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EXTREME DEGENERATIONS FOR SOME GENERIC BIFURCATIONS AND NEW TRANSVERSALITY CONDITIONS

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Abstract. For a family of discrete dynamical systems $f(x, \mu)$, it can be found sufficient conditions in terms of the partial derivatives with respect to the state variable x to have (local) generic bifurcations of fixed points. In this paper we analyze what happens when those conditions degenerate in the sense that all those partial derivatives vanish. Also we study some others transversality conditions for the family to undergo the same bifurcations.

1. Introduction.

Let us consider one-parameter families of maps

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

or

$$f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2,$$

where $\mu \in \mathbb{R}$ will represent a parameter and $x \in \mathbb{R}$ or $\mathbb{R}^2$ will be the variable in the state space under consideration. One can think of f in two different ways: one, like a family of discrete dynamical systems

$$(x, \mu) \to f(x, \mu)$$

and two, like the field map which provides us the family of continuous dynamical systems given by the O.D.E.'s

$$x' = f(x, \mu).$$

We are interested in the local changes on the dynamics of the systems of the family around a fixed/critical point when the parameter is varied. Roughly speaking, when there is a qualitative change at $\mu_0 \in \mathbb{R}$, one says that μ_0 is a *bifurcation value* or that a *bifurcation* occurs at μ_0.

Although in most cases the results are valid for both discrete and continuous dynamical systems, we will concentrate in the discrete case.

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We recall that a fixed point $x \in \mathbb{R}^n$ of f in $C^r(\mathbb{R}^n)$ is *hyperbolic* if the linearization $Df(x)$ has no eigenvalues of unit modulus. From the Hartman-Grobman's theorem it follows (see [7] or [12]) that to study local bifurcations of fixed points in parametric families $f_\mu(x)$ it suffices to consider those parameters μ_0 for which the corresponding map has a non-hyperbolic fixed point x_0. As in [2] or [3], we will only consider the simplest ways in which a fixed point of a map can be a non-hyperbolic and without loss of generality, we will assume that $\mu_0 = 0$ and $x_0 = 0$.

In [2], we called *non-degenerated conditions* to those which are sufficient for the appearance of a local bifurcation in a family with a non-hyperbolic fixed point and generalized them. Some of these conditions can be expressed in terms of partial derivatives with respect to x evaluated in (x_0, μ_0), and are called *nondegeneracy conditions*. All the other conditions, in which partial derivatives with respect to the parameter are involved, are called *transversality conditions*. In [3], we analyzed what happens when one of the nondegeneracy conditions fails.

The purpose of this paper is to show what phenomena occur when the partial derivatives which represent nondegeneracy conditions of some typical bifurcations vanish for all order, what we will call *extreme degenerations*, and give some new transversality conditions under which these bifurcations appear.

Of course, all the results we describe for bifurcation of fixed point can be applied to periodic points by considering the relevant iterate of the map. On the other hand, by using *center manifold theory* (see [4], [11] or [13]), they can also be seen to describe the bifurcations of maps in $\mathbb{R}^n$, or more generally in the setting of Banach spaces.

The analogous cases of bifurcation in continuous dynamical systems can be generalized, in the same way, following a similar sketch of our proofs.

1.1. Extreme Degenerations. For the cases related to the *fold*, *transcritical* and *pitchfork* bifurcations a new kind of phenomena can occur, although the common characteristic will continue being the change in the number of fixed points when we cross the bifurcation value.

In [2], the sufficient conditions to the standard *fold* bifurcation appears are generalized and renamed *non-degenerated conditions*. In particular, the following theorem is obtained:

Theorem 1. *Suppose that a one-parameter family $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of C^{2n} maps has at $\mu_0 = 0$ the fixed point $x_0 = 0$ and let $f_x(0,0) = 1$.*

Assume that the following non-degenerated conditions are satisfied:

*F*1.) $f_{xx}(0,0) = f_{xxx}(0,0) = \cdots = f_{x^{2n-1}}(0,0) = 0,\ f_{x^{2n}}(0,0) \neq 0$

*F*2.) $f_\mu(0,0) \neq 0$

Then the family undergoes a fold *bifurcation.*

In [3], we analyze the cases in which the first nonzero partial derivatives with respect to x at the origin has odd order and obtain that the bifurcation does not occur.

Then, it is natural to wonder what happens if the derivatives, which provides us sufficient conditions, vanish. In that situation, if f is analytic, the family undergoes a new type of bifurcation: while at $\mu = 0$ all points in an sufficiently small neighborhood of $x = 0$ are non-hyperbolic fixed points, at $\mu \neq 0$ there are not fixed points near $x = 0$.

Effectively, by the Implicit Function Theorem, as $f_\mu(0,0) \neq 0$, we can assure the existence and uniqueness of an analytic curve $\mu(x)$, which verifies the equation of fixed points of f, i.e., $f(x, \mu(x)) - x = 0$.

Taking into account the corresponding Taylor series of the analytic function f, one can see that $\mu(x) = 0$ is that unique curve and the bifurcation diagram follows.

Remark 1. It is crucial to know that f is an analytic function, since when f is C^∞, the phenomenon could be different. It is the case of the perturbation of the *Cauchy's function* given by

$$f(x,\mu) = \begin{cases} x + \mu + e^{-1/x^2} & \text{if } x \neq 0 \\ \mu & \text{if } x = 0 \end{cases}$$

which presents a *fold* bifurcation, in spite of satisfying $f_{x^r}(0,0) = 0$ for all $r \in \mathbb{N}$, $r \geq 2$.

As in the previous case, similar generalizations of the nondegeneracy conditions were obtained for the *transcritical*, *pitchfork*, *flip* and *Hopf-Neimark-Sacker* bifurcations (see [2]), and now we are going to analyze what happens when the derivatives of all orders with respect to x vanish.

In classical bifurcation theory, it is often assumed that there is a trivial fixed point which appears for all parameter values, i.e., the family is assumed to satisfy $f(0, \mu) = 0$. Since the *fold* families contain parameter values for which there are no fixed point, this situation is qualitatively different. This is the setting for the appearance of *transcritical* or *pitchfork* bifurcations.

To formulate an appropriate transversality condition, $f_\mu(0,0) \neq 0$ is replaced by the requirement that $f_{x\mu}(0,0) \neq 0$, while $f_\mu(0,0) = 0$ is assumed. More precisely, when the partial derivatives with respect to x, which provide nondegeneracy conditions for the *transcritical* and *pitchfork* bifurcations, are null for an analytic one-parameter family of maps f which satisfies the transversality condition $f_{x\mu}(0,0) \neq 0$, then a new type of bifurcation occurs. Concretely, at $\mu = 0$ all points in an sufficiently small neighbourhood of $x = 0$ are non-hyperbolic fixed points, while at any $\mu \neq 0$, there is a unique fixed point near $x = 0$.

The sketch of the proof is as follows. By the hypothesis, from the Taylor series of f, one can easily see that $\mu = \mu(x) = 0$ is a curve of fixed points of the family. Factoring it, the equation of fixed points of f turns into

$$f(x,\mu) - x = \mu F(x,\mu) = 0$$

where F continues being an analytic function, which satisfies that

$$\begin{cases} F(0,0) = 0 \\ F_x(0,0) = \neq 0 \end{cases}$$

Hence, one can apply the Implicit Function Theorem in order to obtain the existence and uniqueness of an analytic curve $x(\mu)$, which verifies both previous equations for all μ sufficiently small. This curve shows the existence of a unique fixed point for each parameter value $\mu \neq 0$.

For the nonzero parameter values, the stability of the unique fixed point depends on the sign of the transversality condition, $f_{x\mu}(0,0) \neq 0$, and changes when the bifurcation value is crossed.

Remark 2. As before, the analyticity of f is required since, even if f is C^∞, the bifurcation can be different. This is the case of the family

$$f(x,\mu) = \begin{cases} x + \mu x + e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

which is a C^∞ function verifying all the mentioned conditions and, however, it undergoes a *transcritical* bifurcation.

For the appearance of the *flip* or *period doubling* bifurcation, the nondegeneracy conditions can be given using the partial derivatives with respect to x of the second iterate of the family under consideration f (see [17] and [2]).

If f is analytic and the non-hyperbolicity condition $f_x(0,0) = -1$ is assumed, again, the Implicit Function Theorem provides a unique curve $x(\mu)$, which we can assume is $x(\mu) = 0$.

But, for the period two points, if we suppose that all those partial derivatives with respect to x of the second iterate of the family vanish, i.e.,

$$(f^2)_{xxx}(0,0) = (f^2)_{x^4}(0,0) = \cdots = (f^2)_{x^{2n}}(0,0) = \cdots = 0$$

and the transversality condition $f_{x\mu}(0,0) \neq 0$ remains, then the family undergoes a new type of bifurcation: while at $\mu = 0$ all point (except the origin) in an sufficiently small neighborhood of $x = 0$ are non-hyperbolic 2-periodic points, at $\mu \neq 0$ there are not 2-periodic points near $x = 0$. The proof is easily obtained from the fact that the second iterated of the family verifies the conditions related to a extreme degeneration in the *transcritical* or *pitchfork* nondegeneracy conditions.

In the case of the families of maps on the plane, the normal form of one of them, analytic, verifying the conditions of the *Hopf-Neimark-Sacker* bifurcation, and with a pair of complex conjugated eigenvalues of modulus one at the bifurcation point, $e^{\pm 2\pi i\theta}$, where θ is irrational, is in polar coordinates (see [5])

$$g(r,\theta;\mu) = \left((1+\mu)r - \sum_{m=1}^{\infty} a_{2m+1}(\mu)r^{2m+1}, \theta + \phi(\mu) + \sum_{m=1}^{\infty} b_{2m}(\mu)r^{2m}\right) \tag{1}$$

Observe that the first component is a one-parameter family as those examined for the *pitchfork* bifurcation. Because of that, if all the coefficients $a_{2m+1}(0)$ are zero, we have a family of this type with a extreme degeneration. So, one can easily deduce from this normal form that, at $\mu = 0$ all points in a sufficiently small neighborhood of the origin remains in an invariant circle, while at $\mu \neq 0$ there is not any invariant circle near the origen.

1.2. **New Transversality Conditions.** When the imposed transversality conditions for the appearance of a bifurcation involve partial derivatives with respect to x (e.g., in the *transcritical*, *pitchfork* or *flip* cases), it can be generalized in the sense that we can obtain the same changes in the qualitative behavior with conditions of higher order. In particular, we have proved the following results.

Theorem 2. *Suppose that a one-parameter family* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *of analytic maps has the fixed point* $x = 0$ *for all* μ *and let* $f_x(0,0) = 1$.

Assume that the following conditions are satisfied:

1.) $f_{xx}(0,0) = f_{xxx}(0,0) = \cdots = f_{x^{2n-1}}(0,0) = 0,\ f_{x^{2n}}(0,0) \neq 0.$

2.) $f_{x^r\mu}(0,0) \neq 0$ *is the lower-order derivative, involving the parameter, which does not vanish, i.e., any other derivative involving the parameter is an* $\mathcal{O}(x^r\mu^2, x^{r+1}\mu)$.

Then

- *If* $2n - r$ *is an odd (positive) number, then* f *undergoes a transcritical bifurcation.*
- *If* $2n-r$ *is an even (positive) number, then* f *undergoes a pitchfork bifurcation.*

Note that if $2n - r$ is not a positive number, the parametric perturbation does not differ from the original map $f(\cdot, 0)$, for μ sufficiently small.

As in previous cases the analyticity condition is required because the changes could be different even if f is a C^∞ function. It occurs for the family

$$f(x,\mu) = \begin{cases} x + \mu x^2 + xe^{-1/\mu^2} + x^3 & \text{si } \mu \neq 0 \\ x + x^3 & \text{si } \mu = 0 \end{cases}$$

which is a C^∞ function and verifies all the conditions of the above theorem, but, it does not undergo a *transcritical* bifurcation. Concretely, two additional fixed points appear near $x = 0$, for any value of μ sufficiently small.

A similar reasoning leads to the following result:

Corollary 1. *Suppose that a one-parameter family* $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *of analytic maps has the fixed point* $x = 0$ *for all* μ *and let* $f_x(0,0) = 1$.

Assume that the following conditions are satisfied:

1.) $f_{xx}(0,0) = f_{xxx}(0,0) = \cdots = f_{x^{2n}}(0,0) = 0,\ f_{x^{2n+1}}(0,0) \neq 0$.

2.) $f_{x^r\mu}(0,0) \neq 0$ *is the lower-order derivative, involving the parameter, which does not vanish, i.e., any other derivative involving the parameter is an* $\mathcal{O}(x^r\mu^2, x^{r+1}\mu)$.

Then

- *If* $2n-r$ *is an odd (positive) number, then* f *undergoes a pitchfork bifurcation.*
- *If* $2n - r$ *is an even (positive) number, then* f *undergoes a transcritical bifurcation.*

A natural context for the appearance of the *pitchfork* bifurcation in a family f is f having a reflection symmetry, i.e., $f(-x,\mu) = -f(x,\mu)$. In this situation, one can prove, similarly to the previous result, the following:

Corollary 2. *Suppose that a one-parameter family* $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *of analytic maps is an odd function of* x *and let* $f_x(0,0) = 1$.

Assume that the following conditions are satisfied:

1.) $f_{xxx}(0,0) = \cdots = f_{x^{2n-1}}(0,0) = 0,\ f_{x^{2n+1}}(0,0) \neq 0$.

2.) $f_{x^{2m+1}\mu}(0,0) \neq 0$ *is the lower-order derivative, involving the parameter, which does not vanish, i.e., any other derivative involving the parameter is an* $\mathcal{O}(x^{2m+1}\mu^2, x^{2m+2}\mu)$.

Then if $2n - 2m$ *is positive, then* f *undergoes a pitchfork bifurcation.*

As before, if $2n - 2m$ is not a positive number, the parametric perturbation does not differ from the original map, for μ sufficiently small.

If a family presents a non-hyperbolic fixed point with an eigenvalue equals -1, making successive changes of coordinates (*method of the normal forms*, see [1] or [5]) the family turns into one of the form

$$(x,\mu) \to f(x,\mu) = f_1(\mu)x + f_3(\mu)x^3 + \cdots + f_{2n+1}(\mu)x^{2n+1} + \mathcal{O}(x^{2n+3}).$$

Consequently, we will assume in the following theorem that the new transversality condition has odd order in x.

Theorem 3. *Suppose that a one-parameter family $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ of analytic maps has at $\mu_0 = 0$ the fixed point $x_0 = 0$ and let $f_x(0,0) = -1$.*

Assume that the following conditions are satisfied:

1.) $(f^2)_{xxx}(0,0) = (f^2)_{x^4}(0,0) = \cdots = (f^2)_{x^{2n}}(0,0) = 0$, $(f^2)_{x^{2n+1}}(0,0) \neq 0$
2.) $f_{x^{2m+1}\mu}(0,0) \neq 0$ *is the lower-order derivative, involving the parameter, which does not vanish, i.e., any other derivative involving the parameter is an* $\mathcal{O}(x^{2m+1}\mu^2, x^{2m+2}\mu)$.

Then, if $2n - 2m$ is positive f undergoes a flip *or* period doubling *bifurcation.*

Finally note that, the radial component of the expression (1) in polar coordinates of an analytic family of maps on the plane could satisfy some of the new transversality conditions, and it could allow to know the presence of invariant circles near the origin when the parameter is varied. To traduce this conditions to our original family seems difficult.

2. Proofs.

Proof of Theorem 2. By the hypothesis, f has a Taylor series of the form

$$f(x,\mu) = x + \sum_{|k|=2}^{\infty} \frac{1}{k_1!k_2!} f_{x^{k_1}\mu^{k_2}}(0,0)x^{k_1}\mu^{k_2}, \quad |k| = k_1 + k_2$$

with $k_1 \geq r$.

Consider the equation

$$0 = g(x,\mu) = f(x,\mu) - x = \sum_{|k|=2}^{\infty} \frac{1}{k_1!k_2!} f_{x^{k_1}\mu^{k_2}}(0,0)x^{k_1}\mu^{k_2}, \quad |k| = k_1 + k_2$$

which define the fixed points of the family. As $x = 0$ is a fixed point for all value of μ, factoring it, the equation which determine the rest of fixed points is $F(x,\mu) = 0$, where

$$F(x,\mu) = \begin{cases} \dfrac{f(x,\mu) - x}{x^r} & \text{si } x \neq 0 \\[2ex] \dfrac{f_{x^r}(0,\mu)}{r!} & \text{si } x = 0 \end{cases}$$

is also analytic.

As this F verifies

$$\begin{cases} F(0,0) = 0 \\ F_\mu(0,0) = \dfrac{f_{x^r\mu}(0,0)}{r!} \neq 0 \end{cases}$$

the Implicit Function Theorem provides a unique analytic curve $\mu(x)$ satisfying

$$\begin{cases} \mu(0) = 0 \\ F(x,\mu(x)) = 0 \end{cases}$$

which is in fact a curve of fixed points of f.

Differentiating with respect to x the equation

$$F(x, \mu(x)) = 0$$

one can obtain, in a similar way as we have in [2], that

- if $2n - r$ is an odd (positive) number, the curve has neither a maximum nor a minimum at $x = 0$ (although it is a singular point of the curve),
- if $2n-r$ is an even (positive) number, the curve has a maximum or a minimum, depending on the sign of the conditions 1.) and 2.), at $x = 0$

and therefore the respective bifurcation diagram immediately follows.

The stability of the fixed points also depends on the sign of the conditions 1.) and 2.), and it can be evaluated differentiating with respect to x the equation that define them implicitly.

More precisely, if we suppose that

$$\begin{cases} f_{x^{2n}}(0,0) > 0 \\ f_{x^r \mu}(0,0) > 0 \end{cases}$$

we have:

- If $2n - r$ is an odd number:
  - For the fixed points in the curve $\mu(x)$, if $x < 0$, it is stable; if $x > 0$, it is unstable.
  - For the trivial fixed point $x = 0$, if $\mu < 0$, it is stable; if $\mu > 0$ it is unstable.

  Changing the signs of these nondegenerated conditions, which we have supposed, we will arrive at the following different cases:
  1. With a reversal of one of these two inequalities, the slope of the curve $\mu(x)$ reverses. Moreover, if we reverse the inequality corresponding to the derivative with respect to x, then the stability of the fixed points in the curve reverses; and if we reverse the inequality which involves the derivative with respect to μ, then the stability of the trivial fixed points reverses.
  2. With a reversal of both inequalities, only the stability of the fixed points reverses.
- If $2n - r$ is an even number:
  - The fixed points in the curve $\mu(x)$ are unstable
  - For the trivial fixed point $x = 0$, if $\mu < 0$, it is stable; if $\mu > 0$ it is unstable.

  As before, changing the signs of these nondegenerated conditions, which we have supposed, we will obtain four different bifurcation diagrams. Concretely,
  1. With a reversal of one of these two inequalities, the position of the curve $\mu(x)$ with respect to the line $\mu = 0$ reverses. Moreover, if we reverse the inequality corresponding to the derivative with respect to x, then the stability of the fixed points in the curve also reverses; and if we only reverse the inequality which involves the derivative with respect to μ, then the stability of the trivial fixed points reverses.
  2. With a reversal of both inequalities, only the stability of the fixed points reverses.

□

The proofs of the corollaries 1 and 2 are similar to the above one and therefore we will not show it.

Proof of Theorem 3. Considering the second iterate of f, we obtain:

$$\begin{aligned}&(f^2)(0,0)=0\\&(f^2)_x(0,0)=f_x(f(0,0),0)f_x(0,0)=f_x(0,0)^2=1\\&(f^2)_{xx}(0,0)=f_{xx}(0,0)(-1)^2+f_{xx}(0,0)(-1)=0\end{aligned}$$

and thanks to the nondegeneracy conditions

$$(f^2)_{xxx}(0,0)=\cdots=(f^2)_{x^{2n}}(0,0)=0;\ (f^2)_{x^{2n+1}}(0,0)\neq 0$$

As we just say before the theorem, we can assume that our function is of the form

$$(x,\mu)\to f(x,\mu)=f_1(\mu)x+f_3(\mu)x^3+\cdots+f_{2n+1}(\mu)x^{2n+1}+\mathcal{O}(x^{2n+3})$$

where $f_1(0)=-1$. So, analyzing the derivatives involving the parameter of the second iterate

$$\begin{aligned}(f^2)_{x\mu}(x,\mu)=\ &f_{xx}(f(x,\mu),\mu)f_\mu(x,\mu)f_x(x,\mu)+\\&+f_{x\mu}(f(x,\mu),\mu)f_x(x,\mu)+\\&+f_x(f(x,\mu),\mu)f_{x\mu}(x,\mu).\end{aligned}$$

therefore $(f^2)_{x\mu}(0,0)=-2f_{x\mu}(0,0)=0$;

$$\begin{aligned}(f^2)_{xx\mu}(x,\mu)=\ &f_{xxx}(f(x,\mu),\mu)f_\mu(x,\mu)(f_x(x,\mu))^2+\\&+f_{xx\mu}(f(x,\mu),\mu)(f_x(x,\mu))^2+\\&+2f_{x\mu}(x,\mu)f_{xx}(f(x,\mu),\mu)\\&+f_x(f(x,\mu),\mu)f_{xx\mu}(x,\mu),\end{aligned}$$

hence $(f^2)_{xx\mu}(0,0)=0$ and

$$\begin{aligned}(f^2)_{xxx\mu}(x,\mu)=\ &f_{xxxx}(f(x,\mu),\mu)f_\mu(x,\mu)(f_x(x,\mu))^3+\\&+f_{xxx\mu}(f(x,\mu),\mu)(f_x(x,\mu))^+\\&+\ldots\\&+f_x(f(x,\mu),\mu)f_{xxx\mu}(x,\mu),\end{aligned}$$

thus $(f^2)_{xxx\mu}(0,0)=-2f_{xxx\mu}(0,0)=0$, and so on. Analogously, differentiating those previous expressions with respect to μ, we can conclude that the lower-order derivative, involving the parameter of f^2 is $(f^2)_{x^{2m+1}\mu}(0,0)=-2f_{x^{2m+1}\mu}(0,0)\neq 0$. It means that f^2 satisfies the conditions of the corollary 2 and therefore, (f^2) undergoes a *pitchfork* bifurcation, which corresponds to *flip* bifurcation of our original f, since the fixed point, that appear for any parametric value, corresponds to the unique one of the curve that, via the implicit function theorem, f has.

□

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