positive in $\mathbb{C} \setminus \mathbb{R}$.

Let $k > 0$. The purpose of this paper is to investigate the classes $\mathcal{F}^k(\mathbb{C}_+)$ and $\mathcal{F}^k(\mathbb{C} \setminus \mathbb{R})$ of $\mathbb{C}^{n\times n}$–valued functions N analytic the open upper half–plane $\mathbb{C}_+$ or in $\mathbb{C} \setminus \mathbb{R}$ and such that

$$K_N(w,z) = \frac{N(z) - N(w)^*}{z - w^*} - \frac{N(w)^* N(z)}{k} \qquad (5)$$

is positive in the sense of reproducing kernels in $\mathbb{C}_+$ or in $\mathbb{C}\setminus\mathbb{R}$ respectively. Such functions N are in particular Herglotz–Nevanlinna functions.

Remark 1. Dividing both sides of (5) by k we see that the map $N \mapsto N/k$ is a bijection between $\mathcal{F}^k$ and $\mathcal{F}^1$ and we will focus on the case $k = 1$.

Elements of $\mathcal{F}^1$ appear as *generalized resolvents* of closed symmetric relations in Hilbert space; see [12].

2. $\mathcal{L}(N)$ spaces and a representation theorem for elements in $\mathcal{F}^k(\mathbb{C} \setminus \mathbb{R})$.

Some of our analysis relies on the theory of reproducing kernel spaces; see e.g. [17]. We first recall:

Theorem 1. *Associated to a positive $\mathbb{C}^{n\times n}$-valued function is a uniquely defined Hilbert space of $\mathbb{C}^{1\times n}$-valued functions with reproducing kernel $K(w,z)$, which we will denote $H(K)$, and with the following two properties:*

1. *For every $w\in\mathbb{C}_+$ and $\xi\in\mathbb{C}^{1\times n}$, the function $z\mapsto \xi K(w,z)$ belongs to $H(K)$.*
2. *For every $f\in H(K)$ and w,ξ as above*

$$\langle f,\xi K(w,\cdot)\rangle_{H(K)}=f(w)\xi^*.$$

A function $F:\Omega\to\mathbb{C}^n$ is in $H(K)$ with norm $\|F\|_{H(K)}\le m$ if and only if the kernel

$$K(w,z)-\frac{F(w)^*F(z)}{m^2}$$

is positive in Ω.

L. de Branges introduced (see [10]) reproducing kernel Hilbert spaces associated to Herglotz–Nevanlinna functions (more precisely, he considered the case of functions with positive real part). We first recall that every Herglotz–Nevanlinna function admits a representation of the form (3). Furthermore the linear span in of the elements of the form $\xi/(t-w^*)$, where ξ runs through $\mathbb{C}^{1\times n}$ and w runs through $\mathbb{C}\setminus\mathbb{R}$, is dense in the Lebesgue space $\mathbf{L}_2^{1\times n}(d\mu)$.

Theorem 2. *Let $N(z)$ be a Herglotz–Nevanlinna matrix-valued function with representation (3). Then the kernel (4) is positive in $\mathbb{C}_+$. The associated reproducing kernel Hilbert space $\mathcal{L}(N)$ is*

$$\mathcal{L}(N)=\left\{F(z)=Bc+\int_{\mathbb{R}}\frac{f(t)d\mu(t)}{t-z};\quad c\in\mathbb{C}^n,\ f\in\mathbf{L}_2^{1\times n}(d\mu)\right\}$$

with norm

$$\|F\|^2_{\mathcal{L}(N)}=c^*Bc+\|f\|^2_{\mathbf{L}_2^{1\times n}(d\mu)}. \tag{6}$$

With this theorem and Theorem 1, we obtain the following characterization of elements in $\mathcal{F}^k$:

Theorem 3. *Let N be a Herglotz–Nevanlinna matrix-valued function with representation (3). Then, N belongs to $\mathcal{F}^k$ if and only if*

$$N(z)=\int_{\mathbb{R}}\frac{d\mu(t)}{t-z}. \tag{7}$$

with

$$\int_{\mathbb{R}}d\mu(t)\le kI_n. \tag{8}$$

Proof. Let $\xi\in\mathbb{C}^{1\times n}$ of modulus 1. Then, $\xi^*\xi\le I_n$ and in the sense of reproducing kernels we have $N(w)^*\xi^*\xi N(z)\le N(w)^*N(z)$. It follows that the function

$$\frac{N(z)-N(w)^*}{z-w^*}-\frac{N(w)^*\xi^*\xi N(z)}{k}=$$
$$=\frac{N(z)-N(w)^*}{z-w^*}-\frac{N(w)^*N(z)}{k}+\frac{N(w)^*N(z)-N(w)^*\xi^*\xi N(z)}{k}$$

is positive in the open upper half–plane. By Theorem 1 the function $z \mapsto \xi N(z)$ belongs to $\mathcal{L}(N)$ and has norm $\|\xi N\|_{\mathcal{L}(N)} \leq \sqrt{k}$. Thus there exist $c \in \mathbb{C}^{1\times n}$ and $f \in \mathbf{L}_2^{1\times n}(d\mu)$ such that

$$\xi N(z) = cB + \int_{\mathbb{R}} \frac{f(t)d\mu(t)}{t-z}. \tag{9}$$

Applying the operator $R_aF(z) = \frac{F(z)-F(a)}{z-a}$ with some fixed $a \in \mathbb{C}\setminus\mathbb{R}$ on both sides of this equation we obtain

$$\xi B + \int_{\mathbb{R}} \frac{\xi d\mu(t)}{(t-z)(t-a)} = \int_{\mathbb{R}} \frac{f(t)d\mu(t)}{(t-z)(t-a)}.$$

It follows that $\xi B = 0$ and that $\frac{\xi}{t-a} = \frac{f(t)}{t-a}$. Hence, $f(t) = \xi$. Since this holds for every vector $\xi \in \mathbb{C}^{1\times n}$ we deduce that $d\mu$ is summable and from (9) that $B = 0$. From (9) we now obtain

$$\int_{\mathbb{R}} \frac{\xi d\mu(t)}{t-z} + \xi\left(A - \int_{\mathbb{R}} \frac{td\mu(t)}{t^2+1}\right) = \int_{\mathbb{R}} \frac{\xi d\mu(t)}{t-z}.$$

Thus, $\xi\left(A - \int_{\mathbb{R}} \frac{td\mu(t)}{t^2+1}\right) = 0$ for all ξ and so $\left(A - \int_{\mathbb{R}} \frac{td\mu(t)}{t^2+1}\right) = 0$, and N is of the form (7). By (6) and (7), $\|\xi N\|_{\mathcal{L}(N)}\| = \|\xi\|_{\mathbf{L}_2^{1\times n}(d\mu)}$ we get $\int_{\mathbb{R}} \xi d\mu(t)\xi^* \leq k$. Since this inequality holds for all ξ of modulus 1, we conclude that (8) holds.

Conversely, assume that (7) and (8) are in force. Then for every $\xi \in \mathbb{C}^{1\times n}$, the function $z \mapsto \frac{\xi N(z)}{\sqrt{k}}$ belongs to $\mathcal{L}(N)$ and its norm is less than 1. Thus the operator T of multiplication by N on the right is a contraction from $\mathbb{C}^{1\times n}$ into $\mathcal{L}(N)$. Its adjoint is given by $T^*(\xi\frac{N(z)-N(w)^*}{z-w^*}) = \frac{\xi N(w)^*}{\sqrt{k}}$ and is also a contraction. We obtain the positivity of the kernel (5) from the positivity of the operator $I - TT^*$ on $\mathcal{L}(N)$. □

Condition (8) suggests links with the Hamburger moment problem (see [2] for the scalar case).

Remark 2. Herglotz–Nevanlinna matrix–valued functions of the form (7) are characterized in [1, Theorem 3, p. 222] by the condition $\sup_{y>0}\|yN(iy)\| < \infty$. The proof there is in the setting of scalar–valued functions but goes through for matrix-valued functions.

3. **Representation of functions in $\mathcal{F}^k(\mathbb{C}_+)$.**

We now focus on functions defined on $\mathbb{C}_+$ (as opposed to $\mathbb{C}\setminus\mathbb{R}$ in the previous section) and obtain a representation of elements in $\mathcal{F}^k(\mathbb{C}_+)$. This representation will be used to solve the two–sided interpolation problem in these classes. We first recall that a Schur function is a $\mathbb{C}^{n\times n}$–valued function analytic and contractive in the open unit disk.

Theorem 4. *Let $k > 0$. A matrix–valued function N analytic in the open upper half plane is in the class $\mathcal{F}^k(\mathbb{C}_+)$ if and only if it can be written as*

$$N(z) = k(I_n - S(z))(-(i+z)I_n + S(z)(-i+z))^{-1} \tag{10}$$

where S is a Schur function.

Proof of Theorem 4. Let $A(z) = \frac{1}{\sqrt{2}} \cdot (kI_n + (z-i)N(z))$ and $B(z) = \frac{1}{\sqrt{2}} \cdot (kI_n + (z+i)N(z))$. Then,

$$\frac{N(z) - N(w)^*}{z - w^*} - \frac{1}{k}N(w)^*N(z) = \frac{A(w)^*A(z) - B(w)^*B(z)}{-i(z-w^*)k} \geq 0,$$

Since $A(i) = \frac{k}{\sqrt{2}}I_n$, we have $\det A(z) \not\equiv 0$ and $S(z) = B(z)A(z)^{-1}$ is a Schur function. We have

$$(S(z)(z-i) - (i+z)I_n)\, N(z) = k(I_n - S(z)).$$

Since we have $|\frac{-i+z}{i+z}| < 1$ for z in the open upper half–plane we obtain (10). □

It follows from the representation (10) that a bitangential interpolation on N translates into a bitangential interpolation on S. Since this last problem has a well known solution, we can solve the bitangential interpolation problem in the classes $\mathcal{F}^k(\mathbb{C})$.

4. Connections between $\mathcal{F}^k(\mathbb{C}_+)$ and $\mathcal{F}^k(\mathbb{C}\setminus\mathbb{R})$. If a function N belongs to $\mathcal{F}^k(\mathbb{C}\setminus\mathbb{R})$, it is clear that its restriction to the open upper–half plane belongs to $\mathcal{F}^k(\mathbb{C}_+)$. We now prove the converse. If $N \in \mathcal{F}^k(\mathbb{C}_+)$, it can be extended to $\mathcal{F}^k(\mathbb{C}\setminus\mathbb{R})$.

Proposition 1. *Let S be a matrix-valued Schur function. Then the Herglotz–Nevanlinna function N defined by* (10) *satisfies*

$$\sup_{y>0} \|yN(iy)\| < 1. \tag{11}$$

Proof. We first assume that det $(I_n - S(z)) \not\equiv 0$. Then we can rewrite (10) as

$$N(z) = -k\left(z + i(I_n + S(z))(I_n - S(z))^{-1}\right)^{-1}.$$

Thus, $y\ N(iy) = ik\left(I_n + \frac{1}{y}(I_n + S(iy))(I_n - S(iy))^{-1}\right)^{-1}$. The matrix

$$\frac{1}{y}(I_n + S(iy))(I_n - S(iy))^{-1}$$

has a positive real part and therefore $\|(I_n + \frac{1}{y}(I_n + S(iy))(I_n - S(iy))^{-1})^{-1}\| \leq 1$. Thus (11) holds for such S. To prove the result in general, we replace S by ϵS with $0 < \epsilon < 1$ and let ϵ goes to 0. Since the bound (11) is independent of ϵ, the result is proved. □

Theorem 5. *A Herglotz–Nevanlinna matrix–valued function N of the form* (10) *can be written in the form* (7). *Assume that $\int_{\mathbb{R}} d\mu(t) \leq kI_n$. Then N belongs to the class $\mathcal{F}^k(\mathbb{C}\setminus\mathbb{R})$.*

Proof. The first claim is a direct application of Proposition 1 and of the characterization of functions of the form (8) proved in [1, Theorem 3 p. 222], and which we recalled in Remark 2. If now N is of the form (8), we have that $z \mapsto \xi N(z) \in \mathcal{L}(N)$ for every ξ and

$$\|\xi N\|^2_{\mathcal{L}(N)} = \int_{\mathbb{R}} \xi d\mu(t)\xi^*.$$

Thus the operator T defined in the proof of Theorem 3, and we conclude as in the proof of Theorem 3. □

From now on, we write $\mathcal{F}^k$ for either $\mathcal{F}^k(\mathbb{C}_+)$ or $\mathcal{F}^k(\mathbb{C}\setminus\mathbb{R})$.

5. **Bitangential interpolation for Schur functions.** This section is of a review nature. We set

$$J_0 = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}. \tag{12}$$

We follow [7] and introduce the bitangential interpolation problem for Schur functions.

Problem 1. *Given an ordered collection*

$$\Omega = \{C_{+s},\ C_{-s},\ A_{\pi s},\ A_{\zeta s},\ B_{+s},\ B_{-s},\ \Gamma_s\}$$

of seven matrices $C_{+s},\ C_{-s} \in \mathbb{C}^{m\times n_\pi}$, $B_{+s},\ B_{-s} \in \mathbb{C}^{n_\zeta\times m}$, $A_{\pi s} \in \mathbb{C}^{n_\pi\times n_\pi}$, $A_{\zeta s} \in \mathbb{C}^{n_\zeta\times n_\zeta}$ *and* $\Gamma_s \in \mathbb{C}^{n_\zeta\times n_\pi}$, *the pair* $(C_{-s}, A_{\pi s})$ *being assumed observable,*

$$\cap_{k=0}^{\infty} \ker\ C_{-s}A_{\pi s}^k = \{0\},$$

and the pair $(A_{\zeta s}, B_{+s})$ *being controllable,*

$$\cup_{k=0}^{\infty} \mathrm{Im}\ A_{\zeta s}^k B_{+s} = \mathbb{C}^{n_\zeta},$$

find all Schur functions S *such that*

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ (zI_{n_\zeta} - A_{\zeta s})^{-1}B_{+s}S(z) = -B_{-s}, \tag{13}$$

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ S(z)C_{-s}(zI_{n_\pi} - A_{\pi s})^{-1} = C_{+s}, \tag{14}$$

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ (zI_{n_\zeta} - A_{\zeta s})^{-1}B_{+s}S(z)C_{-s}(zI_{n_\pi} - A_{\pi s})^{-1} = \Gamma_s. \tag{15}$$

Remark that the matrix Γ_s satisfies the Sylvester equation

$$\Gamma_s A_{\pi s} - A_{\zeta s}\Gamma_s = B_{+s}C_{+s} + B_{-s}C_{-s}.$$

We note that the bitangential interpolation problem (13)–(15) can be considered in different classes than the Schur class. In the next section we study it in $\mathcal{F}^1$.

Theorem 6. *The bitangential interpolation problem* (13)–(15) *has a solution in the Schur class if and only if the matrix*

$$\Lambda_s = \begin{pmatrix} S_1 & \Gamma_s^* \\ \Gamma_s & S_2 \end{pmatrix}$$

is nonnegative, where

$$\begin{aligned} S_1A_{\pi s} - A_{\pi s}^*S_1 &= i(C_{+s}^*C_{+s} - C_{-s}^*C_{-s}), \\ S_2A_{\zeta s}^* - A_{\zeta s}S_2 &= i(B_{-s}B_{-s}^* - B_{+s}B_{+s}^*). \end{aligned}$$

When $\Lambda_s > 0$, *the set of all solutions is described as follows: set*

$$\begin{aligned} \Theta_s(z) &= I_{2n} + \begin{pmatrix} C_{+s} & -iB_{-s}^* \\ C_{-s} & iB_{-s}^* \end{pmatrix}\begin{pmatrix} (zI - A_{\pi s})^{-1} & 0 \\ 0 & (zI - A_{\zeta s}^*)^{-1} \end{pmatrix}\times \\ &\quad \times\Lambda_s^{-1}\begin{pmatrix} -iC_{+s}^* & iC_{-s}^*. \\ B_{+s} & B_{-s} \end{pmatrix}. \end{aligned} \tag{16}$$

The function Θ_s is J_0–inner. Let V be an arbitrary J_0–unitary constant and let $\Omega_s(z) = \Theta_s(z)V$. Then, the function Ω_s is also J_0–inner and the linear fractional transformation

$$S(z) = (\Omega_{11}(z)G(z) + \Omega_{12}(z))(\Omega_{21}(z)G(z) + \Omega_{22}(z))^{-1}$$

describes the set of all Schur functions which are solution of the interpolation problem (13)–(15) *when G runs through the set of all Schur functions.*

See [7] and [3, Theorem 4.4 p. 512].

6. **Bitangential interpolation in $\mathcal{F}^1$.** In this section we consider the two–sided residue problem 1, where now the unknown is in $\mathcal{F}^1$. To that purpose, we show that the function $S(z)$ defined by (10) satisfies another two–sided interpolation problem. The case of an unknown in $\mathcal{F}^k$ is easily adapted using Remark 1.

Proposition 2. *Let N be a solution of the bitangential interpolation problem* 1, *and assume N is in $\mathcal{F}^1$ and of the form* (10). *Then, S is a solution of the bitangential interpolation problem*

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ (zI_{n_\zeta} - A_\zeta)^{-1}\mathbf{B}_+S(z) = -\mathbf{B}_- \tag{17}$$

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ S(z)\mathbf{C}_-(zI_{n_\pi} - A_\pi)^{-1} = \mathbf{C}_+ \tag{18}$$

and

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ (zI_{n_\zeta} - A_\zeta)^{-1}\mathbf{B}_+S(z)\mathbf{C}_- = \widetilde{\Gamma}$$

where we have set $\widetilde{\Gamma} = 2i(\Gamma + B_-C_+)$ and

$$\begin{aligned}\mathbf{B}_+ &= B_+ + (iI_{n_\zeta} - A_\zeta)B_-, & \mathbf{B}_- &= -(B_+ - (iI_{n_\zeta} + A_\zeta)B_-),\\ \mathbf{C}_+ &= C_- + C_+(iI_{n_\pi} + A_\pi), & \mathbf{C}_- &= (C_- + C_+(-iI_{n_\pi} + A_\pi)).\end{aligned}$$

As in our previous paper [6], we will need the following preliminary lemma:

Lemma 1. *With the notation of Problem* 1, *let F_1 and F_2 be rational matrix–valued functions of appropriate sizes. Then, it holds that*

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ (zI_{n_\zeta} - A_\zeta)^{-1}F_1(z)F_2(z) = \sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ (zI_\zeta - A_\zeta)^{-1}X_{F_1}F_2(z), \tag{19}$$

where

$$X_{F_1} = \sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ (zI_{n_\zeta} - A_\zeta)^{-1}F_1(z) \tag{20}$$

and

$$\sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ F_1(z)F_2(z)(zI_{n_\zeta} - A_\pi)^{-1} = \sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ F_1(z)Y_{F_2}(zI_{n_\pi} - A_\pi)^{-1},$$

where

$$Y_{F_2} = \sum_{\omega\in\mathbb{C}_+} \mathrm{Res}_{z=\omega}\ F_2(z)(zI_{n_\pi} - A_\pi)^{-1} \tag{21}$$

Proof of Proposition 2. We first prove (17). Equation (10) can be rewritten as

$$(-iN(z)+I_n+zN(z))S(z)=iN(z)+I_n+zN(z).$$

Writing $zI_{n_\zeta}=zI_{n_\zeta}-A_\zeta+A_\zeta$ we have:

$$\begin{aligned}
&\sum_{\omega\in\mathbb{C}_+}\mathrm{Res}_{z=\omega}\ (zI_{n_\zeta}-A_\zeta)^{-1}B_+zN(z)=\\
&=\sum_{\omega\in\mathbb{C}_+}\mathrm{Res}_{z=\omega}\ (zI_{n_\zeta}-A_\zeta)^{-1}A_\zeta B_+N(z)+\sum_{\omega\in\mathbb{C}_+}\mathrm{Res}_{z=\omega}\ B_+N(z)\\
&\qquad=\sum_{\omega\in\mathbb{C}_+}\mathrm{Res}_{z=\omega}\ (zI_{n_\zeta}-A_\zeta)^{-1}A_\zeta B_+N(z)=-A_\zeta B_-.
\end{aligned}$$

Using (19) and writing Re_ω instead of $\mathrm{Res}_{z=\omega}$, we have

$$\begin{aligned}
\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}B_+N(z)S(z) &= \sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}X_{B_+N}S(z)\\
&= \sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}(-B_-)S(z)
\end{aligned}$$

since by (20) we have $X_{B_+N}=\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}B_+N(z)=-B_-$. Similarly,

$$\begin{aligned}
&\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}(B_+N(z)zS(z))=\\
&\qquad=\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}(-A_\zeta B_-)S(z).
\end{aligned}$$

(17) is easily deduced from these equations.

Making use of (21), the interpolation condition (18) is proved in much the same way, and we turn to the third condition. The computations are admitedly cumbersome. We start from the equation

$$S(z)\,(-iN(z)+I_n+zN(z))=iN(z)+I_n+zN(z)$$

and multiply it on the left by $(zI_{n_\zeta}-A_\zeta)^{-1}\mathbf{B}_+$ and on the right by $(zI_{n_\pi}-A_\pi)^{-1}C_-$ and take the sum on all the residues in the open upper half plane and obtain

$$\mathbf{1}+\mathbf{2}=\mathbf{3}+\mathbf{4} \tag{22}$$

where

$$\begin{aligned}
\mathbf{1} &= \sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}\mathbf{B}_+S(z)C_-(zI_{n_\pi}-A_\pi)^{-1}\\
\mathbf{2} &= \sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}(-i+z)\mathbf{B}_+S(z)N(z)C_-(zI_{n_\pi}-A_\pi)^{-1}\\
\mathbf{3} &= \sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI-A_\zeta)^{-1}\mathbf{B}_+C_-(zI_{n_\pi}-A_\pi)^{-1}\\
\mathbf{4} &= \sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}(i+z)\mathbf{B}_+N(z)C_-(zI_{n_\pi}-A_\pi)^{-1}.
\end{aligned}$$

The term **3** is clearly equal to 0. We now evaluate **4**; writing

$$(i+z)(zI-A_\zeta)^{-1}=(i+A_\zeta)(zI_{n_\zeta}-A_\zeta)^{-1}+(iI_{n_\zeta}+A_\zeta),$$

we obtain

$$\begin{aligned}\mathbf{4} &= \sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ \mathbf{B}_+N(z)C_-(zI_{n_\pi}-A_\pi)^{-1}+\\ &\quad +(iI_{n_\zeta}+A_\zeta)\cdot\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}\mathbf{B}_+N(z)C_-(zI-A_\pi)^{-1}\\ &= \mathbf{B}_+C_+ +(iI_{n_\zeta}+A_\zeta)\Gamma+\\ &\quad +(iI_{n_\zeta}+A_\zeta)(iI_{n_\zeta}-A_\zeta)\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ B_-N(z)C_-(zI_{n_\pi}-A_\pi)^{-1}.\end{aligned}$$

We note that the last term cannot be expressed in general only in terms of the interpolation data. We now compute **2**:

$$\begin{aligned}\mathbf{2} &= -\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}(-i+z)\mathbf{B}_-(N(z)C_--C_+)(zI_{n_\pi}-A_\pi)^{-1}+\\ &\quad +\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}\mathbf{B}_+S(z)C_+(zI_{n_\pi}-A_\pi)^{-1}(-i+z)\\ &= -\mathbf{2.1}+\mathbf{2.2}.\end{aligned}$$

Furthermore,

$$\begin{aligned}-\mathbf{2.1} &= \sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}(-iI_{n-\zeta}+A_\zeta+(zI_{n_\zeta}-A_\zeta))\mathbf{B}_-\times\\ &\quad \times(N(z)C_--C_+)(zI_{n_\pi}-A_\pi)^{-1}\\ &= \sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ \mathbf{B}_-(N(z)C_--C_+)(zI_{n-\pi}-A_\pi)^{-1}+\\ &\quad +\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ \big((-iI_{n_\zeta}+A_\zeta)(zI_{n_\zeta}-A_\zeta)^{-1}(-B_++(iI_{n_\zeta}+A_\zeta)B_-)\times\\ &\quad \times(N(z)C_--C_+)(zI_{n_\pi}-A_\pi)^{-1}\big)\\ &= (-iI_{n_\zeta}+A_\zeta)\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ \big((zI_{n_\zeta}-A_\zeta)^{-1}(-B_++(iI_{n_\zeta}+A_\zeta)B_-)\times\\ &\quad \times(N(z)C_--C_+)(zI_{n_\pi}-A_\pi)^{-1}\big)\\ &= (-iI_{n_\zeta}+A_\zeta)(-\Gamma)+\\ &\quad +(-iI_{n_\zeta}+A_\zeta)(iI_{n_\zeta}+A_\zeta)\sum_{\omega\in\mathbb{C}_+}\mathrm{Re}_\omega\ B_-N(z)C_-(zI_{n_\pi}-A_\pi)^{-1}.\end{aligned}$$

Since $(-iI_{n_\pi}+A_\pi+(zI_{n_\pi}-A_\pi))=(-i+z)I_n$ we have:

$$
\begin{aligned}
\mathbf{2.2} &= \sum_{\omega\in\mathbb{C}_+} \mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}\mathbf{B}_+S(z)C_+ + \\
&\quad + \sum_{\omega\in\mathbb{C}_+} \mathrm{Re}_\omega\ (zI_{n-\zeta}-A_\zeta)\mathbf{B}_+S(z)C_+(zI_{n_\pi}-A_\pi)^{-1}(-iI_{n_\pi}+A_\pi) \\
&= -\mathbf{B}_-C_+ + \\
&\quad + \sum_{\omega\in\mathbb{C}_+} \mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)\mathbf{B}_+S(z)C_+(zI_{n_\pi}-A_\pi)^{-1}(-iI_{n_\pi}+A_\pi).
\end{aligned}
$$

Using this expression in the above formula for **2**, we obtain:

$$
\begin{aligned}
\mathbf{2} &= \sum_{\omega\in\mathbb{C}_+} \mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)\mathbf{B}_+S(z)C_+(zI_{n_\pi}-A_\pi)^{-1}(-iI_{n_\pi}+A_\pi) + \\
&\quad +(-iI_{n_\zeta}+A_\zeta)\Gamma - \mathbf{B}_-C_+ + \\
&\quad -(-iI_{n_\zeta}+A_\zeta)(iI_{n_\zeta}+A_\zeta)\sum_{\omega\in\mathbb{C}_+} \mathrm{Re}_\omega\ B_-N(z)C_-(zI_{n_\pi}-A_\pi)^{-1}.
\end{aligned}
$$

We now replace these various expressions in (22); the term

$$
(iI_{n_\zeta}+A_\zeta)(iI_{n_\zeta}-A_\zeta)\sum_{\omega\in\mathbb{C}_+} \mathrm{Re}_\omega\ (zI_{n_\zeta}-A_\zeta)^{-1}B_-N(z)C_-(zI_{n_\pi}-A_\pi)^{-1}
$$

appears then on both sides of (22) and is therefore cancelled, and we obtain the third interpolation condition for S. □

Using Theorem 6, we can now solve the bitangential problem in the class $\mathcal{F}^1$.

Theorem 7. *Let S_1 and S_2 be the solutions of the Lyapunov equations*

$$
\begin{aligned}
S_1A_\pi - A_\pi S_1^* &= 2\left((C_+^*C_- - C_-^*C_+) + \left(A_\pi^*C_+^*C_+ - C_+^*C_+A_\pi\right)\right) \\
S_2A_\zeta^* - S_2A_\zeta &= 2\left((B_-B_+^* - B_+B_-^*) + \left(A_\pi B_-B_-^* - B_-B_-^*A_\pi^*\right)\right)
\end{aligned}
$$

and let Λ be defined by

$$
\Lambda = \begin{pmatrix} S_1 & \widetilde{\Gamma}^* \\ \widetilde{\Gamma} & S_2 \end{pmatrix}.
$$

There is a solution to the bitangential interpolation problem in the class $\mathcal{F}^1$ if and only if $\Lambda \geq 0$. When $\Lambda > 0$, the set of all solutions is described as follows: build Θ_s via formula (16) with

$$
B_{+s} = \mathbf{B}_+,\quad B_{-s} = \mathbf{B}_-,\quad C_{-s} = \mathbf{C}_-,\quad C_{+s} = \mathbf{C}_+,\quad \Gamma_s = \widetilde{\Gamma}.
$$

Let $P(z)$ be defined by

$$
P(z) = \begin{pmatrix} -I_n & I_n \\ (-i+z)I_n & -(i+z)I_n \end{pmatrix} \tag{23}
$$

Then, the linear fractional transformation $N(z) = T_{P(z)\Theta_s(z)}(G(z))$ describes the set of all solutions to the interpolation problem in $\mathcal{F}^1$ when G runs through the set of Schur functions.

In the above representation, the parameter and the solution do not vary in the same class. This is easily remedied as follows:

$$N(z) = T_{P(z)}(S(z)) = T_{P(z)\Theta(z)}(G(z)) = T_{P(z)\Theta(z)P(z)^{-1}}\left(T_{P(z)}(G(z))\right). \qquad (24)$$

In view of Theorem 4 and representation (10) we see that in (24) the parameter and the solution are in the same class. Furthermore, the coefficient matrix function is J_1–inner with

$$J_1 = \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix}. \qquad (25)$$

7. Some automorphisms of rational J–unitary functions. This section is written in the spirit of [4]. It is not needed for the understanding of the paper, but we think it is of independent interest, as a study using state space methods of an automorphism of J–unitary rational function for a signature matrix J (i.e. a matrix J such that $J = J^* = J^{-1}$; here J will be J_0 or J_1).

Equation (24) suggests to study the automorphism of rational function

$$\mathcal{A}(W)(z) = P(z)W(z)P(z)^{-1}.$$

Proposition 3. *The automorphism $\mathcal{A}$ sends the set of J_0–unitary rational matrix-valued functions into the set of J_1–unitary rational matrix–valued functions.*

Proof. With P as in (23) and J_0 and J_1 defined by (12) and (25) we have $P(z)J_0P(z)^* = 2\cdot J_1$ for real z. Thus

$$P(z)^{-1}J_1P(z)^{-*} = \frac{1}{2}\cdot J_0, \qquad z\in\mathbb{R},$$

and for a J_0–unitary rational function W we have

$$\begin{aligned} P(z)W(z)P(z)^{-1}J_0P(z)^{-*}W(w)^*P(z)^* &= \frac{1}{2}P(z)W(z)J_0W(z)^*P(z)^* \\ &= \frac{1}{2}P(z)J_0P(z)^* = J_1. \end{aligned}$$

□

Let $M = \frac{1}{\sqrt{2}}\begin{pmatrix} -I_n & I_n \\ -iI_n & -iI_n \end{pmatrix}$. It is useful to note for the sequel that $MJ_0M^* = J_1$ and $Q(z)\cdot M = P(z)$, where $Q(z) = \begin{pmatrix} I_n & 0 \\ -z & I_n \end{pmatrix}$.

Proposition 4. *Let $Q(z) = \begin{pmatrix} I_n & 0 \\ -z & I_n \end{pmatrix}$ Then, $Q(z)J_1Q(z)^* = J_1$ for $z\in\mathbb{R}$ and the automorphism $\mathcal{B}$ defined by $\mathcal{B}(W)(z) = Q(z)W(z)Q(z)^{-1}$ preserves the set of J_1–unitary rational matrices.*

It will be easier to study the automorphism $\mathcal{B}$. We first need to recall a result on the theory of minimal realization and factorization of J–unitary rational functions, taken from [5] (and specialized to the present choice of signature matrix J):

Theorem 8. *Let W be a $\mathbb{C}^{2n\times 2n}$–valued rational function analytic at infinity and let $W(z) = D + C(zI_m - A)^{-1}B$ be a minimal realization of W. The function W*

takes J-unitary values on the real line if and only if the following conditions hold: D is J-unitary and there exists an hermitian and invertible matrix $\mathbb{P}$ such that

$$A^*\mathbb{P} - \mathbb{P}A = C_1^*C_2 - C_2^*C_1 \quad \text{and} \quad B = \mathbb{P}^{-1}(C_2^* - C_1^*)D.$$

The hermitian matrix $\mathbb{P}$ is called the *associated hermitian matrix* to the given minimal realization.

Theorem 9. *Let Θ be a rational J_1-unitary on the line rational function analytic at infinity, and let*

$$\Theta(z) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (zI - A)^{-1}(B_1 \ \ B_2)$$

be a minimal realization of Θ with associated matrix $\mathbb{H}$. Then, the matrix-valued function $P(z)^{-1}\Theta(z)P(z)$ is analytic at infinity if and only $D_{12} = 0$ and

$$D_{22} = D_{11} + C_1B_2. \tag{26}$$

When these conditions are in force, a minimal realization of $Q(z)\Theta(z)Q(z)^{-1}$ is given by

$$Q(z)\Theta(z)Q(z)^{-1} = $$
$$= \begin{pmatrix} D_{22} & 0 \\ X & D_{11} \end{pmatrix} + \begin{pmatrix} C_1 \\ -C_1A + C_2 \end{pmatrix} (zI - A)^{-1}(B_1 + AB_2 \ \ B_2),$$

where

$$X = D_{21} + (C_2B_2 - C_1B_1 - C_1AB_2). \tag{27}$$

The associated hermitian matrix associated to this minimal realization is given by

$$\mathbb{H}^{\triangle} = \mathbb{H} + C_1^*C_1.$$

Proof. The block entries of $Q(z)\Theta(z)Q(z)^{-1}$ are given by

$$(Q(z)\Theta(z)Q(z)^{-1})_{11} = D_{11} + zD_{12} + C_1(zI - A)^{-1}(B_1 + zB_2) \tag{28}$$
$$(Q(z)\Theta(z)Q(z)^{-1})_{12} = D_{12} + C_1(zI - A)^{-1}B_2$$
$$(Q(z)\Theta(z)Q(z)^{-1})_{21} = \tag{29}$$
$$= -z(D_{11} + zD_{12}) + D_{21} + zD_{22} + $$
$$+(-zC_1 + C_2)(zI - A)^{-1}(B_1 + zB_2)$$
$$(Q(z)\Theta(z)Q(z)^{-1})_{22} = (-zD_{12} + D_{22}) + $$
$$+(-zC_1 + C_2)(zI - A)^{-1}B_2.$$

The right side of (28) can be rewritten as $zD_{12} + D_{11} + C_1B_2 + C_1(zI - A)^{-1}(B_1 + AB_2)$, and the analyticity at infinity forces $D_{12} = 0$. An easy computation shows then that the right side of (29) is equal to

$$z(D_{22} - D_{11} - C_1B_2) + X + (-C_1A + C_2)(zI - A)^{-1}(B_1 + AB_2)$$

where X is defined by (27). Analyticity at infinity forces (26).

To prove that $\mathbb{H}^{\triangle}$ is the corresponding associated matrix we have to check that

$$A^*\mathbb{H}^{\triangle} - \mathbb{H}^{\triangle}A = C_1^*(-C_1A + C_2) - (-C_1A + C_2)^*C_1 \tag{30}$$

$$\mathbb{H}^{\triangle}(B_1 + AB_2 \ \ B_2) = (C_2^* - A^*C_1^*, \ -C_1^*)\begin{pmatrix} D_{22} & 0 \\ X & D_{11} \end{pmatrix}. \tag{31}$$

(30) is trivial to check. We now turn to the second block identity in (31), which can be rewritten as $(\mathbb{H} + C_1^* C_1)B_2 = -C_1^* D_{11}$. But $\mathbb{H}B_2 = -C_1^* D_{22}$ and thus we have to prove that $C_1^*(-D_{22} + D_{11} + C_1 B_2) = 0$, but this is plain from (26). In the same way, one can verify the first block in (31). □

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