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ANALYTIC SMOOTHING EFFECT AND GLOBAL EXISTENCE OF SMALL SOLUTIONS FOR THE ELLIPTIC-HYPERBOLIC DAVEY-STEWARTSON SYSTEM

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Abstract. Our purpose is to prove that if the initial data is sufficiently small in a certain weighted Sobolev space, then solutions of the elliptic-hyperbolic Davey-Stewartson system become analytic with respect to x for any $t \neq 0$ and the analytical domain of solutions increases with time.

1. **Introduction.** We consider the Cauchy problem for the Davey-Stewartson (D-S) systems

$$\begin{cases} i\partial_t u + \lambda_1 \partial_{x_1}^2 u + \partial_{x_2}^2 u = \mu_1 |u|^2 u + \mu_2 u \partial_{x_1} \phi, & (t,x) \in \mathbf{R} \times \mathbf{R}^2, \\ \partial_{x_1}^2 \phi + \lambda_2 \partial_{x_2}^2 \phi = \partial_{x_1} |u|^2, \\ u(0,x) = u_0(x), \quad x \in \mathbf{R}^2, \end{cases} \tag{D-S}$$

where $\lambda_1, \lambda_2 \in \mathbf{R}, \mu_1, \mu_2 \in \mathbf{C}$. Let u be a complex-valued function and ϕ be a real-valued function. These systems are classified about the sign of λ_1 and λ_2 by Ghidaglia-Saut [6]. For this pairing, $(+,+)$, $(+,-)$, $(-,+)$ and $(-,-)$ are called elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic respectively.

The Davey-Stewartson systems model the evolution of weakly nonlinear water waves that travel predominantly in one direction, but in which the wave amplitude is modulated slowly in two horizontal directions. Davey and Stewartson (1974) did not think of the effect of surface tension or capillarity. This effect was included by Djordjevic and Reddekopp (1977) who discovered that the parameter λ_2 can become negative when capillary effects are important. The details of the physical background are written in [1], [3] and [5].

Hence, we consider the elliptic-hyperbolic case, namely, (λ_1,λ_2) is $(+,-)$. Here, when we especially determine $\lambda_1 = 1$ and $\lambda_2 = -1$ and use the transformation,

$$\partial_{x_1} = \frac{1}{\sqrt{2}}(\partial_X - \partial_Y), \ \partial_{x_2} = \frac{1}{\sqrt{2}}(\partial_X + \partial_Y)$$

we get the following system

$$\begin{cases} i\partial_t u + (\partial_X^2 + \partial_Y^2) u = \mu_1 |u|^2 u + \mu_2 \dfrac{u}{\sqrt{2}} (\partial_X - \partial_Y)\phi \\ \partial_X \partial_Y \phi = -\dfrac{1}{\sqrt{2}} (\partial_X - \partial_Y)|u|^2. \end{cases}$$

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This is a joint work with N. Hayashi and P.I. Naumkin.

And applying the boundary condition in the infinite point of the function ϕ

$$\lim_{X\to\infty}\phi(t,X,Y)=\lim_{Y\to\infty}\phi(t,X,Y)=0,$$

we obtain the Schrödinger equation with nonlocal nonlinearities.

$$\begin{aligned} &i\partial_t u+\partial_X^2 u+\partial_Y^2 u\\ &=\mu_1|u|^2u+\frac{\mu_2}{2}\left(u\int_Y^\infty \partial_X|u|^2(X,Y')dY'-u\int_X^\infty \partial_Y|u|^2(X',Y)dX'\right).\end{aligned}$$

Hence, we consider the nonlinear Schrödinger equation with nonlocal terms.

$$\begin{cases} i\partial_t u+\partial_{x_1}^2 u+\partial_{x_2}^2 u \\ \quad = c_0|u|^2u+c_1u\displaystyle\int_{x_2}^\infty \partial_{x_1}|u|^2dx_2'+c_2u\int_{x_1}^\infty \partial_{x_2}|u|^2dx_1', \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in\mathbf{R}\times\mathbf{R}^2, \\ u(0,x)=u_0(x), \qquad x\in\mathbf{R}^2, \end{cases} \tag{NLS}$$

where $c_0,\ c_1,\ c_2\in\mathbf{C}$. It is characteristic that the nonlinear terms of this equation satisfy the gauge invariance. More precisely, they have the following properties with the operator J_{x_1} defined by $J_{x_1}(t)=x_1+2it\partial_{x_1}=\mathcal{M}_1 2it\partial_{x_1}\mathcal{M}_1^{-1}$, where $\mathcal{M}_1=\mathcal{M}_1(t)=\exp(ix_1^2/4t)$.

$$J_{x_1}\left(u\int_{x_1}^\infty \partial_{x_2}|u|^2dx_1'\right)=(J_{x_1}u)\int_{x_1}^\infty \partial_{x_2}|u|^2dx_1'-u\left(\overline{u}J_{x_2}u-u\overline{J_{x_2}u}\right) \tag{1}$$

and

$$u\int_{x_1}^\infty \partial_{x_2}|u|^2dx_1'=\frac{u}{2it}\int_{x_1}^\infty\left(\overline{u}J_{x_2}u-u\overline{J_{x_2}u}\right)dx_1'. \tag{2}$$

Especially, the second identity means that we obtain the time decay necessary to show the global existence of solutions. The same theory is also applied to J_{x_1}.

We give some function spaces and notations. We define the weighted Sobolev space $\mathbf{H}^{m,s,p}$ by

$$\begin{aligned} &\mathbf{H}^{m,s,p}=\{f\in L^p\ ;\ \|f\|_{m,s,p}<+\infty\}\\ &\|f\|_{m,s,p}=\|(1+x_1^2+x_2^2)^{s/2}(1-\partial_{x_1}^2-\partial_{x_2}^2)^{m/2}f\|_p,\quad m,s\in\mathbf{R}^+.\end{aligned}$$

And for convenience, we use $\|\cdot\|=\|\cdot\|_2$ and $\|\cdot\|_{m,s}=\|\cdot\|_{m,s,2}$. We consider the spatial dimension as two dimensions. Hence, we use $\mathbf{L}^p_{x_j}=\mathbf{L}^p(\mathbf{R}_{x_j})$, $\mathbf{L}^p_{x_1}\mathbf{L}^q_{x_2}=\mathbf{L}^p(\mathbf{R}_{x_1};\mathbf{L}^q(\mathbf{R}_{x_2}))$ and $\mathbf{H}^{m,s}_{x_j}=\mathbf{H}^{m,s}(\mathbf{R}_{x_j})$ for Lebesgue and Sobolev spaces in one space dimension. We let $\partial^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$, $|\alpha|=\alpha_1+\alpha_2$ and $J^\beta=J_{x_1}^{\beta_1}J_{x_2}^{\beta_2}$, $|\beta|=\beta_1+\beta_2$. We denote $\partial_{x_j}^{-1}=\int_{x_j}^\infty dx_j'$, $j=1,2$. $\mathbf{L}^2$-inner-product is defined by $(f,g)=\int\overline{f}gdx$.

Let $\mathcal{F}_j\phi$ or $\hat{\phi}$ be the Fourier transform of $\phi(x_j)$, namely

$$\hat{\phi}(\xi_j)=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}e^{-i\xi_jx_j}\phi(x_j)dx_j.$$

We also denote by $\mathcal{F}_j^{-1}\psi$ or $\check{\psi}$ the inverse Fourier transform of the function $\psi(\xi_j)$,

$$\check{\psi}(x_j)=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}e^{i\xi_jx_j}\psi(\xi_j)d\xi_j.$$

The free Schrödinger evolution group is defined by

$$\mathcal{U}_j(t)\phi = \mathcal{F}_j^{-1} e^{it|\xi_j|^2} \mathcal{F}_j \phi = \frac{1}{(4\pi i t)^{1/2}} \int_{-\infty}^{\infty} e^{i(x_j - s)^2/4t} \phi(s) ds.$$

The Hilbert transformation with repect to the variable x_1 is defined as following

$$\mathcal{H}_1\phi(x_1, x_2) \equiv \mathcal{H}_{x_1}\phi(x_1, x_2) = \frac{1}{\pi}\text{P.V.} \int_{\mathbf{R}} \frac{\phi(z, x_2)}{z - x_1} dz = i\mathcal{F}_1^{-1} \frac{\xi_1}{|\xi_1|} \mathcal{F}_1 \phi,$$

where P.V. means the principal value of the singular integral. The operator $\mathcal{H}_2$ is defined in the same way.

For each $r > 0$ we denote $S(r)$ the strip $\{z; -r < \text{Im} z_j < r; 1 \le j \le 2\}$ in the complex plane $\mathbf{C}^2$. For $x \in \mathbf{R}^2$, if a complex-valued function $f(x)$ has an analytic continuation to $S(r)$, then we denote this by the same letter $f(z)$ and if $g(z)$ is an analytic function on $S(r)$, then we denote the restriction of $g(z)$ to the real axis by the same letter $g(x)$. We let

$$\mathbf{A}^{m,s}(|\theta|) = \{f(z); f(z) \text{ is analytic on } S(|\theta|), \|f\|_{\mathbf{A}^{m,s}(|\theta|)} < \infty\},$$

where

$$\|f\|_{\mathbf{A}^{m,s}(|\theta|)} = \sup_{y \in (-|\theta|, |\theta|)^2} \|f(\cdot + iy)\|_{m,s}$$

$$\|f(\cdot + iy)\|^2 = \sum_{j,k=1}^{2} \left(\int |f(x_1 + i(-1)^j y_1, x_2 + i(-1)^k y_2)|^2 dx_1 dx_2 \right).$$

We state our main theorem.

Theorem 1. *We assume that*

$$\left\| \left(\prod_{j=1}^{2} \cosh \theta x_j \right) u_0 \right\|_{3,0} + \left\| \left(\prod_{j=1}^{2} \cosh \theta x_j \right) u_0 \right\|_{0,3} = \varepsilon$$

and ε is sufficiently small. Then there exists a unique global solution $u(t,x)$ of (NLS) such that $u(t,x)$ has an analytic continuation to $S(|2\theta t|)$ and

$$\exp(-iz^2/4t)u(t) \in \mathbf{A}^{0,2}(|2\theta t|), \quad \textit{for any } t.$$

Remark 1. From the fact that $\exp(-iz^2/4t)u(t) \in \mathbf{A}^{0,2}(|2\theta t|)$ it shows that the analytical domain of solutions increases with time. This implies the analytic smoothing effect of solutions to (NLS).

2. Linear Smoothing Effect.

We show our main theorem by the contraction mapping principle. However, since the nonlinearities of (NLS) have the so-called derivative-loss, we consider the linear smoothing effect for Schrödinger equations to overcome it. Hence, we study the Cauchy problem for the linear Schrödinger equations

$$\begin{cases} i\partial_t u + (\partial_{x_1}^2 + \partial_{x_2}^2)u = f, & x \in \mathbf{R}^2, \quad t \in \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^2, \end{cases} \tag{LS}$$

where $f = f(t, x_1, x_2)$ is a given function. To get the smooting effect of solutions for (LS), we make use of the opeartor introduced in [8]. The study of this sort of operators is started by Doi [4]. It is researched by Chihara [2], Hayashi and Hirata

[7], Hayashi and Naumkin [8] later. In this study, we use the following operator. We define the special operator $\mathcal{S}(\varphi) = \mathcal{X}(\varphi)\mathcal{Y}(\varphi)$, where

$$\mathcal{X}(\varphi) = \cosh(\varphi_1) + i\sinh(\varphi_1)\mathcal{H}_1 \quad \text{and} \quad \mathcal{Y}(\varphi) = \cosh(\varphi_2) + i\sinh(\varphi_2)\mathcal{H}_2,$$

the vector of the real-valued functions $\varphi(t,x_1,x_2) = (\varphi_1(t,x_1), \varphi_2(t,x_2))$, is such that its components $\varphi_j \in \mathbf{L}^\infty(0,T;\mathbf{H}^{2,0,\infty}_{x_j}) \cap \mathbf{C}^1([0,T];\mathbf{L}^\infty_{x_j})$ are positive. From this definition we easily see that the operator $\mathcal{S}$ acts continuously from $\mathbf{L}^2$ to $\mathbf{L}^2$ with the following estimate

$$\|\mathcal{S}(\varphi)\psi\| \leq 2\exp(\|\varphi\|_\infty)\|\psi\|, \tag{3}$$

where $\|\varphi\|_\infty = \|\varphi_1\|_{\mathbf{L}^\infty_x} + \|\varphi_2\|_{\mathbf{L}^\infty_y}$.

The inverse operator $\mathcal{X}^{-1} = (1 + i\tanh(\varphi_1)\mathcal{H}_1)^{-1}\frac{1}{\cosh(\varphi_1)}$ exists and is continuous in L^2 :

$$\|\mathcal{X}^{-1}(\varphi_1)\psi\| \leq (1 - \tanh(\|\varphi\|_\infty))^{-1}\|\psi\| \leq \exp(\|\varphi\|_\infty)\|\psi\|.$$

The operator $\mathcal{Y}^{-1}$ is similarly considered. Therefore $\mathcal{S}^{-1}$ is well-defined.

The operator $\mathcal{S}$ helps us to obtain a smoothing property of the Schrödinger type equation (LS) by virtue of the usual energy estimates. In Lemma 1 we prepare an energy estimate involving the operator $\mathcal{S}$, in which we have an additional positive term giving us the half derivative of the unknown function u. We assume that $\varphi_j(x_j)$ is written by ω_j as $\varphi_j(x_j) = \partial_{x_j}^{-1}(\omega_j^2)$, so that $\omega_j(x_j) = \sqrt{(\partial_{x_j}\varphi_j)}$.

Here we define the fractional derivative $|\partial_{x_j}|^\gamma$, $\gamma \in (0,1)$ by

$$|\partial_{x_j}|^\gamma\phi = -\mathcal{F}_j^{-1}|\xi_j|^\gamma\mathcal{F}_j\phi = -C\int_{\mathbf{R}}(\phi(x_j+z,x_k) - \phi(x_j,x_k))\frac{dz}{|z|^{1+\gamma}},$$

where $j,k = 1,2$, $k \neq j$ and C is some constant (see [10]).

We make use of the following two lemmas.

Lemma 1 (Hayashi, N. and Naumkin, P.I. [8]). *The following inequality*

$$\begin{aligned}&\frac{d}{dt}\|\mathcal{S}u\|^2 + \left\|\omega_1\mathcal{S}\sqrt{|\partial_{x_1}|}u\right\|^2 + \left\|\omega_2\mathcal{S}\sqrt{|\partial_{x_2}|}u\right\|^2\\ &\leq 2\left|\mathrm{Im}(\mathcal{S}u,\mathcal{S}f)\right| + C\|u\|^2e^{2\|\varphi\|_\infty}\left(\|\omega\|^2_{1,0,\infty} + \|\omega\|^6_\infty + \|\varphi_t\|_\infty\right)\end{aligned}$$

is valid for the solution u of the Cauchy problem (LS).

We also need the following lemma in order to estimate nonlinear terms of equations.

Lemma 2 (Hayashi, N. and Naumkin, P.I. [8]). *We have the following estimate*

$$\begin{aligned}&\left|(\mathcal{S}u,\mathcal{S}\phi\partial_{x_2}^{-1}(\psi\partial_{x_1}w)\right|\\ &\leq 4\exp(4\|\varphi\|_\infty)\left(\left\|\|\phi\|_{\mathbf{L}^2_{x_2}}\mathcal{S}\sqrt{|\partial_{x_1}|}u\right\|^2 + \left\|\|\psi\|_{\mathbf{L}^2_{x_2}}\mathcal{S}\sqrt{|\partial_{x_1}|}w\right\|^2\right)\\ &\quad + C\left(\|u\|^2 + \|w\|^2\right)\exp(6\|\varphi\|_\infty)(\|\phi\|^2_{\mathbf{L}^\infty_{x_1}\mathbf{L}^2_{x_2}} + \|\phi_{x_1}\|^2_{\mathbf{L}^\infty_{x_1}\mathbf{L}^2_{x_2}}\\ &\quad + \|\psi\|^2_{\mathbf{L}^\infty_{x_1}\mathbf{L}^2_{x_2}} + \|\psi_{x_1}\|^2_{\mathbf{L}^\infty_{x_1}\mathbf{L}^2_{x_2}})\left(1 + \|\varphi\|^2_{1,0,\infty}\right),\end{aligned}$$

provided that the right hand sides are bounded.

The nonlinear terms of (NLS) have the derivative-loss. It is difficult to make use of the usual energy method for it. Hence, we apply the above two lemmas to it. When we multiply (NLS) by $J^\beta\partial^\alpha$ and sum over $|\alpha|+|\beta|\le 3$, we have the inequality

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|+|\beta|\le 3}\|J^\beta\partial^\alpha u(t)\|^2+\sum_{j=1}^{2}\sum_{|\alpha|+|\beta|\le 3}\|\omega_j\mathcal{S}\sqrt{|\partial_{x_j}|}J^\beta\partial^\alpha u(t)\|^2\\
&\le 16\exp\left(\frac{8}{\delta}\|u(t)\|^2\right)\sum_{1\le j,k\le 2,k\ne j}\sum_{|\alpha|+|\beta|\le 3}\left\|\|u(t)\|_{\mathbf{L}^2_{x_k}}\mathcal{S}\sqrt{|\partial_{x_j}|}J^\beta\partial^\alpha u(t)\right\|^2\\
&\quad+C(1+|t|)^{-1}\exp\left(\frac{6}{\delta}\|u(t)\|^2\right)\left(\sum_{m+s\le 2}\|u(t)\|^2_{m,s}\right)\left(\sum_{m+s\le 3}\|u(t)\|^2_{m,s}\right),
\end{aligned}$$

where we choose as the function φ in the operator $\mathcal{S}$,

$$\varphi(t,x_1,x_2)=(\varphi_1(t,x_1),\varphi_2(t,x_2)),$$
$$\begin{cases}\varphi_1(t,x_1)=\dfrac{1}{\delta}\partial_{x_1}^{-1}\|u(t,x_1)\|^2_{\mathbf{L}^2_{x_2}},\\[2mm] \varphi_2(t,x_2)=\dfrac{1}{\delta}\partial_{x_2}^{-1}\|u(t,x_2)\|^2_{\mathbf{L}^2_{x_1}}.\end{cases}$$

After this, considering as the positive constant δ,

$$1-16\delta\exp\left(\frac{8}{\delta}\|u(t)\|^2\right)\ge 0,$$

we overcome the derivative-loss and obtain the energy estimate.

In order to show that the solutions for (NLS) which are dependent of the initial data u_0 with an exponential decay become analytic in the domain $S(|2\theta t|)$, we need consider the analytical versions of above lemmas. Here we use the norm equivalence in the following way.

$$\begin{aligned}
&\|(1+x_1^2+x_2^2)^{s/2}\mathcal{U}(t)\cosh\theta(x_1+x_2)\mathcal{U}(-t)(1-\partial_{x_1}^2-\partial_{x_1}^2)^{m/2}u\|^2\\
&\simeq\|e^{i|\cdot|^2/4t}(1-\partial_{x_1}^2-\partial_{x_2}^2)^{m/2}u\|^2_{\mathbf{A}^{0,s}(|2\theta t|)}.
\end{aligned}$$

By this relation, we can consider the problem with the above norm. We state the analytical versions of Lemma 1 and 2.

Lemma 3. *We have*

$$\begin{aligned}
&\frac{d}{dt}\|\mathcal{S}h_{m,s}\|^2+\left\|\omega_1\mathcal{S}\sqrt{|\partial_{x_1}|}h_{m,s}\right\|^2+\left\|\omega_2\mathcal{S}\sqrt{|\partial_{x_2}|}h_{m,s}\right\|^2\\
&\le 2\left|\mathrm{Im}(\mathcal{S}h_{m,s},\mathcal{S}\mathcal{U}(t)(1+x_1^2+x_2^2)^{s/2}e^{\theta x_1+\zeta x_2}\mathcal{U}(-t)(1-\partial_{x_1}^2-\partial_{x_2}^2)^{m/2}f)\right|\\
&\quad+C\|h_{m,s}\|^2e^{2\|\varphi\|_\infty}\left(\|\omega\|^2_{1,0,\infty}+\|\omega\|^6_\infty+\|\varphi_t\|_\infty\right),
\end{aligned}$$

where $h_{m,s}=\mathcal{U}(t)(1+x_1^2+x_2^2)^{s/2}e^{\theta x_1+\zeta x_2}\mathcal{U}(-t)(1-\partial_{x_1}^2-\partial_{x_2}^2)^{m/2}u.$

Proof. Since $\mathcal{U}(-t)x\mathcal{U}(t)$ and $i\partial_t+\partial_{x_1}^2+\partial_{x_2}^2$ are commutable, we obtain our desired result from Lemma 1. □

Lemma 4. *We have the following estimate*

$$
\begin{aligned}
&\left|(\mathcal{S}\mathcal{F}^{-1}e^{-(\theta\xi_1+\zeta\xi_2)}\mathcal{F}u, \mathcal{S}\mathcal{F}^{-1}e^{-(\theta\xi_1+\zeta\xi_2)}\mathcal{F}\phi\partial_{x_2}^{-1}(\psi\partial_{x_1}w))\right| \\
&\leq 4e^{4\|\varphi\|_\infty}\left(\left\| \|\phi(\cdot+i\theta,\cdot+i\zeta)\|_{\mathbf{L}^2_{x_2}}\mathcal{S}\sqrt{|\partial_{x_1}|}u(\cdot+i\theta,\cdot+i\zeta)\right\|^2 \right. \\
&\quad \left. + \left\| \|\psi(\cdot+i\theta,\cdot+i\zeta)\|_{\mathbf{L}^2_{x_2}}\mathcal{S}\sqrt{|\partial_{x_1}|}w(\cdot+i\theta,\cdot+i\zeta)\right\|^2 \right) \\
&\quad + Ce^{6\|\varphi\|_\infty}\left(\|u(\cdot+i\theta,\cdot+i\zeta)\|^2 + \|w(\cdot+i\theta,\cdot+i\zeta)\|^2\right) \\
&\quad \times \left(\|\phi(\cdot+i\theta,\cdot+i\zeta)\|^2_{\mathbf{L}^\infty_{x_1}\mathbf{L}^2_{x_2}} + \|\phi_{x_1}(\cdot+i\theta,\cdot+i\zeta)\|^2_{\mathbf{L}^\infty_{x_1}\mathbf{L}^2_{x_2}}\right. \\
&\quad \left. + \|\psi(\cdot+i\theta,\cdot+i\zeta)\|^2_{\mathbf{L}^\infty_{x_1}\mathbf{L}^2_{x_2}} + \|\psi_{x_1}(\cdot+i\theta,\cdot+i\zeta)\|^2_{\mathbf{L}^\infty_{x_1}\mathbf{L}^2_{x_2}}\right)\left(1+\|\varphi\|^2_{1,0,\infty}\right),
\end{aligned}
$$

where $\mathcal{F} = \mathcal{F}_1\mathcal{F}_2$ *and* $\mathcal{F}^{-1} = \mathcal{F}_2^{-1}\mathcal{F}_1^{-1}$.

Proof. By the identity

$$
\begin{aligned}
&\mathcal{F}^{-1}e^{-(\theta\xi_1+\zeta\xi_2)}\mathcal{F}(\phi(x_1,x_2)\psi(x_1,x_2)\varphi(x_1,x_2)) \\
&= \phi(x_1+i\theta, x_2+i\zeta)\psi(x_1+i\theta, x_2+i\zeta)\varphi(x_1+i\theta, x_2+i\zeta),
\end{aligned}
$$

and Lemma 2, the lemma follows. □

3. **The Outline of the Proof of Main Theorem.** In this section, we state the outline of the proof of Theorem 1. First of all, we introduce the necessary norm defined by

$$
\|u(t)\|_{X^k(t)} = \sum_{m+s\leq k} \|e^{i|\cdot|^2/4t}(1-\partial_{x_1}^2-\partial_{x_2}^2)^{m/2}u\|_{\mathbf{A}^{0,s}(|\theta t|)}
$$

and the closed ball to show it,

$$
\begin{aligned}
\mathbf{B} = \Big\{ & v \in C([0,\infty); \mathbf{H}^{3,0}\cap\mathbf{H}^{0,3}) \ : \\
& \sup_{t\in[0,\infty)} \|v(t)\|_{X^2(t)} \leq \nu, \qquad \sup_{t\in[0,\infty)} (1+|t|)^{-C\rho}\|v(t)\|_{X^3(t)} \leq \rho, \\
& \sup_{t\in[0,\infty)} (1+t)\sum_{0\leq j,k\leq 1} \left\| \partial_t\partial_{x_1}^{-1}\|\mathcal{W}_{j,k}(t)v(t,x_1)\|^2_{L_{x_2}} \right\|_{L^\infty_{x_1}} \leq \sigma, \\
& \sup_{t\in[0,\infty)} (1+t)\sum_{0\leq j,k\leq 1} \left\| \partial_t\partial_{x_2}^{-1}\|\mathcal{W}_{j,k}(t)v(t,x_2)\|^2_{L_{x_1}} \right\|_{L^\infty_{x_2}} \leq \sigma \Big\},
\end{aligned}
\tag{4}
$$

where

$$
\mathcal{W}_{j,k}(t) = \mathcal{U}(t)e^{\theta((-1)^j x_1 + (-1)^k x_2)}\mathcal{U}(-t)
$$

and ν, ρ and σ are some positive constants dependent on the initial data ε.

Since we shall prove our main theorem by the contraction mapping principle in a suitable analytic function space, we consider the following system of equation:

$$
i\partial_t\mathbf{u} + (\partial_{x_1}^2 + \partial_{x_2}^2)\mathbf{u} = c_0\mathbf{F}_0 + c_1\mathbf{F}_1 + c_2\mathbf{F}_2, \tag{5}
$$

where for $j = 1, 2$,

$$\mathbf{u} = \begin{pmatrix} u \\ \partial_{x_j} u \\ \partial_{x_j}^2 u \\ \partial_{x_j}^3 u \\ J_{x_j} u \\ J_{x_j}^2 u \\ J_{x_j}^3 u \end{pmatrix}, \qquad \mathbf{F}_l = \begin{pmatrix} F_{l,0} \\ \vdots \\ F_{l,6} \end{pmatrix}$$

and

$$F_{0,k} = \begin{cases} \partial_{x_j}^k (|v|^2 v), & 0 \le k \le 3, \\ J_{x_j}^{k-3}(|v|^2 v), & 4 \le k \le 6, \end{cases}$$

$$F_{1,k} = \begin{cases} \partial_{x_j}^k (v \partial_{x_2}^{-1} \partial_{x_1} |v|^2), & 0 \le k \le 2, \\ \partial_{x_j}^3 (v \partial_{x_2}^{-1} (v \partial_{x_1} \overline{u} + \overline{v} \partial_{x_1} u)), & k = 3, \\ J_{x_j}^{k-3} (v \partial_{x_2}^{-1} \partial_{x_1} |v|^2), & 4 \le k \le 5, \\ J_{x_j}^3 (v \partial_{x_2}^{-1} (v \partial_{x_1} \overline{u} + \overline{v} \partial_{x_1} u)), & k = 6, \end{cases}$$

$$F_{2,k} = \begin{cases} \partial_{x_j}^k (v \partial_{x_1}^{-1} \partial_{x_2} |v|^2), & 0 \le k \le 2, \\ \partial_{x_j}^3 (v \partial_{x_1}^{-1} (v \partial_{x_2} \overline{u} + \overline{v} \partial_{x_2} u)), & k = 3, \\ J_{x_j}^{k-3} (v \partial_{x_1}^{-1} \partial_{x_2} |v|^2), & 4 \le k \le 5, \\ J_{x_j}^3 (v \partial_{x_1}^{-1} (v \partial_{x_2} \overline{u} + \overline{v} \partial_{x_2} u)), & k = 6, \end{cases}$$

where u is an unknown function and v is a known function in $\mathbf{B}$. For the above system equation, we especially take care about the terms $F_{1,3}$, $F_{1,6}$ for $j = 1$ and $F_{2,3}$, $F_{2,6}$ for $j = 2$ of the above equation. We prove that the mapping

$$u = \mathcal{P} v$$

defined by the linearized equation (5) transforms the closed ball $\mathbf{B}$ into itself.

In order to apply Lemma 3 and 4 to (5), we choose the following function φ of the operator $\mathcal{S}(\varphi)$:

$$\varphi_j(t, x_j) = \frac{1}{\delta} \partial_{x_j}^{-1} (\varphi_{j,\theta,\theta}(t, x_j) + \varphi_{j,\theta,-\theta}(t, x_j) + \varphi_{j,-\theta,\theta}(t, x_j) + \varphi_{j,-\theta,-\theta}(t, x_j)), \tag{6}$$

where $j = 1, 2$ and

$$\varphi_{j,\theta,\varsigma}(t, x_j) = \left\| \mathcal{U}(t) e^{\theta x_1 + \varsigma x_2} \mathcal{U}(-t) v(t, x_1, x_2) \right\|_{\mathbf{L}^2_{x_k}}^2,$$

for $k = 1, 2$, $j \ne k$ so that $\varphi_j \in \mathbf{L}^\infty(0, T; \mathbf{H}_{x_j}^{2,0,\infty}) \cap \mathbf{C}^1([0, T]; \mathbf{L}_{x_j}^\infty)$. We note $\omega = (\omega_1, \omega_2)$, $\omega_1 = \sqrt{\partial_{x_1} \varphi_1}$, $\omega_2 = \sqrt{\partial_{x_2} \varphi_2}$ and $\|\omega\|_\infty = \|\omega_1\|_{\mathbf{L}_{x_1}^\infty} + \|\omega_2\|_{\mathbf{L}_{x_2}^\infty}$. Then, we obtain by Sobolev's inequality,

$$\begin{aligned} \|\omega\|_\infty &\le \frac{C(\rho)}{\sqrt{\delta}} (1+t)^{-1/2}, & \|\omega\|_{1,0,\infty} &\le \frac{C(\rho)}{\sqrt{\delta}} (1+t)^{-1/2}, \\ \|\varphi\|_\infty &\le \frac{C(\rho)}{\delta}, & \|\varphi_t\|_\infty &\le \frac{C(\sigma)}{\delta} (1+t)^{-1}. \end{aligned} \tag{7}$$

After this, by Lemma 3, Lemma 4 and (7), we get the inequality with $\| \cdot \|_{X^3(t)}$

$$\|u(t)\|_{X^3(t)}^2 \le C\varepsilon^2 + C\varepsilon^2 \int_0^t (1+\tau)^{-1} \|u(\tau)\|_{X^3(\tau)}^2 d\tau.$$

Gronwall's inequality yields

$$(1+t)^{-C\rho^2}\|u(t)\|_{X^3(t)} \le \rho.$$

Applying a classical energy method to (5) directly and using the property of the operator J as in (2), we have

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_{X^2(t)} &\le C\nu^2(1+t)^{-2}(\|v(t)\|^2_{X^3(t)} + \|u(t)\|^2_{X^3(t)})\\ &\le C\nu^2\rho^2(1+t)^{-2+C\rho^2}.\end{aligned} \tag{8}$$

Hence we obtain the estimate with respect to the norm $\|\cdot\|_{X^2(t)}$.

We consider the first equation of (5). Multiplying this equation by $\mathcal{W}_{0,0}(t) = \mathcal{U}(t)e^{\theta(x_1+x_2)}\mathcal{U}(-t)$ and using the identity

$$\begin{aligned}&\mathcal{U}(t)e^{\theta x_1+\zeta x_2}\mathcal{U}(-t)(\phi\psi\overline{v})\\ &= (\mathcal{U}(t)e^{\theta x_1+\zeta x_2}\mathcal{U}(-t)\phi)(\mathcal{U}(t)e^{\theta x_1+\zeta x_2}\mathcal{U}(-t)\psi)\overline{(\mathcal{U}(t)e^{-(\theta x_1+\zeta x_2)}\mathcal{U}(-t)v)},\end{aligned}$$

we get

$$\begin{aligned}&i\partial_t\mathcal{W}_{0,0}(t)u + (\partial^2_{x_1}+\partial^2_{x_2})\mathcal{W}_{0,0}(t)u\\ &= c_0(\mathcal{W}_{0,0}(t)v)^2\overline{\mathcal{W}_{1,1}(t)v}\\ &\quad + c_1\mathcal{W}_{0,0}(t)v\int_{x_2}^{\infty}(\overline{\mathcal{W}_{1,1}(t)v}\mathcal{W}_{0,0}(t)\partial_{x_1}v - \mathcal{W}_{0,0}(t)v\overline{\mathcal{W}_{1,1}(t)\partial_{x_1}v})dx_2'\\ &\quad + c_2\mathcal{W}_{0,0}(t)v\int_{x_1}^{\infty}(\overline{\mathcal{W}_{1,1}(t)v}\mathcal{W}_{0,0}(t)\partial_{x_2}v - \mathcal{W}_{0,0}(t)v\overline{\mathcal{W}_{1,1}(t)\partial_{x_2}v})dx_1'.\end{aligned}$$

When we multiply this equation by $\overline{\mathcal{W}_{0,0}(t)u}$, integrate it over $\mathbf{R}_{x_2}$, take the imaginary part of it, integrate it over $[-\infty, x_1]$ and take the supremum with respect to x_1 over $\mathbf{R}$, we have the estimate by the inequalities (7),

$$\left\|\int_{-\infty}^{x_1} i\partial_t\|\mathcal{W}_{0,0}(t)u(t,x_1')\|^2_{\mathbf{L}^2_{x_2}}dx_1'\right\|_{\mathbf{L}^\infty_{x_1}} \le C(\rho)(1+t)^{-1} \le \sigma(1+t)^{-1}.$$

In the same way, we have the estimates

$$\sup_{t\in[0,\infty)}(1+t)\sum_{0\le j,k\le 1}\|\,\partial_t\partial_{x_1}^{-1}\|\mathcal{W}_{j,k}(t)u(t,x_1)\|^2_{L^2_{x_2}}\,\|_{L^\infty_{x_1}} \le \sigma$$

and

$$\sup_{t\in[0,\infty)}(1+t)\sum_{0\le j,k\le 1}\|\,\partial_t\partial_{x_2}^{-1}\|\mathcal{W}_{j,k}(t)u(t,x_2)\|^2_{L^2_{x_1}}\,\|_{L^\infty_{x_2}} \le \sigma.$$

Thus the mapping $\mathcal{P}$ transforms the closed ball $\mathbf{B}$ into itself. Now we show $\mathcal{P}$ is a contraction mapping in the norm $\sup_{t\in[0,\infty)}\|\cdot\|_{X^2(t)}$. Let $\tilde{v}$ be the known function in the ball $\mathbf{B}$, $w_u = u - \tilde{u}$ and $w_v = v - \tilde{v}$. Then we consider

$$i\partial_t\mathbf{w} + (\partial^2_{x_1}+\partial^2_{x_2})\mathbf{w} = c_0\mathbf{G}_0 + c_1\mathbf{G}_1 + c_2\mathbf{G}_2, \tag{9}$$

where

$$\mathbf{w} = \begin{pmatrix} w_u \\ \partial_{x_j}w_u \\ \partial^2_{x_j}w_u \\ J_{x_j}w_u \\ J^2_{x_j}w_u\end{pmatrix}, \qquad \mathbf{G}_l = \begin{pmatrix} G_{l,0}\\ \vdots \\ G_{l,4}\end{pmatrix}$$

and

$$G_{0,k} = \begin{cases} \partial_{x_j}^k(|v|^2 w_v + \tilde{v}\bar{v}w_v + \tilde{v}^2\bar{w}_v), & 0 \le k \le 2, \\ J^{k-2}(|v|^2 w_v + \tilde{v}\bar{v}w_v + \tilde{v}^2\bar{w}_v), & 3 \le k \le 4, \end{cases}$$

$$G_{1,k} = \begin{cases} \partial_{x_j}^k(w_v\partial_{x_2}^{-1}\partial_{x_1}|v|^2 + \tilde{v}\partial_{x_2}^{-1}((\partial_{x_1}\tilde{v})\bar{w}_v + (\partial_{x_1}\bar{\tilde{v}})w_v)) \\ \qquad + \tilde{v}\partial_{x_2}^{-1}((\partial_{x_1}w_v)\bar{v} + (\partial_{x_1}\bar{w}_v)v), & 0 \le k \le 2, \\ J_{x_j}^{k-2}(w_v\partial_{x_2}^{-1}\partial_{x_1}|v|^2 + \tilde{v}\partial_{x_2}^{-1}((\partial_{x_1}\tilde{v})\bar{w}_v + (\partial_{x_1}\bar{\tilde{v}})w_v)) \\ \qquad + \tilde{v}\partial_{x_2}^{-1}((\partial_{x_1}w_v)\bar{v} + (\partial_{x_1}\bar{w}_v)v), & 3 \le k \le 4, \end{cases}$$

$$G_{2,k} = \begin{cases} \partial_{x_j}^k(w_v\partial_{x_1}^{-1}\partial_{x_2}|v|^2 + \tilde{v}\partial_{x_1}^{-1}((\partial_{x_2}\tilde{v})\bar{w}_v + (\partial_{x_2}\bar{\tilde{v}})w_v)) \\ \qquad + \tilde{v}\partial_{x_1}^{-1}((\partial_{x_2}w_v)\bar{v} + (\partial_{x_2}\bar{w}_v)v), & 0 \le k \le 2, \\ J_{x_j}^{k-2}(w_v\partial_{x_1}^{-1}\partial_{x_2}|v|^2 + \tilde{v}\partial_{x_1}^{-1}((\partial_{x_2}\tilde{v})\bar{w}_v + (\partial_{x_2}\bar{\tilde{v}})w_v)) \\ \qquad + \tilde{v}\partial_{x_1}^{-1}((\partial_{x_2}w_v)\bar{v} + (\partial_{x_2}\bar{w}_v)v), & 3 \le k \le 4. \end{cases}$$

In the same way as in the proof of (8), we obtain the inequality

$$\sup_{t\in[0,\infty)} \|w_u\|_{X^2(t)} \le \frac{1}{2} \sup_{t\in[0,\infty)} \|w_v\|_{X^2(t)}. \tag{10}$$

Thus $\mathcal{P}$ is a contraction mapping in the norm $\sup_{t\in[0,\infty)} \|\cdot\|_{X^2(t)}$. This is desired result.

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