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SOME LIMIT-POINT/LIMIT-CIRCLE RESULTS FOR THIRD ORDER DIFFERENTIAL EQUATIONS

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Abstract. The authors consider the third order nonlinear differential equation

$$y''' + q(t)y' + r(t)f(y) = 0 \tag{E}$$

where $q \geq 0$, $r > 0$, and $uf(u) > 0$ for $u \neq 0$. They give two results ensuring that equation (E) is of the nonlinear limit-point type, and for the case $f(u) = u$, they give two results guaranteeing that (E) is of the limit-circle type. Some examples to illustrate the results are included.

1. Introduction We consider the third order nonlinear differential equation

$$y''' + q(t)y' + r(t)f(y) = 0 \tag{1}$$

where $q \in C^1(R_+)$, $r \in C(R_+)$, $f \in C(R)$, $R_+ = [0, \infty)$, $R = (-\infty, \infty)$, $q \geq 0$, $r > 0$, and $uf(u) > 0$ for $u \neq 0$. Throughout the paper we restrict our attention to proper solutions of (1), i.e., to those solutions of (1) which are defined on R_+ and are nontrivial in any neighborhood of infinity. As usual, a continuous function $g : R_+ \to R$ is said to be an *oscillatory function* if there exists a sequence $\{t_n\} \to \infty$ as $n \to \infty$ such that $g(t_n) = 0$ and $\sup_{t \geq T}\{|g(t)|\} > 0$ for every $T \in R_+$. A proper solution is said to be *oscillatory* if it is an oscillatory function, and *nonoscillatory* otherwise. A proper solution y of (1) is said to be of the *nonlinear limit-point (nonlinear limit-circle) type* if $\int_0^\infty y(t)f(y(t))dt = \infty$ ($\int_0^\infty y(t)f(y(t))dt < \infty$). Equation (1) is said to be of the nonlinear limit-circle type if every proper solution is of the nonlinear limit-circle type; equation (1) is said to be of the nonlinear limit-point type if there exists a proper solution that is of the nonlinear limit-point type.

The study of the nonlinear limit-point/limit-circle problem for higher order equations has it roots in the study of this problem for second order nonlinear equations (see, for example, Graef [7, 8] and Graef and Spikes [9]). Results of this type on third order equations can be found in the recent papers of Bartušek and Došlá [3], Bartušek and Graef [4], and Došlá [5]. In Section 2, we obtain some new results of the limit point type for equation (1). Section 3 contains new limit circle results for the case when equation (1) is linear.

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It is convenient to let $\mathcal{O}$ and $\mathcal{N}$ denote the sets of oscillatory and nonoscillatory solutions of (1), respectively, and define the following classes of solutions:

$$W_1 = \{y \in \mathcal{N} : y' \text{ is an oscillatory function}\},$$
$$W_2 = \{y \in \mathcal{N} : y(t)y'(t) < 0 \text{ for large } t \text{ and } y'' \text{ is an oscillatory function}\},$$
$$\mathcal{N}_0 = \{y \in \mathcal{N} : y(t)y'(t) < 0 \text{ and } y(t)y''(t) > 0 \text{ for large } t\},$$
$$\mathcal{N}_1 = \{y \in \mathcal{N} : y(t)y'(t) > 0 \text{ and } y(t)y''(t) > 0 \text{ for large } t\}.$$

Finally, we recall that the solutions in the class $\mathcal{N}_0$ are called Kneser solutions, and those in the classes W_1 or W_2 are called weakly oscillatory solutions.

2. Limit-point Type Results In what follows, we need to impose a growth condition on f, namely, that there exists constants $0 < \alpha_1 \leq \alpha$ such that

$$\alpha_1|u| \leq |f(u)| \leq \alpha|u| \text{ for } u \in R. \tag{H}$$

Observe that this implies that the nonlinear limit-circle property is equivalent to the square integrability of solutions.

In view of condition (H), every solution (1) is defined on R_+ (see [1; Theorem 1.4.]), and moreover, a nontrivial solution is nontrivial in any neighborhood of infinity (see [1; Theorem 1.3]). The structure of the solutions of (1) is described in the following lemma [2; Theorem 1].

Lemma 1. *Let (H) hold and let y be a nontrivial solution of (1). Then $y \in \mathcal{O} \cup W_1 \cup W_2 \cup \mathcal{N}_0 \cup \mathcal{N}_1$.*

Theorem 1. *Suppose (H) holds and let $T > 0$, $M_1 > 0$, and $M > 0$ be such that one of the following conditions holds for $t \geq T$:*

(i) $q(t) \leq M_1 t$ and $q'(t) \geq 2\alpha r(t)$;

(ii) $q \in C^3(R_+)$, $q(t) > 0$, $M \geq 0$,

$$1 + \left(\frac{t}{q(t)}\right)''' - 2\alpha\frac{tr(t)}{q(t)} \geq 0, \quad 0 \leq -\left(\frac{t}{q(t)}\right)' \leq M, \text{ and } \left(\frac{t}{q(t)}\right)'' \leq M_1 t. \tag{2}$$

Then equation (1) is of the nonlinear limit-point type, i.e., there is a solution of (1) that does not belong to $L^2(R_+)$.

Proof. With $R(t) \equiv 1$ and $M = 0$ in case (i), and $R(t) \equiv \frac{t}{q(t)}$ in case (ii), we define

$$Z(t) = Ry'y - \frac{3}{2}\int_T^t R(s)(y'(s))^2 ds - R'y^2 + \int_T^t [R(s)q(s) + 3R''(s)]\frac{y^2(s)}{2} ds.$$

Then,

$$Z'(t) = Ry''y - R\frac{(y')^2}{2} - R'y'y + [Rq + R'']\frac{y^2}{2}$$

and

$$\begin{aligned} Z''(t) &= -rRf(y)y + [(qR)' + R''']\frac{y^2}{2} - \frac{3}{2}R'(y')^2 \\ &\geq [-2\alpha rR + (qR)' + R''']\frac{y^2}{2} - \frac{3}{2}R'(y')^2. \end{aligned} \tag{3}$$

Moreover, (H) and the hypotheses of the theorem show that in both cases we have $Z''(t) \geq 0$ for $t \geq T$. Let y be a solution of (1) such that $Z'(T) > 0$. It is easy to see that y must be nonoscillatory since if there is a sequence $\{t_k\} \to \infty$ of zeros of y, then $Z'(t_k) \leq 0, k = 1, 2, \ldots$. In addition, since Z' is nondecreasing, we would have $\lim_{t\to\infty} Z'(t) \leq 0$, which contradicts $Z'(T) > 0$ and $Z''(t) \geq 0$. Hence, by Lemma 1,

$$y \in W_1 \cup W_2 \cup \mathcal{N}_0 \cup \mathcal{N}_1.$$

First note that if $y \in \mathcal{N}_1$, then $\int_T^\infty y^2(t)dt = \infty$, and so the conclusion of the theorem holds. For the remainder of the possibilities, we proceed by contradiction. Suppose y satisfies

$$\int_0^\infty y^2(t)dt < \infty$$

and say $y(t) > 0$ for $t \geq T_1 \geq T$. Then there exists $\tau \geq T_1$ such that

$$\int_\tau^\infty y^2(s)ds \leq \frac{Z'(T)}{1+3M_1}. \tag{4}$$

Since Z' is nondecreasing for $t \geq T$, (2)–(4) yield

$$\begin{aligned} Z'(T)(t-T) \leq Z(t) - Z(T) &\leq R(t)y'(t)y(t) + My^2(t) + \frac{1+3M_1}{2} t \int_\tau^\infty y^2(s)ds \\ &\quad + \int_T^\tau [R(s)q(s) + 3R''(s)] \frac{y^2(s)}{2} ds - Z(T) \\ &\leq R(t)y'(t)y(t) + My^2(t) + \frac{Z'(T)t}{2} + K, \end{aligned} \tag{5}$$

for $t \geq \tau$ and some constant K. If $y \in \mathcal{N}_0 \cup W_2$, then y is decreasing and $yy' < 0$ for large t, and so we have a contradiction.

Finally, suppose $y \in W_1$. Now (4) implies $\liminf_{t\to\infty} y(t) = 0$ and there exists a sequence $\{t_k\} \to \infty$ of zeros of y' such that $\lim_{k\to\infty} y(t_k) = 0$ (e.g., a sequence of local minimums of y). We again obtain a contradiction by taking $t = t_k$ in (5) with k large. □

Example 1. Consider the equation

$$y''' + qt^m y' + rt^n f(y) = 0 \tag{6}$$

where $q > 0$ and $r > 0$ are constants and (H) holds. By parts (i) (for $0 < m \leq 1$) and (ii) (for $m > 1$) of Theorem 1, we see that equation (6) is of the nonlinear limit-point type if any of the following conditions hold:

(a) $m > n + 1$;
(b) $m = n + 1$, $0 < m \leq 1$, and $mq > 2\alpha r$;.
(c) $m = n + 1$, $m > 1$, and $q > 2\alpha r$.

The following theorem gives us a limit-point type result for small q and r.

Theorem 2. *Let (H) hold and suppose there exist constants $T \geq 0$, $M \geq 0$, and $M_1 \geq 0$ such that for $t \geq T$*

$$q(t) \leq M, r(t) \leq M_1 t^{\frac{1}{2}}, \textit{and } 2\alpha_1 r(t) \geq q'(t).$$

Then equation (1) is of the nonlinear limit-point type.

Proof. For a solution y of (1) we define

$$Z(t) = -2y'(t)y(t) + 3\int_T^t (y'(s))^2\,ds - \int_T^t q(s)y^2(s)ds \tag{7}$$

Then

$$Z'(t) = -2y''y + (y')^2 - qy^2 \tag{8}$$

and

$$Z''(t) = 2ryf(y) - q'y^2 \ge (2\alpha_1 r - q')y^2 \ge 0$$

for $t \ge T$. Thus Z' is nondecreasing.

Let y be a solution of (1) such that $Z'(T) > 0$. Lemma 1 implies $y \in \mathcal{O} \cup W_1 \cup W_2 \cup \mathcal{N}_0 \cup \mathcal{N}_1$. Integrating, we obtain

$$\frac{Z'(T)}{2}t \le Z'(T)(t-T) \le Z(t) - Z(T) \le -2y'(t)y(t) + 3\int_T^t (y'(s))^2\,ds \tag{9}$$

for $t \ge T_1$ for some $T_1 \ge T$. Let $y \in W_2 \cup \mathcal{N}_0$; then $y(t)y'(t) < 0$ for $t \ge T_2 \ge T_1$, and there exists a sequence $\{t_k\} \to \infty$ such that $\lim_{k\to\infty} y'(t_k) = 0$ and $y''(t_k)y(t_k) \ge 0$. (If $y \in \mathcal{N}_0$, then any sequence tending to ∞ will work, and if $y \in W_2$, then take a sequence of local maximas of $y'(t)sgn\ y(t)$.) Hence, $\lim_{k\to\infty} Z'(t_k) \le 0$, and since Z' is nondecreasing, $Z'(t) \le 0$ for large t. This contradicts $Z'(T) > 0$ and $Z'' \ge 0$.

If $y \in \mathcal{N}_1$, the conclusion of the theorem clearly holds. Now let $y \in \mathcal{O} \cup W_1$ and let $\{t_k\} \to \infty$, be a sequence of zeros of y'. Then (9) yields

$$\frac{Z'(T)}{2}t_k \le 3\int_T^{t_k} (y'(s))^2\,ds, \quad k = 1, 2, \dots \tag{10}$$

for $t_k \ge T_2$. Let $J_k = [T, t_k], k = 1, 2, \dots, v \in C^0[T, \infty)$, and $\|v\|_k = \left(\int_T^{t_k} v^2(s)ds\right)^{\frac{1}{2}}$. Suppose, to the contrary, that $\int_T^\infty y^2(s)ds < \infty$. Then (H) yields the existence of constants K and K_1 such that

$$\int_T^\infty y^2(s)ds = K^2 < \infty \ \text{ and } \int_T^\infty [f(y(s))]^2\,ds = K_1^2 < \infty. \tag{11}$$

Set $\varepsilon = \varepsilon_1/(2M_1K_1 + M\varepsilon_1)$, where $\varepsilon_1 = \sqrt{Z'(T)/6}$. By [12; Theorem 1.2], there exists a constant K_2 such that

$$\|y'\|_k \le \varepsilon \|y'''\|_k + K_2 \|y\|_k, \quad k = 1, 2, \dots. \tag{12}$$

From (1) and Minkowski's inequality, we obtain

$$\|y'''\|_k = \|-qy' - rf(y)\|_k \le \|qy'\|_k + \|rf(y)\|_k. \tag{13}$$

From (11) – (13), we have

$$\|y'\|_k \le \varepsilon\left(M\|y'\|_k + M_1 t_k^{\frac{1}{2}}\|f(y)\|_k\right) + K_2\|y\|_k,$$

so, using (10),

$$2\varepsilon M_1 K_1 t_k^{\frac{1}{2}} \le (1 - \varepsilon M)\|y'\|_k \le \varepsilon M_1 K_1 t_k^{\frac{1}{2}} + K_2 K.$$

This contradiction for large k completes the proof of the theorem. □

As an example of Theorem 2.2, we have the following.

Example 2. If $m \le 0$ and $n \le 1/2$, then equation (6) is of the nonlinear limit-point type.

3. Limit-circle Type Results Under appropriate assumptions on q, Appel's equation

$$y''' + 4q(t)y' + 2q'(t)y = 0 \tag{14}$$

is a simple example of a third order linear equation of the limit-circle type (see [3; Corollary 4.2] or [5; Corollary 1]). In order to generalize this result, we first introduce some notation and a lemma.

Consider the linear differential equation

$$h'' + q(t)h = 0. \tag{15}$$

Let $T \in R_+$ and let h_1 and h_2 be solutions of (15) with the Cauchy initial conditions $h_1(T) = 1$, $h_1'(T) = 0$, $h_2(T) = 0$, $h_2'(T) = 1$. Also, we set

$$a(t) = \max(|h_1(t)|, |h_2(t)|). \tag{16}$$

Recall that the second order linear equation (15) is said to be of the limit-circle type if every solution of (15) belongs to $L^2(R_+)$.

The following lemma is a special case of a result of Staněk [13; Theorem 1]. It will be used to prove our limit circle results.

Lemma 2. Let $\varepsilon > 0$, $T \in R_+$, $I = [T, \infty)$, $q \in C^1(I)$, and let $F \in C^0(I \times R)$ satisfy

$$|F(t,x)| \le \omega(t, |x|) \quad \text{on} \quad I \times R,$$

where $\omega \in C^0(I \times R_+)$ is nondecreasing with respect to the second variable for any fixed $t \in R_+$. Let $\beta \in C^0(R_+)$ be positive and satisfy

$$\varepsilon + 2\int_T^t a^2(s)\omega\left(s, \beta(s)a^2(s)\right) ds \le \beta(t), \quad t \in I. \tag{17}$$

If y is a solution of

$$y''' + qy' + \frac{q'}{2}y = F(t,y) \tag{18}$$

with the Cauchy initial conditions $y(T) = C_1$, $y'(T) = C_2$, $y''(T) = C_3$ and

$$|C_1|(1 + 4q(0)) + |C_2| + \left|\frac{C_3}{2}\right| < \varepsilon,$$

then y is defined on R_+ and

$$|y(t)| \le \beta(t)a^2(t), \quad t \in I.$$

Using this lemma, we can prove the following limit-circle theorem for the linear equation

$$y''' + q(t)y' + r(t)y = 0 \tag{19}$$

where $q \in C^1(R_+)$, $r \in C^0(R_+)$, $q \ge 0$, and $r > 0$ on R_+.

Theorem 3. Let $q' \geq 0$ on R_+. Assume that (15) is of the limit-circle type and define the function a by (16). If there exist $T \in R_+$ and $\delta > 0$ such that

$$\left| r(t) - \frac{q'(t)}{2} \right| \leq K = \frac{1}{(2+\delta)\int_T^\infty a^4(s)ds}, \quad t \in [T, \infty), \tag{20}$$

then equation (19) is of the limit-circle type.

Proof. Let $\bar{y}$ be an arbitrary solution of (19). Then $\bar{y}(T) = C_1$, $\bar{y}'(T) = C_2$, and $\bar{y}''(T) = C_3$ for some constants C_1, C_2, and C_3, and we set $\varepsilon = |C_1|(1 + 4q(T)) + |C_2| + |C_3|/2 + 1$. Since $q' \geq 0$, it is well known that all solutions of (15) are bounded (see [1; Consequence 3.12], for example). Hence,

$$|h_1(t)| \leq C \text{ and } |h_2(t)| \leq C \tag{21}$$

for some constant $C > 0$. Moreover, since (15) is of the limit-circle type, we have

$$\int_T^\infty h_1^2(t)dt < \infty \quad \text{and} \quad \int_T^\infty h_2^2(t)dt < \infty.$$

From this and (21), it follows that

$$\int_T^\infty a^4(s)ds < \infty. \tag{22}$$

Clearly, (19) is equivalent to (18) on $[T, \infty)$ where

$$F(t,x) = \left(-r(t) + \frac{q'(t)}{2} \right) x.$$

Then, with

$$\omega(t,x) = Kx \quad \text{and} \quad \beta(t) \equiv \beta_0 = \frac{2+\delta}{\delta}\varepsilon$$

for $t \in [T, \infty)$ and $x \in R_+$, we see that

$$|F(t,x)| = \left| \left(-r(t) + \frac{q'(t)}{2} \right) x \right| \leq K|x| = \omega(t, |x|).$$

In addition,

$$\varepsilon + 2\int_T^t a^2(s)\, \omega\left(s, \beta(s)a^2(s)\right) ds = \varepsilon + 2K\beta_0 \int_T^t a^4(s)ds \leq \varepsilon + \beta_0 \frac{2}{2+\delta} = \beta_0,$$

so (17) holds. By Lemma 2, $|\bar{y}(t)| \leq \beta_0 a^2(t)$, so (22) yields

$$\int_T^\infty \bar{y}^2(s)ds \leq \beta_0^2 \int_T^\infty a^4(s)ds < \infty,$$

and this completes the proof of the theorem. □

To show the applicability of Theorem 3, we have the following corollary.

Corollary 1. Suppose $q \in C^2(R_+)$ and $q' \geq 0$. If

$$\int_0^\infty \left| \frac{q''(t)}{q^{\frac{3}{2}}(t)} - \frac{5[q'(t)]^2}{4q^{\frac{5}{2}}(t)} \right| dt < \infty, \quad \int_0^\infty \frac{1}{\sqrt{q(t)}} dt < \infty, \tag{23}$$

and

$$\lim_{t\to\infty}\left(r(t)-\frac{q'(t)}{2}\right)=0, \tag{24}$$

then (19) is of the limit-circle type.

Proof. By a well known result of Dunford and Schwartz [6; p. 1414], (23) guarantees that equation (15) is of the limit-circle type. Condition (24) ensures the existence of $T \geq 0$ so that (20) holds. □

Remark 1. A result similar to Corollary 1 above was proved by Došlá in [5; Corollary 1] under the conditions that $q \in C^2(R_+)$, (23) holds, $\int_0^\infty |r(t)-\frac{q'(t)}{2}|dt < \infty$, and $\int_0^\infty \frac{q'_-(t)}{q(t)}dt < \infty$, where $q_-(t) = \max(-q(t), 0)$. It is easy to see that these two results are independent of each other.

If we apply Corollary 1 to equation (6), we obtain that (6) is of the limit-circle type provided $m > 2$, $n = m-1$, $mq = 2r$, and $f(x) \equiv x$. Observe that under these restrictions on n and m, (6) has the form of Appel's equation.

The next lemma will be used to prove our final limit circle result for linear equations, namely, Theorem 4 below.

Lemma 3. Let (H) hold and assume that:

(i) $\varepsilon > 0$ and $2r(t)\alpha_1 \geq q'(t) + \varepsilon$ for $t \geq T$;

(ii) either (a) $\int_0^\infty q(t)dt = \infty$, (b) $\int_0^\infty tr(t)dt = \infty$, (c) $\int_0^\infty tq(t)dt < \infty$ and $\alpha_1 r(t) \geq \frac{\sigma}{t^3}$ for $t \geq T$ and some $\sigma > 1$, or (d) $\int_0^\infty q(t)dt < \infty$ and $q(t) + \alpha_1 tr(t) \geq \frac{\sigma}{t^2}$ for $t \geq T$ and some $\sigma > 4$;

(iii) either $\mathcal{O} = \emptyset$, or y' is bounded for every $y \in \mathcal{O}$, or $y \in L^2(R_+)$ for every $y \in \mathcal{O}$.

Then equation (1) is of the nonlinear limit-circle type.

Proof. Let y be a nontrivial solution of (1); by Lemma 1, $y \in \mathcal{O}\cup W_1 \cup W_2 \cup \mathcal{N}_0 \cup \mathcal{N}_1$. If $y \in W_1 \cup W_2$, then Theorem 3 in [4] implies $y \in L^2(R_+)$. If $y \in \mathcal{N}_0$, then by Theorem 5 in [4], $y \in L^2(R_+)$.

If the first or second condition in (ii) holds, then $\mathcal{N}_1 = \emptyset$ by Theorem 7(ii) in [4]. If (ii)(c) holds, then Corollary 3 in [2] implies $\mathcal{N}_1 = \emptyset$. Theorems 1 and 5 in [2] imply $\mathcal{N}_1 = \emptyset$ if (ii)(d) holds.

The last case is that $y \in \mathcal{O} \neq \emptyset$ and $y \notin L^2(R_+)$. Define the function $Z(t) = -2y''y + (y')^2 - qy^2$. Then

$$Z'(t) = 2ryf(y) - q'y^2 \geq (2\alpha_1 r - q')y^2 \geq \varepsilon y^2,$$

for $t \geq T$, and so Z is nondecreasing. Let $\{t_k\} \to \infty$ be a sequence of zeros of y. Then,

$$Z(t_k) = (y'(t_k))^2 \leq M, k = 1, 2, \ldots$$

by (iii), so $Z(t) \leq M$ for $t \geq T$. Integrating Z', we obtain

$$\int_T^\infty y^2(t)dt \leq \frac{1}{\varepsilon}\int_T^\infty Z'(t)dt = \frac{1}{\varepsilon}(Z(\infty) - Z(T)) < \infty.$$

Thus, in all cases, $y \in L^2(R_+)$, and the lemma is proved. □

Remark 2. Lemma 3 guarantees that every nonoscillatory solution is in $L_2(R_+)$ if (H), (i), and (ii) hold. A special case of this assertion is proved in [11, Theorem 9].

Theorem 4. Let $\int_0^\infty \frac{2r(t)-q'(t)}{q(t)}dt < \infty$, and suppose there are constants $\varepsilon > 0$ and $T > 0$ such that

$$2r(t) - q'(t) \geq \varepsilon, \quad \text{and} \quad q^{-\frac{1}{4}} \text{ is convex on } [T, \infty).$$

Then equation (19) is of the limit-circle type.

Proof. First, we prove that

$$\lim_{t\to\infty} q(t) = \infty. \tag{25}$$

Since $q^{-\frac{1}{4}}$ is convex on $[T,\infty)$, $\lim_{t\to\infty} q(t) = q_0 \geq 0$ exists. Moreover, the hypotheses of the theorem imply

$$\infty > \int_T^\infty \frac{2r(t) - q'(t)}{q(t)} dt \geq \int_T^\infty \frac{\varepsilon dt}{q(t)},$$

so $q_0 = \infty$. Now (25) and Theorem 3.18 in [10] imply that every oscillatory solution y of (1) has a bounded derivative y' on R_+, and so the conclusion of the theorem follows from Lemma 3. □

We conclude this section with an application of Theorem 4.

Corollary 2. Suppose there are positive constants ε, ε_1, q_0, b_0, and m and a function $B \in C^0(R_+)$ such that

$$q(t) = \frac{2q_0 t^{m+1}}{m+1},\ r(t) = q_0 t^m + B(t), \text{ and } \varepsilon \leq B(t) \leq b_0 t^{m-\varepsilon_1}.$$

Then equation (19) is of the limit-circle type.

Clearly, it would be desirable to relax condition (H) to allow for more general nonlinear functions f. Also, extending the results in this section to nonlinear equations would be of interest. The result of Staněk could continue to be useful in this regard.

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