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STOCHASTIC BEHAVIOR OF ASYMPTOTICALLY EXPANDING MAPS

José F. Alves

Department of Pure Mathematics
University of Porto
4099-002 Porto, Portugal

1. **Introduction.** In these notes we present some recent results about the stochastic stability of maps exhibiting some asymptotically expanding behavior. This is part of the joint work [2] with V. Araújo, mostly motivated by the previous work [3] together with C. Bonatti and M. Viana.

We consider the setting of non-uniformly expanding maps introduced in [3], and give both sufficient conditions and necessary conditions for the stability of the statistical properties of those maps under small perturbations.

As an application of these results we prove the stochastic stability of the non-uniformly expanding maps introduced in [17].

2. **SRB measures.** Let $f : M \to M$ be a smooth map defined on a finite dimensional compact Riemannian manifold M, and fix some normalized volume form m on M that we call *Lebesgue measure*. If μ is a probability measure defined on the Borel subsets of M that is invariant by f, meaning that $\mu(f^{-1}(A)) = \mu(A)$ for every Borel set $A \subset M$, we define $B(\mu)$ the *basin of* μ as

$$B(\mu) = \left\{ x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \to \mu \right\}.$$

Here δ_x represents the Dirac measure supported on a point $x \in M$, and the convergence is taken in the weak* topology. So, we have that $x \in B(\mu)$ if and only if

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^n(x)) = \int \varphi \, d\mu$$

for every continuous map $\varphi : M \to \mathbf{R}$. We say that μ is a *Sinai-Ruelle-Bowen measure* (SRB measure, for short), if the basin of μ has positive Lebesgue measure.

Example 2.1. (Uniformly expanding maps) *Let f be an expanding map in a Riemannian manifold M, meaning that there is some constant $\sigma < 1$ for which*

$$\|Df(x)^{-1}\| \le \sigma \quad \text{for all } x \in M.$$

It follows from the work of Ruelle [15] *that there is a unique SRB measure whose basin cover the whole manifold M.*

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Example 2.2. (Quadratic maps) *Consider the family of quadratic transformations* f_a *from the interval* $[-1,1]$ *into itself given by*

$$f_a(x) = 1 - ax^2.$$

Jakobson [12] *showed the existence of a positive Lebesgue measure set of parameters* $a \in (1,2)$ *for which* f_a *exhibits a unique ergodic absolutely continuous invariant measure.*

An easy application of Birkhoff's ergodic theorem shows that the measure obtained in the previous example is an SRB measure. The uniqueness is a consequence of good mixing properties of those maps.

3. **Non-uniform expansion.** The existence of SRB measures was established in [3] for certain classes of maps exhibiting some asymptotically expanding behaviour. We say that f is a *non-uniformly expanding map* if there is some constant $c > 0$ for which

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -c < 0, \tag{3.1}$$

for Lebesgue almost every $x \in M$. In the case of f having a nonempty set $\mathcal{C}$ of critical points, we also assume that the map f *behaves like a power of the distance* close to $\mathcal{C}$,

$$\|Df(x)\| \approx \operatorname{dist}(x,\mathcal{C})^{\beta}$$

for some $\beta > 0$ and x in a neighborhood of $\mathcal{C}$ (see [3] for the precise statement), and orbits have *slow approximation* to $\mathcal{C}$: given small $\gamma > 0$ there is $\delta > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x),\mathcal{C}) \leq \gamma, \tag{3.2}$$

where $\operatorname{dist}_\delta(x,\mathcal{C}) = \operatorname{dist}(x,\mathcal{C})$ if $\operatorname{dist}_\delta(x,\mathcal{C}) < \delta$, and $\operatorname{dist}_\delta(x,\mathcal{C}) = 1$ otherwise. In all that follows we implicitly assume that a non-uniformly expanding map with critical set $\mathcal{C}$ (if it exists) behaves like a power of a distance close to $\mathcal{C}$ and orbits have slow approximation to $\mathcal{C}$.

Theorem 3.1. **[Alves, Bonatti, Viana]** *Let* f *be a* C^2 *non-uniformly expanding map. There is a finite number of SRB measures* $\mu_1,\dots,\mu_p$ *for which*

$$B(\mu_1)\cup\cdots\cup B(\mu_p) = M$$

mod 0 Lebesgue measure. Moreover, every absolutely continuous f*-invariant probability measure* μ *may be written as a convex linear combination of* $\mu_1,\dots,\mu_p$.

A probability measure μ is said to be a *convex linear combination* of $\mu_1,\dots,\mu_p$, if there are $w_1,\dots,w_p \geq 0$ with $w_1+\cdots+w_p = 1$ for which $\mu = w_1\mu_1+\cdots+w_p\mu_p$.

4. **Random perturbations.** We are interested in considering random perturbations of a non-uniformly expanding map f. Roughly speaking, this concept may be formulated in the following terms: assume that instead of iterates $x_j = f^j(x_0)$ of points $x_0 \in M$, we consider perturbed iterates $\tilde{x}_j$, where at each iteration we take $\tilde{x}_{j+1}$ close to $f(\tilde{x}_j)$ with an error bounded by some small $\epsilon > 0$. Two important questions naturally arise in this situation:

Question 1: Do the sequences $\frac{1}{n}\sum_{j=0}^{n-1}\delta_{\tilde{x}_j}$ converge to any asymptotic probability measure μ^ϵ?

Question 2: How are these measures μ^ϵ related with the SRB measures of the system?

Let us now be more precise in the formulation of the random perturbations of the map f. We consider the space $C^2(M,M)$ of C^2 maps from M into itself, and take

1. a continuous map $F : T \ni t \mapsto f_t \in C^2(M,M)$ with $f = f_{t^*}$ for some $t^* \in T$;
2. a family $(\theta_\epsilon)_{\epsilon>0}$ of probability measures on T whose supports approach $\{t^*\}$ when ϵ goes to 0.

We also assume some nondegeneracy conditions on the family $(\theta_\epsilon)_{\epsilon>0}$, which essentially correspond to the absolute continuity with respect to the Lebesgue measure on M of the push-forwards of θ_ϵ via $T \ni t \to f_t(x)$, for each $x \in M$. We will refer to $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ as a *random perturbation* of f.

Example 4.1. *Let $M = \mathbf{R}$ and $f : M \to M$ a C^2 map. Take $T = \mathbf{R}$ and define, for each $\epsilon > 0$, the probability measure*

$$\theta_\epsilon = \frac{1}{2\epsilon}\big(m \mid [-\epsilon, \epsilon]\big).$$

Considering $f = f + t$ for each $t \in T$ we obtain a simple example of a random perturbation of f with $t^* = 0$. As a generalization of this, it was shown in [5, Examples 1 & 2] *that given any smooth map $f : M \to M$ of a compact manifold M we can always construct a (nondegenerate) random perturbation of f, if we take $T = \mathbf{R}^d$, $t^* = 0$ and θ_ϵ equal to the normalized restriction of the Lebesgue measure to the ball of radius ϵ around 0, for sufficiently big number $d \in \mathbf{N}$ of parameters.*

Given $x \in M$ and $\underline{t} = (t_1, t_2, t_3, \dots) \in T^{\mathbf{N}}$, we define

$$f^n_{\underline{t}} = f_{t_n} \circ \cdots \circ f_{t_1}$$

for $n \geq 1$ and call the sequence $(f^n_{\underline{t}}(x))_{n\geq 1}$ a *random orbit* of x. We say that a Borel probability measure μ^ϵ on M is a *physical measure* if, for a positive Lebesgue measure set of points $x \in M$, the time averaged sequence of Dirac probability measures along random orbits $(f^n_{\underline{t}}(x))_{n\geq 0}$ converges in the weak* topology to μ^ϵ, for $\theta_\epsilon^{\mathbf{N}}$ almost every $\underline{t} \in T^{\mathbf{N}}$. That is, for a positive Lebesgue measure set of points $x \in M$ we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f^n_{\underline{t}}(x)\big) = \int \varphi \, d\mu^\epsilon \tag{4.1}$$

for all continuous φ: $M \to \mathbf{R}$ and $\theta_\epsilon^{\mathbf{N}}$ almost every $\underline{t} \in T^{\mathbf{N}}$. In this setting, we define $B(\mu^\epsilon)$, the *basin of* μ^ϵ, as the set of those points $x \in M$ for which (4.1) holds for every continuous φ and $\theta_\epsilon^{\mathbf{N}}$ almost every $\underline{t} \in T^{\mathbf{N}}$.

Remark 4.2. *The systems presented in Examples 2.1 and 2.2 have a unique* physical measure μ^ϵ *that converges to the unique SRB measure of the unperturbed system. See* [13] *and* [14] *for expanding maps. For quadratic maps, see* [9] *for convergence in the weak* topology, and* [6] *for convergence in the L^1 norm.*

The map f is said to be *stochastically stable* if the weak* accumulation points (when $\epsilon > 0$ goes to zero) of the physical measures μ^ϵ are convex linear combinations of the SRB measures of f. We have that the maps of Examples 2.1 and 2.2 are stochastically stable (cf. Remark 4.2).

We are able to prove a result on the finiteness of the number of physical measures for a non-uniformly expanding C^2 map.

Theorem 4.3. **[Alves, Araújo]** *Let $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ be a random perturbation of a non-uniformly expanding C^2 map f. For $\epsilon > 0$ small enough there are physical measures $\mu_1^\epsilon, \dots, \mu_l^\epsilon$ (with l not depending on ϵ) such that*

1. *for each $x \in M$ and $\theta_\epsilon^{\mathbf{N}}$ almost every $\underline{t} \in T^{\mathbf{N}}$ there is some μ_i^ϵ for which*
$$\frac{1}{n}\sum_{j=1}^{n-1}\delta_{f_{\underline{t}}^j(x)} \to \mu_i^\epsilon.$$
2. *for each $i = 1, \dots, l$ we have*
$$\frac{1}{n}\sum_{j=0}^{n-1}\int (f_{\underline{t}}^j)_*(m \mid B(\mu_i^\epsilon))\, d\theta_\epsilon^{\mathbf{N}}(\underline{t}) \to \mu_i^\epsilon.$$

Here $m \mid B(\mu_i^\epsilon)$ denotes the normalized restriction of Lebesgue measure to $B(\mu_i^\epsilon)$ and convergence is always meant in the sense of weak* topology. Since the number of physical measures stabilizes for small enough noise levels $\epsilon > 0$, it is natural to ask if there is any relation between p, the number of SRB measures of f and l, the number of physical measures of the random perturbation. Actually we have $l \le p$, with the possibility of occurring $l < p$ or $l = p$, see [2].

5. **Stochastic stability.** In this section we assume that the critical set of f is equal to the empty set, that is, f is a non-uniformly expanding C^2 a local diffeomorphism. Let $\mu_1^\epsilon, \dots, \mu_l^\epsilon$ be the physical measures given by Theorem 4.3 for small $\epsilon > 0$. Since we are taking f a local diffeomorphism, then $\log \|(Df)^{-1}\|$ is a continuous map. Thus, we have for each $x \in M$ and $\theta_\epsilon^{\mathbf{N}}$ almost every $\underline{t} \in T^{\mathbf{N}}$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df(f_{\underline{t}}^j(x))^{-1}\| = \int \log\|(Df)^{-1}\|d\mu_i^\epsilon,$$

for some physical measure μ_i^ϵ with $1 \le i \le l$. Assuming that f is stochastically stable, then we have a probability measure $\mu_0 = \omega_1\mu_1 + \dots + \omega_p\mu_p$ (possibly depending on ϵ and i) with $\omega_1, \dots, \omega_p \ge 0$ and $\omega_1 + \dots + \omega_p = 1$, such that

$$\int \log\|(Df)^{-1}\|d\mu_i^\epsilon \approx \int \log\|(Df)^{-1}\|d\mu_0 \tag{5.1}$$

(see [2] for a precise statement and proof of this assertion). Since each $\mu_1, \dots \mu_p$ is an SRB measure, then we have for $1 \le k \le p$ and $z \in B(\mu_k)$

$$\int \log\|(Df)^{-1}\|d\mu_k = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df(f^j(z))^{-1}\| \le -c < 0$$

(recall that f is non-uniformly expanding and $B(\mu_k)$ has positive Lebesgue measure). Now, using the fact that μ_0 is a convex linear combination of $\mu_1, \dots, \mu_p$, then we also have

$$\int \log\|(Df)^{-1}\|d\mu_0 \le -c < 0.$$

Putting this last inequality together with (5.1) we deduce that f *is non-uniformly expanding for random orbits*: there is some constant $\hat{c} > 0$ such that for $\epsilon > 0$ small enough we have

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df(f_{\underline{t}}^j(x))^{-1}\| \le -\hat{c} < 0, \tag{5.2}$$

for $\theta_\epsilon^{\mathbf{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbf{N}} \times M$. Thus we have

Theorem 5.1. **[Alves, Araújo]** *Let $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ be a random perturbation of a non-uniformly expanding C^2 local diffeomorphism f. If f is stochastically stable, then f is non-uniformly expanding for random orbits.*

We are also able to prove a kind of (weaker) converse to the theorem above. For that we introduce the following definition: given $0 < \alpha < 1$, we say that $n \in \mathbf{Z}^+$ is a *α-hyperbolic time* for $(\underline{t}, x) \in T^{\mathbf{N}} \times M$ if

$$\prod_{j=n-k}^{n-1} \|Df(f_{\underline{t}}^j(x))^{-1}\| \leq \alpha^k$$

for every $1 \leq k \leq n$. Condition (5.2) permits us to prove that $\theta_\epsilon^{\mathbf{N}} \times m$ almost every point $(\underline{t}, x) \in T^{\mathbf{N}} \times M$ has a α-hyperbolic time (see [2]) and define a map

$$h_\epsilon \colon T^{\mathbf{N}} \times M \to \mathbf{N}$$

by taking $h_\epsilon(\underline{t}, x)$ the first α-hyperbolic time for the point $(\underline{t}, x) \in T^{\mathbf{N}} \times M$. Assuming the integrability of h_ϵ with respect to $\theta_\epsilon^{\mathbf{N}} \times m$, then we have

$$\|h_\epsilon\|_1 = \sum_{k=0}^{\infty} k\,(\theta_\epsilon^{\mathbf{N}} \times m)\big(\{h_\epsilon(\underline{t}, x) = k\}\big) < \infty. \tag{5.3}$$

We say that the family $(h_\epsilon)_{\epsilon>0}$ has *uniform L^1-tail*, if the series in (5.3) converges uniformly to $\|h_\epsilon\|_1$ (as a series of functions on the variable ϵ). The basic properties of hyperbolic times are stated in the result below.

Lemma 5.2. *There are $\delta_1 > 0$ and $C_1 > 0$ such that if n is a α-hyperbolic time for x, then there is a neighborhood $V_n(\underline{t}, x)$ of x in M for which*

1. *$f_{\underline{t}}^n$ maps $V_n(\underline{t}, x)$ diffeomorphically onto the ball of radius δ_1 around $f_{\underline{t}}^n(x)$;*
2. *if $y, z \in V_n(\underline{t}, x)$, then*

$$\frac{1}{C_1} \leq \frac{|\det Df_{\underline{t}}^n(y)|}{|\det Df_{\underline{t}}^n(z)|} \leq C_1 .$$

Now we define for each $\underline{t} \in T^{\mathbf{N}}$ and $n \geq 1$,

$$H_n(\underline{t}) = \{x \in B(\mu^\epsilon) \colon \ n \text{ is a } \alpha\text{-hyperbolic time for } (\underline{t}, x)\ \}.$$

Using the bounded distortion property given by Lemma 5.2, we are able to prove uniform bounded density for the iterates of Lebesgue measure at hyperbolic times.

Proposition 5.3. *There is a constant $C_2 > 0$ such that for every $\underline{t} \in T^{\mathbf{N}}$ and $n \geq 0$ we have*

$$\frac{d}{dm}(f_{\underline{t}}^n)_*(m \mid H_n(\underline{t})) \leq C_2.$$

In order to control the densities of the push-forwards of Lebesgue measure between consecutive hyperbolic times, we also define for $\underline{t} \in T^{\mathbf{N}}$ and $n, k \geq 1$,

$$R_{n,k}(\underline{t}) = \Big\{x \in H_n(\underline{t}) :\ k \text{ is the first } \alpha\text{-hyperbolic time for } f_{\underline{t}}^n(x)\ \Big\}.$$

Defining for each $n \geq 1$

$$\nu_n^\epsilon = \int (f_{\underline{t}}^n)_*(m \mid H_n(\underline{t}))\, d\theta_\epsilon^{\mathbf{N}}(\underline{t})$$

and

$$\eta_n^\epsilon = \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\int (f_{\underline{t}}^{n+j})_*(m \mid R_{n,k}(\underline{t}))d\theta_\epsilon^{\mathbf{N}}(\underline{t}),$$

it is straightforward to check that

$$\mu_n^\epsilon \leq \frac{1}{n}\sum_{j=0}^{n-1}(\nu_j^\epsilon + \eta_j^\epsilon).$$

As an easy consequence of Proposition 5.3 we deduce that

$$\frac{d\nu_n^\epsilon}{dm} \leq C_2$$

for every $n \geq 0$. Our goal now is to bound the density of the measures η_n^ϵ in such a way that we may ensure the absolute continuity of the weak* accumulation points of the measures μ^ϵ when ϵ goes to zero. Using that $(h_\epsilon)_\epsilon$ has uniform L^1-tail we are able to prove:

Proposition 5.4. *Given $\zeta > 0$, there is $C_3(\zeta) > 0$ such that for every $n \geq 0$ and $\epsilon > 0$ we may bound η_n^ϵ by the sum of two non-negative measures, $\eta_n^\epsilon \leq \omega^\epsilon + \rho^\epsilon$, with*

$$\frac{d\omega^\epsilon}{dm} \leq C_3(\zeta) \quad \textit{and} \quad \rho^\epsilon(M) < \zeta.$$

It follows from these last two propositions that the weak* accumulation points of $(\mu^\epsilon)_\epsilon$, when $\epsilon \to 0$, cannot have singular part, thus being absolutely continuous with respect to the Lebesgue measure. Since weak* accumulation points of physical measures are f-invariant, then, applying Theorem 3.1 we obtain:

Theorem 5.5. **[Alves, Araújo]** *Let $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ a random perturbation of a non-uniformly expanding C^2 local diffeomorphism f. If f is non-uniformly expanding for random orbits and $(h_\epsilon)_\epsilon$ has uniform L^1-tail, then f is stochastically stable.*

6. **Maps with critical sets.** In the proof of Theorem 5.1 we used in an important way that $\log \|Df^{-1}\|$ is a continuous map. Therefore, we cannot apply the same method for maps with critical sets. However, we are still able to obtain a result similar to Theorem 5.5 with a very interesting application to the class of maps introduced in [17]. In order to do that we restrict the class of perturbations we are going to consider: we take maps f_t with the same critical set $\mathcal{C}$ of f, and impose that

$$Df_t(x) = Df(x) \quad \text{for every } x \in M \setminus \mathcal{C} \text{ and } t \in T. \tag{6.1}$$

This holds, for instance, in the random perturbations of Example 4.1 and may be generalized for parallelizable manifolds such as tori $\mathbf{T}^d$ or cylinders $\mathbf{T}^{d-k} \times \mathbf{R}^k$.

For the case of maps with critical sets we also impose an analog of condition (3.2) for random orbits; we assume *slow approximation of random orbits to the critical set*: given any small $\gamma > 0$ there is $\delta > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f_{\underline{t}}^j(x), \mathcal{C}) \leq \gamma \tag{6.2}$$

for $\theta_\epsilon^{\mathbf{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbf{N}} \times M$ and small $\epsilon > 0$. In the case we are also able to prove finiteness of physical measures obtaining the same conclusions of Theorem 4.3.

The property of non-uniform expansion for random orbits together with the slow approximation of random orbits to the critical set permit us to introduce a notion of hyperbolic times for points in $(\underline{t}, x) \in T^{\mathbf{N}} \times M$, and define a map

$$h_\epsilon \colon T^{\mathbf{N}} \times M \to \mathbf{Z}^+,$$

taking $h_\epsilon(\underline{t}, x)$ the first hyperbolic time for the point $(\underline{t}, x) \in T^{\mathbf{N}} \times M$. Then we define *uniform L^1-tail* exactly in the same way as before.

For the next result we take $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ a random perturbation of a non-uniformly expanding C^2 map f satisfying (6.1).

Theorem 6.1. [**Alves, Araújo**] *Let $f\colon M \to M$ be non-uniformly expanding C^2 map behaving like a power of the distance close to its critical set $\mathcal{C}$ and whose orbits have slow approximation to $\mathcal{C}$. Assume that f is non-uniformly expanding for random orbits and random orbits have slow approximation to $\mathcal{C}$. If $(h_\epsilon)_\epsilon$ has uniform L^1-tail, then f is stochastically stable.*

As a major application of the previous theorem we are thinking of a class of maps on the cylinder $S^1 \times \mathbf{R}$ introduced in [17]. Subsequent works [1] and [4] showed that such systems are topologically mixing (thus transitive) and have a unique SRB measure.

Example 6.2. (Viana maps) *We describe a class of non-uniformly expanding maps (with critical sets) introduced in* [17] *that satisfy the hypotheses of Theorem 3.1 and Theorem 6.1.*

Let $a_0 \in (1,2)$ be such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $b : S^1 \to \mathbf{R}$ be a Morse function defined on the unit circle $S^1 = \mathbf{R}/\mathbf{Z}$. For fixed small $\alpha > 0$, consider the map

$$\begin{array}{rccc} \hat{f}: & S^1 \times \mathbf{R} & \longrightarrow & S^1 \times \mathbf{R} \\ & (s,x) & \longmapsto & (\hat{g}(s), \hat{q}(s,x)) \end{array}$$

where $\hat{g}$ is the uniformly expanding map of the circle defined by $\hat{g}(s) = ds$ (mod $\mathbf{Z}$) for some $d \geq 16$, and $\hat{q}(s,x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$. It is easy to check that for $\alpha > 0$ small enough there is an interval $I \subset (-2,2)$ for which $\hat{f}(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map f sufficiently close to $\hat{f}$ in the C^0 topology has $S^1 \times I$ as a forward invariant region. We consider maps f close to $\hat{f}$ restricted to the forward invariant region $S^1 \times I$.

We may deduce from the results in [17] *(see also* [2]*) that if f is a map close to $\hat{f}$ in the C^2 topology, then f is a non-uniformly expanding map with a critical set, in the sense that f behaves like a power of the distance close to the critical set and orbits of f have slow approximation of to the critical set.*

Fixing some random perturbation as before, it is shown in [2] *that*

- *f is non-uniformly expanding for random orbits;*
- *random orbits have slow approximation to the critical set $\mathcal{C}$;*
- *the family of hyperbolic time maps $(h_\epsilon)_\epsilon$ has uniform L^1-tail.*

Hence we may apply Theorem 6.1 and conclude that any f close enough to $\hat{f}$ is stochastically stable.

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E-mail address: `jfalves@fc.up.pt`