PROCEEDINGS OF THE INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
May 18 – 21, 2000, Atlanta, USA pp. 174–181

SUBHARMONIC OSCILLATIONS FOR SOME SECOND-ORDER DIFFERENTIAL EQUATIONS WITHOUT LANDESMAN-LAZER CONDITIONS

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Abstract. We prove the existence of a sequence of kT-periodic solutions of equation

$$u''(t) + g(t, u(t)) = e(t),$$

with amplitudes and minimal periods tending to infinity, without assuming Landesman-Lazer conditions. We study also the existence of an infinite number of solutions of the above equation assuming that the non-linearity g has a one-sided growth restriction.

1. **Introduction.** We consider the scalar second-order differential equation

$$u''(t) + g(t, u(t)) = e(t) \tag{1}$$

assuming that

- the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is T-periodic in its first variable, satisfies a Charathéodory condition and, for every positive constant R, there is a $h_R \in L^1_{loc}(\mathbb{R})$ with $|g(t,x)| \leq h_R(t)$ for a.e. $t \in \mathbb{R}$ and all $|x| \leq R$.
- the function $e : \mathbb{R} \to \mathbb{R}$ is locally integrable and T-periodic.

We denote by $\bar{e}$ the mean value of e, i.e., $\bar{e} = \frac{1}{T}\int_0^T e(t)\,dt$ and by $G(t,x) = \int_0^x g(t,s)ds$, a primitive of $g(t,.)$.

Our first aim is to study the existence of a sequence of kT-periodic solutions of equation (1), $(u_k)_{k\geq 0}$, with amplitudes and minimal periods tending to infinity, without assuming Landesman-Lazer conditions. Our second aim concerns the existence of an infinite number of solutions of (1) in the case where the non-linearity g has a one-sided growth restriction. The theorems we prove generalize some results contained in [2]. In fact, it was proved in [2] that, equation (1) has a sequence $(u_k)_{k\geq 0}$ of kT-periodic solutions, with amplitudes and minimal periods tending to infinity, assuming that the non-linearity g satisfies the condition

(H_1) for every $\epsilon > 0$ there exist two functions $\alpha_\epsilon(t)$, $\beta_\epsilon(t) \in L^1(0,T)$ such that, for a.e. $t \in [0,T]$ and every $x \in \mathbb{R}$,

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1991 *Mathematics Subject Classification.* Primary: 34B15, 34C25.
Key words and phrases: Subharmonic solutions, truncation procedure, *a priori* estimates .
[1]Supported by CMAF, Fundação para a Ciência e a Tecnologia.
[2]Supported by Dep. de Matemática, FCT, Universidade Nova de Lisboa.

$$G(t,x) \le \epsilon x^2 + \alpha_\epsilon(t)|x| + \beta_\epsilon(t)$$

and the Landesman-Lazer conditions (H_2)–(H_3)

(H_2) there exists $h(t) \in L^1(0,T)$ such that, for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$,

$$sgn(x)g(t,x) \ge h(t),$$

(H_3)

$$\frac{1}{T}\int_0^T \limsup_{x\to-\infty} g(t,x)\,dt < \bar{e} < \frac{1}{T}\int_0^T \liminf_{x\to+\infty} g(t,x)\,dt.$$

It was also remarked that conditions (H_2) and (H_3) can be replaced by

(H_4) there exists $d > 0$ such that, for a.e. $t \in [0,T]$ and all $|x| > d$,

$$sgn(x)[g(t,x) - \bar{e}] > 0;$$

and the Ahmad-Lazer-Paul condition

(H_5)

$$\lim_{|x|\to+\infty}\left[\frac{1}{T}\int_0^T G(t,x)\,dt - x\bar{e}\right] = +\infty.$$

These ones are better conditions when we are dealing with a non-linearity $g(t,x)$ that does not depend on t. On the other hand, the Landesman-Lazer conditions allow the non-linearity g to assume negative values for some points t. In Theorem 1, we obtain a result under weaker conditions that replace the previous ones.

In Theorem 2, we prove the existence of an infinite number of solutions of (1) in the case where the non-linearity has a one-sided growth restriction. To deal with this situation we use some energy estimates, used in [6], and a truncation of squareroot type that allows us to consider non-linearities that do not satisfy $(H_2) - (H_3)$ nor $(H_4) - (H_5)$. Under our assumptions, the function $\Upsilon(x) = \int_0^T g(t,x) - \bar{e}\,dt$ can vanish for arbitrarily large negative values of x and is assumed to be non-negative on some arbitrarily large intervals of positive x. The last condition was used in [3] where a function that satisfies it was called "one-sided generalized expansive"; it was also used in [7] to treat a singular problem. We note that such statements on the function $\Upsilon(x)$ were not possible under conditions $(H_2) - (H_3)$ nor $(H_4) - (H_5)$

Throughout the paper we denote by H^1_{kT} the space of functions in the Sobolev space H^1 that are kT-periodic, where we consider the usual norm $\|u\|_{kT} = \int_0^{kT} [u^2(t) + \dot{u}^2(t)]\,dt$. We consider the decomposition $H^1_{kT} = \mathbb{R} \oplus \widetilde{H}^1_{kT}$ where we identify the set of constant functions with $\mathbb{R}$ and $\widetilde{H}^1_{kT}$ is the subspace of functions with mean value zero over $[0,kT]$. For each $u \in H^1_{kT}$ we write $u = \bar{u} + \tilde{u}$, where $\bar{u} \in \mathbb{R}$ and $\tilde{u} \in \widetilde{H}^1_{kT}$.

2. **Symmetric conditions.** Let us introduce the following assumption

(H_6) there exist $h_+, h_- \in L^1(0,T)$, with $\int_0^T h_+ \ge 0$, $\int_0^T h_- \le 0$, and $M \in \mathbb{R}^+$ such that for a.e. $t \in [0,T]$

$$g(t,x) - \bar{e} > h_+(t) \quad \text{for all} \quad x > M,$$
$$g(t,x) - \bar{e} < h_-(t) \quad \text{for all} \quad x < -M.$$

We have the following result.

Theorem 1. *Suppose that g satisfies the conditions (H_1), (H_5) and (H_6). Then the equation (1) has a sequence $(u_k)_{k\in\mathbb{N}}$ of kT-periodic solutions with amplitudes and minimal periods tending to infinity.*

We observe that condition (H_6) is clearly weaker than (H_4), just take $h_+, h_- \equiv 0$. It is known that the Landesman-Lazer conditions (H_2), (H_3) imply (H_5) (See [8], [4]). The following proposition holds.

Proposition 1. *The Landesman-Lazer conditions $(H_2) - (H_3)$ imply (H_6).*

Proof: We will prove it for $x > 0$. The case $x < 0$ is analogous. Consider

$$\Omega = \{t \in [0,T] : \liminf_{x\to+\infty} g(t,x) = +\infty\}.$$

Suppose that $|\Omega| > 0$. Fix $\epsilon > 0$ and define for each $n \in \mathbb{N}$

$$\Omega_n = \left\{ t \in \Omega : g(t,x) \geq \frac{2}{|\Omega|} \int_0^T |h(t) - (\epsilon + \bar{e})|\, dt + \epsilon + \bar{e},\ \forall x > n \right\},$$

where h is given by (H_2). We have that Ω_n is measurable and $\Omega_n \subset \Omega_{n+1}$, for every $n \in \mathbb{N}$, and $\cup_{n\in\mathbb{N}}\Omega_n = \Omega$. So, $|\Omega_n| \to |\Omega|$, when $n \to +\infty$. Fix $n_0 \in \mathbb{N}$ such that $|\Omega_{n_0}| > |\Omega|/2$. Define

$$h_+(t) = \begin{cases} h(t) - (\epsilon + \bar{e}) & \text{if } t \in [0,T] \setminus \Omega_{n_0}, \\ \frac{2}{|\Omega|} \int_0^T |h(t) - (\epsilon + \bar{e})|\, dt & \text{if } t \in \Omega_{n_0}. \end{cases}$$

Then if we take $M = n_0$ we have

$$g(t,x) \geq h_+(t) + \bar{e} + \epsilon > h_+(t) + \bar{e}, \quad \text{for all } x > M, \text{ a.e. } t \in [0,T].$$

So, (H_6) holds. Consider now the case $|\Omega| = 0$. By (H_3), we can fix $\epsilon > 0$ in such a way that

$$\int_0^T \left[\liminf_{x\to+\infty} g(t,x) - \bar{e} - 2\epsilon \right] dt > 0.$$

Define for each $n \in \mathbb{N}$

$$\Omega'_n = \{t \in [0,T] : g(t,x) \geq \liminf_{x\to+\infty} g(t,x) - \epsilon,\ \forall x > n\}.$$

It is clear that Ω'_n is measurable and $\Omega'_n \subset \Omega'_{n+1}$, for every $n \in \mathbb{N}$, and $\cup_{n\in\mathbb{N}}\Omega'_n = [0,T] \setminus \Omega$. So, $|\Omega'_n| \to T$ when $n \to +\infty$. Let us fix $n_0 \in \mathbb{N}$ in such a way that

$$\int_{\Omega'_{n_0}} \left[\liminf_{x\to+\infty} g(t,x) - \bar{e} - 2\epsilon \right] dt + \int_{[0,T]\setminus\Omega'_{n_0}} [h(t) - (\epsilon + \bar{e})]\, dt > 0$$

and define

$$h_+(t) = \begin{cases} h(t) - (\epsilon + \bar{e}) & \text{if } t \in [0,T] \setminus \Omega'_{n_0}, \\ \liminf_{x\to+\infty} g(t,x) - \bar{e} - 2\epsilon & \text{if } t \in \Omega'_{n_0}. \end{cases}$$

If we take $M = n_0$, it follows as above that condition (H_6) is satisfied, which completes the proof of the claim.

Example: As an example of a non-linearity that satisfies (H_5) and (H_6) we have

$$g(t,x) = t - \frac{T}{2} + \frac{x}{1+x^2} + \bar{e}.$$

Observe that this function does not satisfy (H_3) nor (H_4).

Proof of Theorem 1: It is clear that (H_6) implies (H_2) and, as was remarked in [2], conditions (H_1), (H_2) and (H_5) are enough to prove the existence of a sequence of solutions $(u_k)_{k\in\mathbb{N}}$ such that $\|u_k\|_{k\in\mathbb{N}}$ are unbounded. So, (H_6) will provide conditions to guarantee the existence of subharmonic solutions. Following the proof of Theorem 2.1 in [2], we see that in order to reach the desired conclusion it is enough to rule out the possibility of the existence of a sequence of kT-periodic solutions $(u_k)_{k\in\mathbb{N}}$ of (1) in such a way that $\min_{t\in[0,kT]}|u_k| \to +\infty$ as $k \to +\infty$. Suppose that there is a subsequence, that we denote in the same way, with $\min_{t\in[0,kT]} u_k \to +\infty$ (the other case can be treated similarly). If $k_0 \in \mathbb{N}$ is such that $u_{k_0}(t) > M$ for every $t \in [0, kT]$ then

$$\int_0^{k_0T} g(t, u_{k_0}(t))\, dt > \int_0^{k_0T} h^+(t) + \bar{e}\, dt \geq k_0T\bar{e}.$$

However, integration of (1) shows that the left hand side of the above inequality is $k_0T\bar{e}$ which results in a contradiction.

3. **One-sided sublinear non-linearities.** Let us consider the following condition

(H_6') there exist $h_+, h_- \in L^1(0,T)$, with $\int_0^T h_+ \geq 0$, $\int_0^T h_- \leq 0$, and a constant $M \in \mathbb{R}^+$ such that

$$g(t,x) - \bar{e} \leq h_-(t) \text{ for all } x < -M, \text{ for a.e. } t \in [0,T]$$

and that for all $\rho > 0$ there are $d, D > M$ satisfying $D - d > \rho$ and

$$g(t,x) - \bar{e} \geq h_+(t) \text{ for all } x \in [d,D] \text{ a.e. } t \in [0,T].$$

The following theorem states the existence of infinitely many solutions of (1) such that, under some more conditions on g, are subharmonic.

Theorem 2. *Suppose that g satisfies (H_5), (H_6'). Moreover, assume that*

(H_7) *there exists a differentiable function $F :]-\infty, 0] \to \mathbb{R}$ satisfying*

$$\lim_{x\to-\infty} \frac{F(x)}{x^2} = 0$$

and such that, for a.e. $t \in \mathbb{R}$ and all $x \leq 0$,

$$g(t,x) \geq F'(x),$$

(H_8) *there exists $h \in L^1_{loc}(\mathbb{R})$ such that for a.e. $t \in \mathbb{R}$ and for every $x > 0$*

$$g(t,x) \geq h(t).$$

Then, equation (1) has a sequence of kT-periodic solutions, $(u_k)_{k\in\mathbb{N}}$, such that $\|u_k\|_\infty$ is unbounded (so an infinite number of such solutions are distinct).

If in addition g verifies (H_6) then the amplitudes and the minimal periods tend to infinity.

Remark 1. By subtracting $\bar{e}$ to both members of equation (1), it is easy to see that we can assume without loss of generality that $\bar{e} = 0$ (and use in the proofs the assumptions with the obvious modifications).

The following truncation will be used throughout the paper. Given a function $g : \mathbb{R}^2 \to \mathbb{R}$ L^1-Carathéodory and $r < 0 < R$, define

$$g_{r,R}(t,x) = \begin{cases} g(t,r) + \sqrt{-r} - \sqrt{-x} & \text{if } x < r \\ g(t,x) & \text{if } r \le x \le R \\ g(t,R) - \sqrt{R} + \sqrt{x} & \text{if } x > R. \end{cases}$$

We note that $g_{r,R}$ is L^1-Carathéodory. For each $r, R > 0$, define $G_{r,R}(t,x) = \int_0^x g_{r,R}(t,s)ds$.

The following lemma gives us some estimates for the periodic solutions of

$$u''(t) + g_{r,R}(t, u(t)) = e(t). \tag{2}$$

The above equation will be referred throughout the paper as $(2)_{r,R}$.

Lemma 1. *Suppose that g satisfies the assumptions of Theorem 3.1. Then for each $k \in \mathbb{N}$ there exist $r_k < -M < 0 < R_k$ so that every kT-periodic solution of $(2)_{r_k,R_k}$ satisfies $r_k < u(t) < R_k$, $\forall t \in \mathbb{R}$. (So it is a kT-periodic solution of (1) too.)*

Proof: Taking the above remark into account, assume $\bar{e} = 0$. Conditions (H_6') and (H_7) give us sequences $M \le d_n < D_n$ and $D_n^- < -M$ such that $(D_n - d_n) \to +\infty$, $D_n^- \to -\infty$ as n tends to infinity, and

$$g(t,x) \ge h_+(t) \quad \text{for} \quad x \in [d_n, D_n], \text{ a.e. } t \in [0,T]; \tag{3}$$

$$\lim_{n\to+\infty} \frac{F'(D_n^-)}{D_n^-} = 0. \tag{4}$$

In order to prove the assertion we argue by contradiction. So, let us suppose the existence of $k \in \mathbb{N}$ in such a way that for each $n \in \mathbb{N}$ there exists a kT-periodic solution u_n of $(2)_{D_n^-,D_n}$ verifying

$$\{u_n(t) : t \in \mathbb{R}\} \not\subset [D_n^-, D_n]. \tag{5}$$

We claim that the following statements hold:

(C *1*) Passing to a subsequence, if necessary, we have $\min u_n \to -\infty$ as $n \to +\infty$.
(C *2*) For each $n \in \mathbb{N}$ there exists $t_n^1 \in [0, kT]$ such that $u_n(t_n^1) > -M$.

Suppose that (C *1*) does not hold. Then, writing $u_n = \bar{u}_n + \tilde{u}_n$ with $\bar{u}_n \in \mathbb{R}$ and $\tilde{u}_n \in \widetilde{H}_{kT}^1$, we can conclude arguing as in [2], [4], that there exists $l > 0$ such that $\|\tilde{u}_n\|_\infty < l$ and $\min u_n \to +\infty$ as $n \to +\infty$. By (5), there exists $p \in \mathbb{N}$ such that if $n > p$ then $\max u_n > D_n$. On the other hand, if $n > p$,

$$\min u_n \ge \bar{u}_n - l \ge D_n - 2l.$$

So, if we choose $n_0 > p$ such that $(D_{n_0} - d_{n_0}) > 2l$, we will have

$$u_{n_0}(t) \ge d_{n_0}, \forall t \in [0,T]$$

and

$$\{t \in \mathbb{R} : u_{n_0}(t) > D_{n_0}\} \ne \emptyset.$$

Now, using (H_6') and (3)

$$\begin{aligned}
\int_0^{kT} g_{D_{n_0}^-,D_{n_0}}(t,u_{n_0}(t))\,dt &\geq \int_{d_{n_0}\leq u_{n_0}(t)\leq D_{n_0}} g(t,u_{n_0}(t))\,dt \\
&+ \int_{u_{n_0}(t)>D_{n_0}} g(t,D_{n_0}) - \sqrt{D_{n_0}} + \sqrt{u_{n_0}(t)}\,dt > 0, \\
&\geq \int_{d_{n_0}\leq u_{n_0}(t)\leq D_{n_0}} h_+(t)\,dt + \int_{u_{n_0}(t)>D_{n_0}} h_+(t) - \sqrt{D_{n_0}} + \sqrt{u_{n_0}(t)}\,dt \\
&= \int_0^{kT} h_+(t)\,dt + \int_{u_{n_0}(t)>D_{n_0}} -\sqrt{D_{n_0}} + \sqrt{u_{n_0}(t)}\,dt > 0
\end{aligned}$$

which is a contradiction with the result obtained by integration of equation $(2)_{D_{n_0}^-,D_{n_0}}$ Therefore, $(C\ 1)$ is proved.

As for $(C\ 2)$, assume by contradiction that for some $n_0 \in \mathbb{N}$ we have $u_{n_0}(t) \leq -M$, $\forall t \in \mathbb{R}$. By (H_6') and since, by (5),

$$\{t \in \mathbb{R} : u_{n_0}(t) < D_{n_0}^-\} \neq \emptyset,$$

we have

$$\begin{aligned}
\int_0^{kT} g_{D_{n_0}^-,D_{n_0}}(t,u_{n_0}(t))\,dt &\leq \int_{D_{n_0}^-\leq u_{n_0}(t)\leq -M} g(t,u_{n_0}(t))\,dt \\
&+ \int_{u_{n_0}(t)<D_{n_0}^-} g(t,D_{n_0}^-) + \sqrt{-D_{n_0}^-} - \sqrt{-u_{n_0}(t)}\,dt \\
&\leq \int_0^{kT} h_+(t)\,dt + \int_{u_{n_0}(t)<D_{n_0}^-} \sqrt{-D_{n_0}^-} - \sqrt{-u_{n_0}(t)}\,dt < 0
\end{aligned}$$

which results in a contradiction as in the proof of $(C\,1)$.

By $(C\ 1)$ and $(C\ 2)$ we can conclude that for every sufficiently large $n \in \mathbb{N}$ there exists an interval $[\alpha_n, \beta_n]$ with $(\beta_n - \alpha_n) \leq kT$ and $t_n^2 \in [\alpha_n, \beta_n]$ such that $\min\ u_n(t) = u_n(t_n^2)$ and

$$\begin{aligned}
u_n(\alpha_n) &= -M = u_n(\beta_n), \\
u_n(t_n^2) &\leq u_n(t) \leq -M, \quad \forall t \in [\alpha_n, \beta_n].
\end{aligned}$$

Defining

$$v_n'(t) := -g_{D_n^-,D_n}(t,u_n(t)) \quad \text{and} \quad v_n(\alpha_n) := u_n'(\alpha_n)$$

by integration of $(2)_{D_n^-,D_n}$ we have

$$u_n'(t) = v_n(t) + \int_{\alpha_n}^t e(s)\,ds. \tag{6}$$

Since, by (H_6'),

$$v_n'(t) + h_-(t) \geq 0, \ \forall t \in [\alpha_n, \beta_n]$$

then $v_n(.) + \int_{\alpha_n}^{\cdot} h_-(s)ds$ is increasing. Note also that by Lagrange theorem

$$u_n(t_n^2) - u_n(\alpha_n) = u_n'(\xi)(t_n^2 - \alpha_n),$$

with $\xi \in [t_n^2, \alpha_n]$ and satisfying moreover $u_n'(\xi) \leq 0$. So, we deduce, using also (6), that

$$\min u_n + M = u_n(t_n^2) - u_n(\alpha_n) \geq kT(v_n(\alpha_n) - k\|e - h_-\|_1). \tag{7}$$

So, $v_n(\alpha_n) \to -\infty$ when $n \to +\infty$ and, for n large enough, $v_n(\alpha_n) < -k\|e\|_1$. On the other hand, by (6), $v_n(t_n^2) \geq -k\|e\|_1$. Thus, there exists the smallest value $t_n^3 \in [\alpha_n, t_n^2]$ such that $v_n(t_n^3) = -k\|e\|_1$.

For $r < 0$, consider the truncation

$$F_r'(x) = \begin{cases} F'(r) + \sqrt{-r} - \sqrt{-x} & \text{if } x < r \\ F'(x) & \text{if } r \leq x \leq 0 \end{cases}$$

and denote by $F_r(x) = F(0) + \int_0^x F_r'(s)\,ds$ one of the primitives.

By (H_7) and (H_6'), we observe that $F'(x) \leq h_-(t)$ for $x < -M$ and a.e. $t \in [0, T]$. But since $h_-(t)$ must be non-positive on a subset of $[0, T]$ of positive measure, it follows that $F'(x) \leq 0$ if $x < -M$. Then, for every $t \in [\alpha_n, t_n^3]$ we have, using (6)

$$\begin{aligned} &\frac{d}{dt}\left[F_{D_n^-}(u_n(t)) + \frac{1}{2}(v_n(t) + k\|e\|_1)^2\right] \\ = \;& F_{D_n^-}'(u_n(t))\left(v_n(t) + \int_{\alpha_n}^t e(s)\,ds\right) - g_{D_n^-, D_n}(t, u_n(t))(v_n(t) + k\|e\|_1) \\ \geq \;& F_{D_n^-}'(u_n(t))(v_n(t) + k\|e\|_1) - g_{D_n^-, D_n}(t, u_n(t))(v_n(t) + k\|e\|_1) \geq 0. \end{aligned}$$

We conclude that $F_{D_n^-}(u_n(.)) + \frac{1}{2}(v_n(.) + k\|e\|_1)^2$ is increasing in $[\alpha_n, t_n^3]$. Therefore, for n large, since $D_n^- < -M$,

$$F(-M) + \frac{1}{2}(v_n(\alpha_n) + k\|e\|_1)^2 \leq F_{D_n^-}(u_n(t_n^3)). \tag{8}$$

For every $x \leq D_n^- < -M < 0$,

$$\begin{aligned} F_{D_n^-}(x) &= F(D_n^-) - \int_x^{D_n^-} [F'(D_n^-) + \sqrt{-D_n^-} - \sqrt{-s}]\,ds \\ &\leq F(x) + (x - D_n^-)(F'(D_n^-) + \sqrt{-D_n^-}) - \frac{2}{3}(-D_n^-)^{\frac{3}{2}} + \frac{2}{3}(-x)^{\frac{3}{2}} \\ &\leq F(x) + xF'(D_n^-) + \frac{1}{3}(-D_n^-)^{\frac{3}{2}} + \frac{2}{3}(-x)^{\frac{3}{2}} \\ &\leq F(x) + x^2\frac{F'(D_n^-)}{D_n^-} + (-x)^{\frac{3}{2}}. \end{aligned}$$

Due to the definition of $F_{D_n^-}$, we note that the last inequality also holds for $D_n^- < x \leq -M$. Now by (H_7), (4) and the previous inequalities we conclude that for every $\epsilon > 0$ there exists C_ϵ and n_0 such that

$$F_{D_n^-}(x) \leq \epsilon x^2 + C_\epsilon\ , \ \forall n > n_0,\ x \leq -M. \tag{9}$$

Finally, using (7), (8) and (9), we have that if $n > n_0$

$$\begin{aligned} F(-M) + \frac{1}{2}(v_n(\alpha_n) + k\|e\|_1)^2 &\leq \epsilon(u_n(t_n^3))^2 + C_\epsilon \\ &\leq \epsilon(kT(v_n(\alpha_n) - k\|e - h^-\|_1) - M)^2 + C_\epsilon, \end{aligned}$$

which gives a contradiction for small ϵ when $n \to +\infty$ and finishes the proof.

Proof of Theorem 2: For every $k \in \mathbb{N}$ we consider $r_k < 0 < R_k$ given by Lemma 1 and define the following functional defined in H^1_{kT}

$$\varphi_k(u) := \int_0^{kT} [\frac{1}{2}\dot{u}^2(t) - G_{r_k,R_k}(t,u(t)) + e(t)u(t)]\, dt.$$

It is well known that this is a continuously differentiable functional and its critical points coincide with the weak solutions of $(1)_{r_k,R_k}$ (See [5]). As in the proof of [2] (Theorem 2.1.) we can show that the conditions of Saddle point theorem are satisfied. So, we conclude that equation $(1)_{r_k,R_k}$ has a kT-periodic solution u_k. By Lemma 1, u_k is a kT-solution of (1).

The following estimate similar to that on [1] holds

$$\lim_{k\to+\infty} \frac{1}{k}\varphi_k(u_k) = -\infty. \tag{10}$$

Its proof follows generically some ideas used in [2] together with some computations carefully modified accordingly to the case considered here.

If there exists $L > 0$ such that $\|u_k\|_\infty < L$, by the assumptions on g, we obtain

$$\begin{aligned} \frac{1}{k}\varphi_k(u_k) &\geq \frac{1}{k}\int_0^{kT} [-G(t,u(t)) + e(t)u(t)]\, dt \\ &\geq -\|u_k\|_\infty \frac{1}{k}\int_0^{kT} [|h_L(t)| + |e(t)|]\, dt \end{aligned}$$

for $h \in L^1(0,kT)$, which gives a contradiction with (10) and completes the proof of the first part of the conclusion.

For the rest of the proof we can follow the argument presented in the proof of Theorem 1.

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